OPTIMISED STATISTICAL PERFORMANCE IN AN ADAPTIVE BEAMFORMER

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ABSTRACT

A modified least squares adaptive steered beamformer that aims to eliminate the excess weight noise induced by a strong desired signal is proposed. It functions by alternating projection of the array data onto a subspace orthogonal to both the desired signal and interferences. The desired signal can be finally recovered, while retaining interference cancellation, by the use of a commutation property of the projections. The new method generalizes previous solutions to this problem which have relied on ideal plane wave signal models and it can operate with imperfect knowledge of the array manifold while still achieving the appropriate Cramér-Rao lower bounds on signal estimation.

Index Terms— constrained least squares beamformer, blind source separation, excess weight jitter, Cramér Rao bound

1. INTRODUCTION

In least squares adaptive filters the weight solution takes the form $W = R^{-1}C$ where *R* is the multivariate data covariance and *C* is the correlation vector between the data and the target signal, all using *the same data set*. Matrix *R* can be ill-conditioned and approximations to this solution generally work badly. For example, in a blind adaptive beamformer no target signal exists since the desired signal waveform is not known *a priori* and here it is commonplace to replace *C* by an external vector derived from the beam pointing direction, substituting $W_S = \hat{R}_{N+S}^{-1}\tilde{S}$ where $\tilde{S}$ is the precomputed steering vector and R_{N+S} is the sampled target - signal + noise covariance. This approximation, variously called sample matrix inversion (SMI), Capon's estimate, Constrained Least Squares (CLS) or Minimum Variance Distortionless Response (MVDR), is often accepted as the best that can be done in the circumstances. However, it is an effective approach only in the absence of a target signal. In applications like passive sonar, primary radar and certain communication systems, the desired signal strength may be appreciable and covariance estimate $\hat{R}_{N+S}$ becomes corrupted by 2nd order intermodulation products which increase the weight covariance $Cov(W_S)$ and boost unwanted noise output, a phenomenon termed *excess weight noise*. CLS adaptive solutions output suboptimum SNR's and a strong desired signal can even drive the array into a negative gain region unless there is increased training.

The statistical nature of the problem was described by Boroson [1] who derived probability distributions for CLS array gains when using limited training. He showed that with zero desired signal the directivity relative to optimum, $\rho = g/g_{OPT}$, is a random variable which follows the Beta distribution $P_G(\rho) = B\rho^{T-N+1}(1-\rho)^{N-2}$ where *T* is the number of samples used for estimating $\hat{R}_{N+S}$, *N* is the number of array elements and *B* is a normalization. However, in the presence of a desired signal causing enhanced weight noise, the reduced directivity is g' and the normalized gain becomes $\rho' = g'/g_{OPT}$. If the signal is Gaussian, arrives from the matched direction θ_0, $S_0 = S(\theta_0)$, and has asymptotic output SNR metric $\gamma = p_S S_0^H \tilde{R}_N^{-1} S_0$ where $\tilde{R}_N$ is the interference covariance and p_S is the desired signal power, the PDF of the finite sample reduced gain is functionally related to that of ρ via $\rho' = \rho/[1+\gamma(1-\rho)]$ and the PDF of ρ' can be derived from $P_G(\rho)$ by the standard change of variate method using Jacobian $|d\rho'/d\rho| = [1+\gamma(1-\rho)]^2/(1+\gamma)$. Since $\rho' \le \rho$ there is always a performance loss in CLS which increases as the desired signal level gets larger.

To illustrate the magnitude of the problem, fig. 1 shows the theoretical average gain impairment of a 5 element array which estimates $W_S = \hat{R}_{N+S}^{-1}\tilde{S}$ from limited numbers of samples *T* when the data consists of a single look direction signal plus uncorrelated noise. Some simulation points are shown as circles.

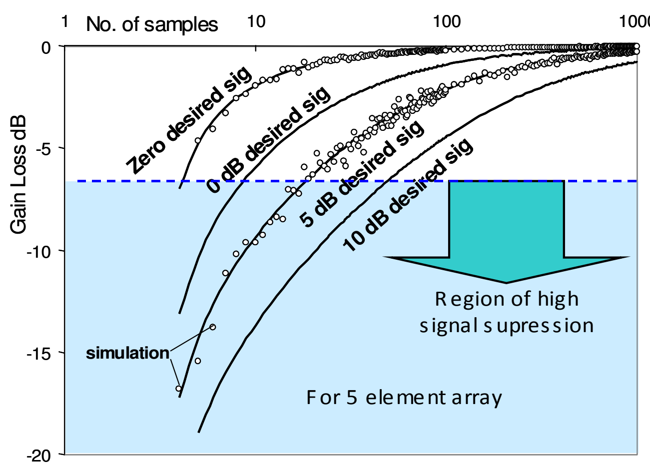

Fig 1: Average gain loss of a 5 element least squares steered array vs number of samples, with look-direction signals of -∞, 0, +5, +10 dB element SNR.

For *T*=7 samples and zero signal a 3.01 dB gain loss occurs but, if the input SNR is increased to 5 dB (per sensor value), γ takes value 5√10 =15.8 and the degradation increases to 10.7 dB: the array actively suppresses the signal, becoming less sensitive than an omnidirectional sensor. Alternatively, more samples (~63) are required just to maintain the original 3 dB loss. The region where the loss exceeds conventional array gain and CLS performance is worse than an omni single sensor is shaded grey. Of course for $T \to \infty$, CLS performance is asymptotically optimum, even with a desired signal present and, here, equivalent to a conventional steered array but protracted training does not allow good results for time varying data.

In direct contradiction to these findings, it is known from such statistical theory as the Cramér-Rao Lower Bound (CRLB) [2], [5], and discussed further later, that there should be little or no impairment of optimum beamformer performance in the presence of a strong desired signal and this, in turn, implies the CLS solution is not a *statistically efficient estimator* in relation to the CRLB. This is a great difficulty with CLS whose foundations seem questionable. Capon's estimate was long ago dismissed as *inconsistent* [8] and it is rare to find any signal processing system being fielded that is statistically suboptimum, failing to achieve the CRLB or a near equivalent.

Searching for alternatives to CLS to get a better adaptive array throws up a number of candidates and some well-known blind source separation (BSS) systems already approach CRLB optimality with limited training. Independent component analysis (ICA) achieves near optimum SNR's but only for non-Gaussian signals. Direct likelihood search operations in weight vector space [4] work well with Gaussian signals but use a difficult error minimization procedure. The MUSIC algorithm [2], [3] is known to be a near-optimum estimator of signal directions which can be used to set up adaptive weight solutions and estimate source amplitudes but its one-dimensional search method requires the array manifold $S(\theta)$ be known accurately. The Davies sequential null-steering tree works with performance approaching MUSIC for limited training data in a digital environment, provided the extra step is done of eliminating the desired signal as well as the interferences during training [6], [7]. The desired signal is recovered later simply by removing its null.

The new method described below is an attempt to generalize the latter ideas to the case where the array element locations are not known accurately. The desired signal being the cause of the weight noise, according to [1], the ideal is to filter it out at the beamformer inputs *before* cancelling interferences, then subsequently restore it without disturbing the nulls of the latter. But this goal is near-impossible to achieve. Given the desired signal direction θ_0, and assuming the array manifold is known, the signal could be removed from the array inputs by projecting the data onto a subspace orthogonal to space vector S_0 but the projection has the side effect of distorting the interference space vectors and the nulls subsequently formed in their vector directions evaporate if the signal-orthogonal projection is later removed to restore the original array data. So the goal of eliminating the interferences is not achieved.

The solution is to commute the order of signal and interference elimination. It is shown that it is possible to remove the interferences *before* the signal and still get the desired effects as follows:

(i) project the array data onto a subspace orthogonal to all the interferences
(ii) project the residual data onto a subspace orthogonal to the desired signal $\tilde{S}$
(iii) Repeat at (i) until convergence is achieved
(iv) Finally remove the desired signal projection

Projections (i) and (ii) interact since the interferences and desired signal do not lie in orthogonal spaces so the process has to be essentially iterative. It would seem hard to distinguish desired signal and unknown interference in reverse order, but it proves successful if the desired signal space vector is exactly known and a suitable optimizing metric is selected. The algorithm is described below.

2. THE ALGORITHM

The proposed method is represented in fig 2. The interference is structured as an array of space vectors forming $N \times K$ matrix $Q = [S_1 \cdots S_K]$, initially unknown, whose signals are removed from the input data X by aiming for linear projection $P_N = I - Q(Q^H Q)^{-1} Q^H$ through minimization of an error metric.

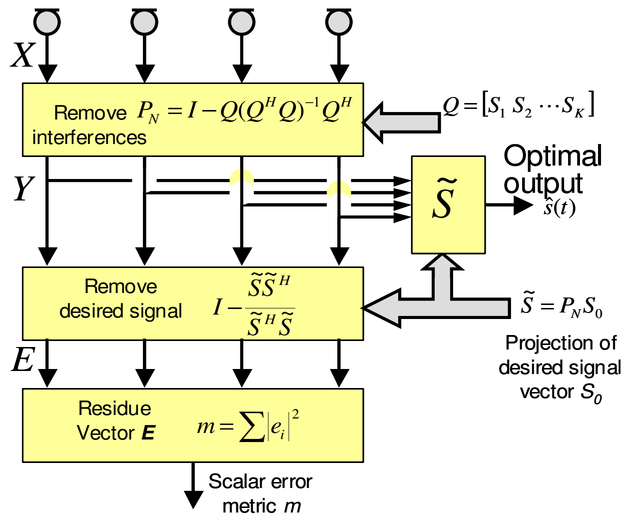

Fig. 2: The alternating projection algorithm

The modified desired-signal space vector remaining after projection P_N is $\tilde{S} = P_N S_0$ and has two further functions: (i) the desired signal output waveform estimate is $\hat{s}(t) = \tilde{S}^H P_N X(t)$ (ii) a second projection $P_S = I - \tilde{S}(\tilde{S}^H \tilde{S})^{-1} \tilde{S}^H$ is applied to remove the desired signal leaving a residual uncorrelated noise vector $E = P_S P_N X$ in the orthogonal subspace. The error metric is the L_2 norm $m = |E|^2$ summed over the training data. The two projections are iteratively applied to the data set, updating and fine tuning $[S_1 \cdots S_K]$ by minimization of m.

It may not be obvious that reversed order projections have the required effect on excess weight noise and to validate it we need to invoke a commutation property. The alternating projections converge to a projection $P_Z = I - Z(Z^H Z)^{-1} Z^H$ based on estimation of the union Z of signal and interference subspaces S_0 and Q, and this projection is invariant whether we partition Z as $Z = [S_0, Q]$ or as $Z = [Q, S_0]$, either choice having an equivalent like fig. 2. It is concluded that the converged-state residual vector $E = P_Z X$ remains the same irrespective of the order of removing desired signal and interference. Moreover, either signal or interference can be restored to the output without affecting cancellation of the other by deleting S_0 or Q from Z. In the practical algorithm we simply bypass the second projection and in fig. 2 this outputs the desired signal.

Training the beamformer is equivalent to estimating interference subspace Q and is done by minimizing scalar metric m over variation of the complex elements of $[S_1 \cdots S_K]$, a total of $2NK$ real & imaginary variables. A steepest descent method is quite reliable as m is almost quadratic near optimality. Alternatively, if the array manifold is known, postulate directions θ_k for the interferences giving space vectors $S_i = S(\theta_i)$ and do a K-dimensional search over the real vector $\Theta = [\theta_1 \cdots \theta_K]$ with a side constraint that no θ_k must get too close to the desired signal. This reduces the problem to the null steering tree of [7] for demonstration purposes, without loss of performance.

3. EXAMPLES OF PERFORMANCE

Simulation results are shown below for

a) CLS adaptive beamformer using finite-sample matrix inverse solution $W_S = \hat{R}_{N+S}^{-1} S_0$.

b) CLS performance prediction by Boroson as in [1]

c) MUSIC, converting estimates of interference directions to a null steering solution

d) The new alternating projection method

e) Asymptotic least squares performance, $T \to \infty$, equivalent to having prior knowledge of covariances and close to the CRLB

In the MUSIC algorithm, the estimated signal and interference directions are used to set up an adaptive beamformer with finite gain for the desired signal and nulls pointing to space vectors in the interference subspace matrix Q. Some prior knowledge and logic is needed here to decide which source directions belong to S_0 and Q.

The line antenna has 5 elements at locations {-0.5, 0.5, 1.0, 1.5, 3.5} wavelengths, two interferences are located at 71.6^o and 90^o with SNR's of 15 dB and 11 dB (per element) and the desired signal is located exactly at the steering direction of 0^o, and has variable power. After convergence of the projection algorithm, using parameterization $\Theta = [\theta_1 \cdots \theta_K]$, output SNR is plotted firstly *vs* input SNR with T = 20 samples in fig. 3 and secondly *vs* number of samples with a fixed input SNR = 5 dB in fig 4. The new algorithm and MUSIC show near-perfect performances close to signal-absent results and very near the CRLB. Traditional CLS performance agrees closely with the theory of [1] and has an output SNR degradation of 5 dB at input SNR=10 dB in the first test and its convergence rate is much slower in the second test.

4. CONCLUSION

It has been shown that a Constrained Least Squares blind beamformer suppresses its gain below theoretical CRLB predictions if there is a desired signal of high power present and, at worst, the desired signal can be highly suppressed even though the array is accurately steered. A new alternating projection algorithm eliminates the problem, improving adaptive array gain to be near-consistent with the CRLB which defines the theoretical performance limit. One-dimensional search algorithms like MUSIC already achieve similar results but are closely tied to plane wave signal models and need accurate array calibration whereas the new method also has the potential to operate with uncertain array manifolds, within sensible limits.

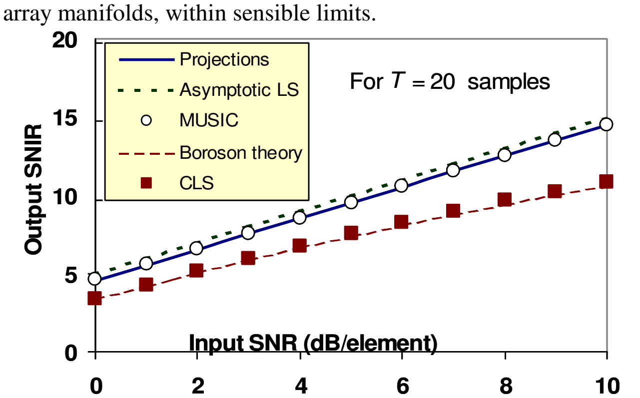

Fig. 3: Performances vs target-signal SNR 20 data samples

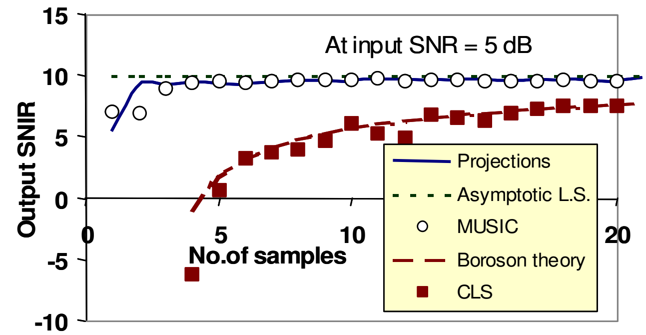

Fig. 4: Performances vs number of samples at 5 dB target SNR

The technique offers a very significant improvement in output SNR and/or allows reduced training in relation to the dynamics of the scenario, giving an advantage in various blind adaptation applications in sonar, communication and radar where there is a significant desired signal. The compressed training is also important in wideband adaptive systems. When these are based on multiple frequency domain filters the real-time sampling rate becomes very low and training is slow.

The penalty for the improved performance is an order of magnitude increase in computation load over linear least squares solutions, roughly equivalent to a new LS solution being required for each iteration of the alternating projections. In partial compensation, the target metric m is near-quadratic, well behaved and there is good control over which subspaces constitute desired signal and interference.

The method remains sensitive to desired signal direction errors since the second projection P_S fails to remove the desired signal if look direction θ_0 is in error. However the pointing direction can be iteratively tuned by minimizing m in the second projection. Using more complexity, the desired signal space vector S_0 can be determined in N-vector space when the array manifold is uncertain.

5. REFERENCES

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APPENDIX – CRAMÉR-RAO LOWER BOUNDS

The prediction of signal estimation accuracy by antenna arrays has generally focused on two signal models: a sum of constant amplitude sinusoids arriving from different directions and a similar sum of zero mean Gaussian signals. Extensive discussion of the Cramér Rao Lower Bounds for such models is found in [5].

In the first case, sometimes called the Swerling I model, the data consists of multiple plane wave signals in Gaussian noise and is appropriate for blind MIMO communications channels with slow trackable fading. The PDF of the real-valued array data for each data sample is

$$P(X \mid V) = \frac{1}{(2\pi)^{N/2}\left|R_0\right|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(X-V)^T R_0^{-1}(X-V)\right]$$

Here the $2N$-vector of sampled real and imaginary data components is $X \in R^{2N}$, the array sum of sinusoids $\{V_i\}$ is $V = \sum a_k S_k$ where a_k is the amplitude, and $S_k = \{\exp(i\theta_{k,1} \cdots i\theta_{k,N})\}$ is the space vector, of the k^{th} source and the background noise covariance is R_0. The Cramér Rao lower bounds on parameter estimates are the diagonal elements of the inverse of the Fisher information matrix given by

$$J_{ab} = -\mathrm{E}\left|\frac{\partial^2}{\partial\alpha_a \partial\alpha_b}\log P\right| = \frac{\partial V}{\partial \alpha_a} R_0^{-1} \frac{\partial V}{\partial \alpha_b}$$

Here α_j is any one of the parameters $a_k, \theta_k \quad k = 1,2,..$ The CRLB for T iid samples is the single sample variance divided by T.

While we cannot have a CRLB for complex parameters, there is no difficulty with using complex data, so long as the relevant estimated parameters are real. Thus in the narrow-band complex data case we can write

$$P(X \mid V) = \frac{1}{\pi^N \left|R_0\right|}\exp\left[-(X-V)^H R_0^{-1}(X-V)\right]$$

where $(.)^H$ is an Hermitian transpose and this model can still be used for real-valued CRLB parametersIn [5] it is shown that, when signals are reasonably spaced apart in bearing, their amplitude estimates interact very little and best output SNR's are similar to when the sources are present only one at a time. This indicates that the SNR of a weak signal is not significantly affected by the appearance of strong interferences in directions differing by more than half a beamwidth. Additionally, amplitude estimation accuracy does not depend on amplitude. Therefore the "weight jitter" effect and SNR degradation that strong signals cause in CLS, discussed earlier, do not make an appearance in the Cramér Rao bounds,.In the second case of independent and identically distributed (iid) Rayleigh-distributed fast-fading signals in Gaussian noise, sometimes called the Swerling II model and appropriate for noise-like signals, the sample covariance matrix $\sum X_t X_t^H$ follows the complex Wishart distribution and is a sufficient statistic for parameter estimation. By analogy with Swerling I, inserting zero means, we get the sample PDF

$$P(X_1 \ldots X_T \mid R) = \frac{1}{\pi^{TN}\left|R\right|^T}\prod_{t=1\cdots T}\exp\left[-X_t^H R^{-1} X_t\right]$$

$$= \frac{1}{\pi^{TN}\left|R\right|^T}\exp\left[-Tr\left(R^{-1}\sum_t X_t X_t^H\right)\right]$$

The asymptotic complex covariance matrix is $R = R_0 + \sum_{k=1..K} p_k S_k S_k^H$ where p_k is the k^{th} source power, S_k is the associated space vector, R_0 is the background noise covariance and the Fisher information matrix for estimation of the real-valued source powers, when their space vectors are known, is

$$J_{a,b} = Tr\left[R^{-1}\frac{\partial R}{\partial p_a}R^{-1}\frac{\partial R}{\partial p_b}\right] = Tr\left[R^{-1}S_a S_a^H R^{-1} S_b S_b^H\right]$$

Numerical computations and asymptotic arguments in [5] show that again the accuracy of source power estimation is little affected by mutual interference between sources. In the high signal power case, Swerling II amplitude estimation accuracy converges to Swerling I, after allowance for the source amplitudes intrinsically having Rayleigh distributions.