Morphisms for Inhibitor Nets and Related Transition Systems

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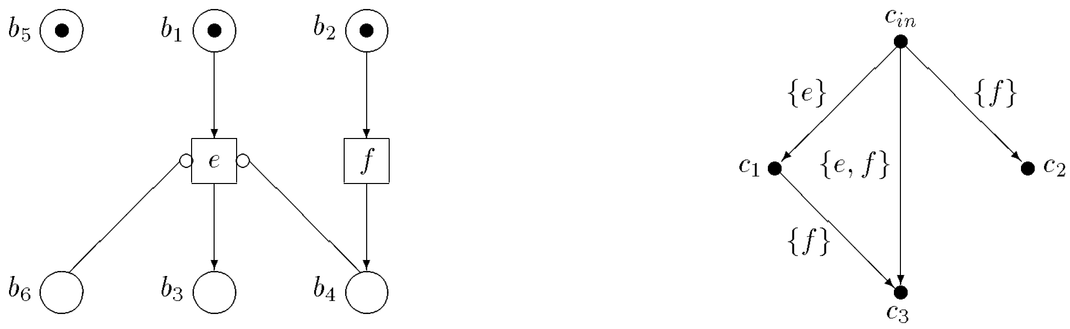
Abstract

We here consider Elementary Net Systems (ENI-systems) with inhibitor arcs executed according to the *a-priori* semantics, and transition systems generated by them (TSENI). The relationship between nets and their transition systems is established via the notion of a region. We introduce morphisms both for the ENI-systems, and for the TSENI transition systems. We then define the category of ENI-systems ($\mathcal{CAT}_{ENI}$), and the category of TSENI transition systems ($\mathcal{CAT}_{TSENI}$), as well as functors between them. Finally, we prove that the functors between categories $\mathcal{CAT}_{ENI}$ and $\mathcal{CAT}_{TSENI}$ form an adjunction.
Keywords: *Petri nets, concurrency, transition systems, category theory, regions.*

1 Introduction

In this paper we define morphisms for the Elementary Net systems with Inhibitor arcs (ENI-systems), and morphisms for the related class of transition systems - Transition Systems of Elementary Nets with Inhibitor Arcs (TSENI) [19].
TSENI transition systems are essentially a subset of general *step transition systems* of [17] as their arcs are labelled by sets of events rather than by single events (see Figure 1(b)). Examples of other work on transition system models include [2, 3, 5, 7, 10, 11, 12, 14, 18, 20]. The type of Petri nets we are interested in is shown in Figure 1(a). The meaning of all the elements of $\mathcal{N}$ is standard except for the inhibitor arc like the one between condition b_4 and event e which is represented by an edge ending with a small circle, and indicates that e can only be fired if b_4 is empty. This has a clear interpretation if one considers purely interleaving net semantics: $\mathcal{N}$ can execute e or f or ef (i.e. e followed by f). However, when we consider a non-interleaving semantics based on step sequences, then one is faced with the problem whether or not the concurrent step $\{e, f\}$ should be allowed. Basically, both interpretations are possible, as discussed in [8]. The one in which it is possible to execute $\{e, f\}$ is called there the *a-priori* semantics, and that in which this is disallowed is called the *a-posteriori* semantics. In this paper we will interpret all inhibitor arcs using the a-priori semantics. Other work on nets with inhibitor arcs include [6, 9, 16].
The net morphisms defined in this paper are of the form $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$, where α is a partial function mapping places of $\mathcal{N}_2$ into places of $\mathcal{N}_1$ and β is a partial function which maps events of $\mathcal{N}_1$ into events of $\mathcal{N}_2$, and are similar to the N-morphisms of [18]. They preserve the



(a) $\mathcal{N}$ (b) $TS_{\mathcal{N}}$

Figure 1: ENI-system $\mathcal{N}$ and its TSENI transition system $TS_{\mathcal{N}}$.

environments of events and initial markings in the sense that places of $\mathcal{N}_1$ and $\mathcal{N}_2$ related by α either both belong or both do not belong to their respective initial markings. The crucial difference is due to the presence of inhibitor arcs. We require that the net $\mathcal{N}_2$ exhibits at least the same degree of concurrency as the net $\mathcal{N}_1$. Our net morphisms, unlike the N-morphisms of [18], do not enjoy the property of being uniquely determined by the way they map events. Two net morphisms $(\alpha_i, \beta_i) : \mathcal{N}_1 \to \mathcal{N}_2$, $(i = 1, 2)$ which satisfy $\beta_1 = \beta_2$ can be different. This is due to the fact that we do not require nets to be simple and allow isolated places.

Transition system morphisms are of the form $(\sigma, \eta) : TS_1 \to TS_2$, where σ is a total function mapping states of TS_1 into states of TS_2 and η is a partial function which maps events of TS_1 into events of TS_2, and are similar to the transition system morphisms defined in [20]. They preserve initial states, and step transitions if a step in TS_1 is mapped into a non-empty set of events of TS_2. The partiality of η allows for some events of TS_1 to be treated as internal, and such events are not required to be mapped into any events of TS_2. Their execution in TS_1 is simulated in TS_2 by remaining in the same state. The mapping for events, η, is injective on every step u of TS_1. When steps are mapped it is necessary to preserve the uniqueness of some special events of every step u (called *r-crossing* elements of a step in section 2.2). Transition system morphisms defined in this paper preserve the independence of events locally within the steps, due to a condition which involves steps instead of individual events. Labelling transitions with steps, rather than with individual events only, can be viewed as embedding an independence relation on events explicitly in the graph of the transition system. Morphisms between TSENI transition systems enjoy the property of being uniquely determined by the way they map the events. Two transition system morphisms $(\sigma_i, \eta_i) : \mathcal{N}_1 \to \mathcal{N}_2$, $(i = 1, 2)$ which satisfy $\eta_1 = \eta_2$ are the same.

The paper is organised as follows. In section 2, we recall from [19] definitions and properties of ENI-systems and TSENI transition systems. Section 3 introduces transition systems morphisms, and section 4 introduces a class of morphisms for the ENI-systems. In section 5, we define the category of ENI-systems ($\mathcal{CAT}_{ENI}$) and the category of TSENI transition systems ($\mathcal{CAT}_{TSENI}$). In sections 6 and 7, functors between the categories $\mathcal{CAT}_{TSENI}$ and $\mathcal{CAT}_{ENI}$ are introduced, and section 8 contains the proof that the two functors form an adjunction.

2 Preliminaries

For any (partial or total) function $f : X \to Y$ we will denote by $dom(f)$ the domain of f, by $codom(f)$ the codomain of f, and by $\widehat{f}$ the lifting of f to a total function $\widehat{f} : 2^X \to 2^Y$ defined, for every $X' \subseteq X$, by

$$\widehat{f}(X') = f(X' \cap dom(f)).$$

In the rest of this paper, we will denote the composition of two functions or partial functions by "$\circ$". For two partial functions $f : X \to Y$ and $g : Y \to Z$ the composition $g \circ f : X \to Z$ will be understood as follows. For all $x \in X$,

$$g \circ f(x) = \begin{cases} z & \text{if there is } y \in Y \text{ such that } f(x) = y \text{ and } g(y) = z \\ \text{undefined} & \text{otherwise.} \end{cases}$$

2.1 Nets with inhibitor arcs

In this section we recall (with only few notational adjustments) the definition of ENI-systems from [13]. We first define their syntax.

Let $\mathcal{E}$ be a non-empty set of *events* fixed throughout this paper. A *net with inhibitor arcs* is a tuple $N = (B, E, F, I)$ such that B and $E \subseteq \mathcal{E}$ are finite disjoint sets, $F \subseteq (B \times E) \cup (E \times B)$ and $I \subseteq B \times E$. The meaning and graphical representation of B (conditions), E (events) and F (flow relation) is the same as in the standard net theory. An *inhibitor* arc $(s, e) \in I$ means that e can be enabled only if s is not marked (in the diagrams, it is represented by an edge ending with a small circle). We denote, for every $x \in B \cup E$,

$$\begin{array}{rcll} {}^\bullet x & = & \{y \mid (y, x) \in F\} & \text{(pre-elements)}, \\ x^\bullet & = & \{y \mid (x, y) \in F\} & \text{(post-elements)}, \\ \text{and } \overset{\blacksquare}{x} & = & \{y \mid (x, y) \in I \cup I^{-1}\} & \text{(I-elements)}. \end{array}$$

The dot-notation extends in the usual way to sets, for example, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$. It is assumed that for every $e \in E$,

$$\begin{array}{rcl} e^\bullet & \neq & \emptyset, \\ {}^\bullet e & \neq & \emptyset, \\ e^\bullet \cap {}^\bullet e & = & \emptyset, \\ \text{and } (e^\bullet \cup {}^\bullet e) \cap \overset{\blacksquare}{e} & = & \emptyset. \end{array} \tag{1}$$

An *elementary net system with inhibitor arcs* (ENI-system) is a tuple

$$\mathcal{N} = (B, E, F, I, c_{in})$$

such that $N_{\mathcal{N}} = (B, E, F, I)$ is the (underlying) net with inhibitor arcs and $c_{in} \subseteq B$ is the *initial case* (in general, any subset of B is a *case*). We will assume that $\mathcal{N}$ is fixed until the end of this section.

The concurrency semantics of ENI-systems will be based on steps of simultaneously executed events. We first define valid steps.

$$V_{\mathcal{N}} = \{u \subseteq E \mid u \neq \emptyset \wedge \forall e, f \in u : (e \neq f \Rightarrow ({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset)\}. \tag{2}$$

The transition relation of $N_{\mathcal{N}}$, denoted by $\rightarrow_{N_{\mathcal{N}}}$, is given by:

$$\rightarrow_{N_{\mathcal{N}}} = \{(c, u, c') \in 2^B \times V_{\mathcal{N}} \times 2^B \mid c \setminus c' = {}^\bullet u \wedge c' \setminus c = u^\bullet \wedge \overset{\blacksquare}{u} \cap c = \emptyset\}. \tag{3}$$

The *state space* of $\mathcal{N}$, denoted by $C_{\mathcal{N}}$, is the least subset of 2^B containing c_{in} such that if $c \in C_{\mathcal{N}}$ and $(c, u, c') \in \rightarrow_{N_{\mathcal{N}}}$ then $c' \in C_{\mathcal{N}}$.
The *transition relation* of $\mathcal{N}$, denoted by $\rightarrow_{\mathcal{N}}$, is then defined as $\rightarrow_{N_{\mathcal{N}}}$ restricted to $C_{\mathcal{N}} \times V_{\mathcal{N}} \times C_{\mathcal{N}}$.
The set of *active steps* of $\mathcal{N}$ is given by $U_{\mathcal{N}} = \{u \mid (c, u, c') \in \rightarrow_{\mathcal{N}}\}$.
We will use $c \xrightarrow{u}_{\mathcal{N}} c'$ to denote that $(c, u, c') \in \rightarrow_{\mathcal{N}}$. Also, $c \xrightarrow{u}_{\mathcal{N}}$ if $(c, u, c') \in \rightarrow_{\mathcal{N}}$, for some c'.
The above definition of the operational semantics of $\mathcal{N}$ is what is referred to as the *a-priori* semantics in [8].

Proposition 2.1 Let $c \in C_{\mathcal{N}}$ and $u \in V_{\mathcal{N}}$.

1. $c \xrightarrow{u}_{\mathcal{N}}$ if and only if ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$.
2. If $c \xrightarrow{u}_{\mathcal{N}} c'$ then $c' = (c \setminus {}^\bullet u) \cup u^\bullet$.
3. If $c \xrightarrow{u}_{\mathcal{N}} c'$ and $d \xrightarrow{u}_{\mathcal{N}} d'$ then $c \setminus c' = d \setminus d'$ and $c' \setminus c = d' \setminus d$.
4. If $c \xrightarrow{u}_{\mathcal{N}} c'$ and $c \xrightarrow{u}_{\mathcal{N}} c''$ then $c' = c''$. □

A *step sequence* is a sequence of sets $\varrho = u_1 \ldots u_n$ of $U_{\mathcal{N}}$ for which there are cases $c_1, \ldots, c_n$ satisfying $c_{in} \xrightarrow{u_1}_{\mathcal{N}} c_1$, $c_1 \xrightarrow{u_2}_{\mathcal{N}} c_2$, ... , $c_{n-1} \xrightarrow{u_n}_{\mathcal{N}} c_n$. We will write $c_{in}[\varrho\rangle c_n$. For the ENI-system $\mathcal{N}$ in Figure 1(a), we have:

$$\{b_1, b_2, b_5\}[\{e\}\rangle\{b_2, b_3, b_5\} \qquad \{b_1, b_2, b_5\}[\{f\}\rangle\{b_1, b_4, b_5\}$$
$$\{b_1, b_2, b_5\}[\{e\}\{f\}\rangle\{b_3, b_4, b_5\} \qquad \{b_1, b_2, b_5\}[\{e, f\}\rangle\{b_3, b_4, b_5\}.$$

2.2 Transition system of nets with inhibitor arcs

In this section we recall from [19] the main definitions and results concerning the TSENI transition systems. A *transition system* is a quadruple $TS = (S, U, T, s_{in})$, where:

TS1 S is a non-empty finite set of *states*,

TS2 $U \subseteq 2^{\mathcal{E}}$ is a set of *steps*; u is finite and non-empty, for every $u \in U$,

TS3 $T \subseteq S \times U \times S$ is the *transition relation*,

TS4 $s_{in} \in S$ is the *initial state*.

We will denote $s \xrightarrow{u} s'$ whenever $(s, u, s') \in T$. By $E_{TS} = \bigcup_{u \in U} u$ we will denote all the events which can appear in steps labelling transitions in TS. The notion of a region links nodes of a transition system (global states) with conditions in the corresponding net (local states). A set of states $r \subseteq S$ is a *region* if the following two conditions are satisfied:

R1 If $s \xrightarrow{u} s'$ and $s \in r$ and $s' \notin r$ then there is $e \in u$ such that

(a) if $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \in r$,

(b) if $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.

R2 If $s \xrightarrow{u} s'$ and $s \notin r$ and $s' \in r$ then there is $e \in u$ such that

(a) if $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \notin r$,

(b) if $q \xrightarrow{v} q'$ and $e \in v$ then $q \notin r$ and $q' \in r$.

The event $e \in u$ which satisfies the conditions in (R1) (or (R2)) is unique. Such an event will be called *r-crossing* in u. It can also be shown that a compliment of a region is also a region. The set of *non-trivial* regions (i.e. those different from S and $\emptyset$) will be denoted by R_{TS}. Moreover, for every state $s \in S$, we will denote by R_s the set of non-trivial regions containing s,

$$R_s = \{r \in R_{TS} \mid s \in r\}.$$

The sets of pre-regions, $^\circ u$, and post-regions, u°, of a step $u \in U$ are defined as:

$$\begin{array}{rrcl} & ^\circ u & = & \{r \in R_{TS} \mid \exists (s,u,s') \in T : s \in r \wedge s' \notin r\} \\ \text{and} & u^\circ & = & \{r \in R_{TS} \mid \exists (s,u,s') \in T : s \notin r \wedge s' \in r\}. \end{array}$$

We will use $^\circ e$ and e° instead of respectively $^\circ\{e\}$ and $\{e\}^\circ$, for every $e \in E_{TS}$.
The set which comprises sets of events which are potential steps in the transition system (they do not share neither pre- nor post-regions) is denoted V_{TS}, and defined by:

$$V_{TS} = \{u \subseteq E_{TS} \mid u \neq \emptyset \wedge \forall e,f \in u : \Big(e \neq f \Rightarrow (^\circ e \cup e^\circ) \cap (^\circ f \cup f^\circ) = \emptyset\Big)\}.$$

In the ENI-system constructed from a TSENI transition system, pre-regions will constitute pre-places and post-regions will constitute post-places of events. We also define inhibitor-regions, which in the constructed net will play the role of places connected with events by means of inhibitor arcs. We start with an auxiliary definition. Let $e \in E_{TS}$ be an event, and $r \in R_{TS}$ be a non-trivial region. Then

$$\mathcal{B}^e_r = \{(s,\{e\},s') \in T \mid s \in r \wedge s' \in r\}$$

is the set of all the transitions labelled by $\{e\}$ which are totally included in r, and the set of *inhibitor-regions* (I-regions) of e is defined as follows:

$$\overset{\square}{e} = \{r \in R_{TS} \mid \mathcal{B}^e_r = \emptyset \wedge \mathcal{B}^e_{S \setminus r} \neq \emptyset\}.$$

We can extend the last notion to a set of events $u \in U$, as follows:

$$\overset{\square}{u} = \bigcup_{e \in u} \overset{\square}{e}.$$

We now can define the class of transition systems which will be the subject of our investigation throughout this paper. A transition system TS is a *TSENI transition system* if it satisfies the following six axioms:

A1 For every $(s,u,s') \in T$, $s \neq s'$.

A2 For every $u \in U$, there are $s, s' \in S$ such that $(s,u,s') \in T$.

A3 For every $s \in S \setminus \{s_{in}\}$, there are $(s_0,u_0,s_1),(s_1,u_1,s_2),\ldots,(s_{n-1},u_{n-1},s_n) \in T$ such that $s_0 = s_{in}$ and $s_n = s$.

A4 If $s \xrightarrow{u}$ and $e \in u$ then $s \xrightarrow{\{e\}}$.

A5 For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.

A6 Let $s \in S$ and $u \in V_{TS}$ be such that, for every $e \in u$, $^\circ e \subseteq R_s$ and $\overset{\Box}{e} \cap R_s = \emptyset$. Then $s \xrightarrow{u}$.

We will now recall some facts concerning TSENI transition systems proved in [19]. Assuming that $s \xrightarrow{u} s'$, the following hold:

$$r \in {}^\circ u \text{ implies } s \in r \text{ and } s' \notin r \tag{4}$$

$$r \in u^\circ \text{ implies } s \notin r \text{ and } s' \in r \tag{5}$$

$$u = \{e\} \text{ and } r \in \overset{\Box}{e} \text{ implies } s, s' \notin r \tag{6}$$

$$s \xrightarrow{u} s'' \text{ implies } s' = s'' \tag{7}$$

$$\left.\begin{array}{rclrclrcl} R_s \setminus R_{s'} & = & {}^\circ u & R_{s'} \setminus R_s & = & u^\circ & R_{s'} & = & (R_s \setminus {}^\circ u) \cup u^\circ \\ {}^\circ u & \subseteq & R_s & u^\circ \cap R_s & = & \emptyset & \overset{\Box}{u} \cap R_s & = & \emptyset. \end{array}\right\} \tag{8}$$

Moreover, all the individual events from E_{TS} are steps in TS:

$$\forall e \in E_{TS} : \{e\} \in U \tag{9}$$

and if $u \in U$ then

$$u \in V_{TS} \tag{10}$$

$$u^\circ = \{S \setminus r \mid r \in {}^\circ u\} \tag{11}$$

$${}^\circ u = \bigcup_{e \in u} {}^\circ e \text{ and } u^\circ = \bigcup_{e \in u} e^\circ. \tag{12}$$

In Figure 1(b), the regions of the transition system $TS_{\mathcal{N}}$ are:

$$\begin{array}{rclrcl} r_1 & = & \{c_{in}, c_1\} & r_2 & = & \{c_{in}, c_2\} \\ r_3 & = & \{c_1, c_3\} & r_4 & = & \{c_2, c_3\} \end{array}$$

and the pre-regions, post-regions and I-regions of events are given by:

$$\begin{array}{rclrclrcl} {}^\circ e & = & \{r_2\} & e^\circ & = & \{r_3\} & \overset{\Box}{e} & = & \{r_4\} \\ {}^\circ f & = & \{r_1\} & f^\circ & = & \{r_4\} & \overset{\Box}{f} & = & \emptyset. \end{array}$$

2.3 Translations between ENI-systems and TSENI transition systems

We now recall from [19] how to construct a TSENI transition system from a given ENI-system, and an ENI-system from a given TSENI transition system. The first construction is straightforward.

Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI-system. Then

$$TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \rightarrow_{\mathcal{N}}, c_{in})$$

is the *transition system generated* by $\mathcal{N}$.

Theorem 2.2 $TS_{\mathcal{N}}$ is a TSENI transition system. Moreover, for every $b \in B$, $r_b = \{c \in C_{\mathcal{N}} \mid b \in c\}$ is a (possibly trivial) region in $TS_{\mathcal{N}}$. □

The reverse translation is based on the pre- post- and I-regions of events appearing in a transition system. Let $TS = (S, U, T, s_{in})$ be a TSENI transition system. The net system *associated* with TS is defined as

$$\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$$

where F_{TS} and I_{TS} are defined thus:

$$\begin{aligned} F_{TS} &= \{(r, e) \in R_{TS} \times E_{TS} \mid r \in {}^{\circ}e\} \;\cup\; \{(e, r) \in E_{TS} \times R_{TS} \mid r \in e^{\circ}\}, \\ \text{and} \quad I_{TS} &= \{(r, e) \in R_{TS} \times E_{TS} \mid r \in \overset{\square}{e}\}. \end{aligned} \tag{13}$$

Directly from the definition of $\mathcal{N}_{TS}$ we obtain that, for every $e \in E_{TS}$,

$${}^{\circ}e = {}^{\bullet}e, \; e^{\circ} = e^{\bullet} \text{ and } \overset{\square}{e} = \overset{\blacksquare}{e} \,. \tag{14}$$

Theorem 2.3 $\mathcal{N}_{TS}$ is an ENI-system. Moreover, for every $r \in R_{TS}$, there exists an event $e \in E_{TS}$ such that $r \in {}^{\bullet}e$ or $r \in e^{\bullet}$ (in $\mathcal{N}_{TS}$). □

The last result states that the ENI-system associated with a TSENI transition system TS generates a transition system which is isomorphic to TS.

Theorem 2.4 Let $TS = (S, U, T, s_{in})$ be a TSENI transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI-system associated with it.

1. $C_{\mathcal{N}} = \{R_s \mid s \in S\}$.
2. $\rightarrow_{\mathcal{N}} = \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.
3. $TS_{\mathcal{N}}$ is isomorphic to TS. □

3 Transition systems morphisms

In this section we introduce morphisms between TSENI transition systems.

Definition 3.1 Let $TS_i = (S_i, U_i, T_i, s_{in}^i)$ (for $i = 1, 2$) be TSENI transition systems. A *transition system morphism* from TS_1 to TS_2 is a pair of functions $f = (\sigma, \eta) : TS_1 \to TS_2$ such that the following hold.

MTS1 $\sigma : S_1 \to S_2$ is a total function satisfying $\sigma(s_{in}^1) = s_{in}^2$.

MTS2 $\eta : E_{TS_1} \to E_{TS_2}$ is a partial function, which is injective on every $u \in U_1$.

MTS3 For every $(s, u, s') \in T_1$, either $\widehat{\eta}(u) = \emptyset$ and $\sigma(s) = \sigma(s')$, or $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$. □

Note that transition system morphisms defined above are similar to the ones defined in [20]. They preserve the initial states, and step transitions if a step in TS_1 is mapped into a non-empty set of events of TS_2. Directly from (MTS3), (A2), and (10), we obtain the following.

Corollary 3.2 For every $u \in U_1$, $\widehat{\eta}(u) \in V_{TS_2} \cup \{\emptyset\}$. □

This means that transition system morphisms preserve the independence of events locally within the steps, due to (MTS3) where steps are used instead of individual events. There is no need to assume this separately, like it was done for the morphisms of asynchronous transition systems in [20]. Notice that labelling transitions with steps, rather than with individual events only, can be viewed as embedding an independence relation on events explicitly in the graph of a transition system. The first result we prove shows that a transition system morphism $f : TS_1 \to TS_2$ is determined by the way in which steps in TS_1 have been transformed into steps in TS_2.

Proposition 3.3 Let TS_1 and TS_2 be TSENI transition systems, and $f = (\sigma_f, \eta_f)$ and $g = (\sigma_g, \eta_g)$ be transition system morphisms from TS_1 to TS_2 such that $\eta_f = \eta_g$. Then $f = g$.

Proof. We prove that $\sigma_f(s) = \sigma_g(s)$, for every $s \in S_1$, by induction on the smallest number k of transitions it takes to reach s from s_{in}^1 (such an induction is valid due to (A3)).
The base case is $k = 0$. Then $s = s_{in}^1$ and we have, by (MTS1), $\sigma_f(s_{in}^1) = s_{in}^2 = \sigma_g(s_{in}^1)$.
Suppose that $k > 0$. Let ϱ be a path of length k defined by the following sequence of transitions in T_1:

$$(s_{in}^1, u_1, s_1), (s_1, u_2, s_2), \ldots, (s_{k-2}, u_{k-1}, s_{k-1}), (s_{k-1}, u_k, s).$$

By the induction hypothesis, $\sigma_f(s_{k-1}) = \sigma_g(s_{k-1})$. We consider two cases (recall that $\widehat{\eta}_f = \widehat{\eta}_g$).
Case 1: $\widehat{\eta}_f(u_k) = \widehat{\eta}_g(u_k) = \emptyset$. Then from (MTS3) it follows that

$$\sigma_f(s) = \sigma_f(s_{k-1}) = \sigma_g(s_{k-1}) = \sigma_g(s).$$

Case 2: $\widehat{\eta}_f(u_k) = \widehat{\eta}_g(u_k) \neq \emptyset$. Then from (MTS3) it follows that

$$\begin{array}{rcl} (\sigma_f(s_{k-1}), \widehat{\eta}_f(u_k), \sigma_f(s)) & \in & T_2 \\ (\sigma_g(s_{k-1}), \widehat{\eta}_g(u_k), \sigma_g(s)) & \in & T_2. \end{array}$$

As TS_2 is a TSENI transition system, it is deterministic, by (7). Hence, by $\sigma_f(s_{k-1}) = \sigma_g(s_{k-1})$ and $\widehat{\eta}_f(u_k) = \widehat{\eta}_g(u_k)$, we obtain $\sigma_f(s) = \sigma_g(s)$. □

In the next proposition we show how transition system morphisms preserve regions of transition systems.

Proposition 3.4 Let TS_1 and TS_2 be TSENI transition systems and $f = (\sigma, \eta) : TS_1 \to TS_2$ be a transition system morphism from TS_1 to TS_2. If $r \subseteq S_2$ is a region in TS_2 then $\sigma^{-1}(r)$ is a region in TS_1. Moreover, the following hold.

1. For all $u \in U_1$,
$$\sigma^{-1}(r) \in {}^\circ u \;\;\Leftrightarrow\;\; \widehat{\eta}(u) \neq \emptyset \;\wedge\; r \in {}^\circ\widehat{\eta}(u)$$
$$\sigma^{-1}(r) \in u^\circ \;\;\Leftrightarrow\;\; \widehat{\eta}(u) \neq \emptyset \;\wedge\; r \in \widehat{\eta}(u)^\circ.$$

2. For all $e \in dom(\eta)$,
$$\sigma^{-1}(r) \in \overset{\Box}{e} \;\;\Leftarrow\;\; r \in \overset{\Box}{\eta(e)} \;\wedge \sigma^{-1}(r) \neq \emptyset.$$

Proof. We first show that $r' = \sigma^{-1}(r)$ is a region in TS_1.
Suppose that $(s, u, s') \in T_1$, $s \in r'$ and $s' \notin r'$. Then $\sigma(s) \in r$ and $\sigma(s') \notin r$. Hence $\sigma(s) \neq \sigma(s')$ and so, by (MTS3), $\widehat{\eta}(u) \neq \emptyset$ and $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$. Let $d \in \widehat{\eta}(u)$ be the r-crossing event in $\widehat{\eta}(u)$. From (A4) it follows that $\sigma(s) \xrightarrow{\{d\}}$ in TS_2. Let $e \in u$ be the unique event such that $d = \eta(e)$[1]. Again, from (A4) it follows that $s \xrightarrow{\{e\}}$ in TS_1. We will show that e is the r'-crossing event in u and thus that r' is a region (since the argument is symmetric if $s \notin r'$ and $s' \in r'$).
Consider $w \subseteq u \setminus \{e\}$ such that $(s, w, q) \in T_1$. To show that $q \in r'$ we consider two cases:
Case 1: $\widehat{\eta}(w) = \emptyset$. Then, by (MTS3), $\sigma(s) = \sigma(q)$. Hence $\sigma(q) \in r$ and so $q \in r'$.
Case 2: $\widehat{\eta}(w) \neq \emptyset$. Then, by (MTS3), $(\sigma(s), \widehat{\eta}(w), \sigma(q)) \in T_2$. Since η is injective on steps in TS_1, $d \notin \widehat{\eta}(w)$. Hence, since d is the r-crossing event in $\widehat{\eta}(u)$ and r is a region, $\sigma(q) \in r$. Thus $q \in r'$.
Consider now $(q, v, q') \in T_1$ such that $e \in v$. We need to show that $q \in r'$ and $q' \notin r'$. Since $\eta(e) = d \in \widehat{\eta}(v) \neq \emptyset$, by (MTS3), $(\sigma(q), \widehat{\eta}(v), \sigma(q')) \in T_2$. Hence, since d is the r-crossing event in $\widehat{\eta}(u)$ and r is a region, $\sigma(q) \in r$ and $\sigma(q') \notin r$. Thus $q \in r'$ and $q' \notin r'$.
We now move on to the second part of the proposition. Let $u \in U_1$. To show the ($\Rightarrow$) implication, we proceed as follows. By $\sigma^{-1}(r) \in {}^\circ u$, there exists $(s, u, s') \in T_1$ such that $s \in \sigma^{-1}(r)$ and $s' \notin \sigma^{-1}(r)$. Hence $\sigma(s) \in r$ and $\sigma(s') \notin r$ which means $\sigma(s) \neq \sigma(s')$. Thus, by (MTS3), $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$. Hence $\widehat{\eta}(u) \neq \emptyset$ and $r \in {}^\circ\widehat{\eta}(u)$.
To show the reverse ($\Leftarrow$) implication, assume that $\widehat{\eta}(u) \neq \emptyset$ and $r \in {}^\circ\widehat{\eta}(u)$. By $u \in U_1$ and (A2), there is a transition $(q, u, q') \in T_1$. From (MTS3) it follows that $(\sigma(q), \widehat{\eta}(u), \sigma(q')) \in T_2$. Moreover, from (4) and $r \in {}^\circ\widehat{\eta}(u)$, we have $\sigma(q) \in r$ and $\sigma(q') \notin r$. Hence $q \in \sigma^{-1}(r)$ and $q' \notin \sigma^{-1}(r)$, and thus $\sigma^{-1}(r) \in {}^\circ u$.
A similar argument applies to post-regions.

Finally, we will prove that, for every $e \in dom(\eta)$, $r \in \overset{\Box}{\eta(e)}$ and $\sigma^{-1}(r) \neq \emptyset$ imply $\sigma^{-1}(r) \in \overset{\Box}{e}$. Let $d = \eta(e)$. From (A2) and (9), we have $(s, \{e\}, s') \in T_1$, for some s and s'. Hence, by (MTS3), $(\sigma(s), \{d\}, \sigma(s')) \in T_2$. This, (6), and $r \in \overset{\Box}{d}$ yield $\sigma(s), \sigma(s') \notin r$. Hence $s, s' \notin \sigma^{-1}(r)$. Moreover, $\sigma^{-1}(r) \neq \emptyset$, so $\sigma^{-1}(r)$ is non-trivial and $\mathcal{B}^e_{S_1 \setminus \sigma^{-1}(r)} \neq \emptyset$. Suppose now that $\mathcal{B}^e_{\sigma^{-1}(r)} \neq \emptyset$. Then there is $(s, \{e\}, s') \in T_1$ such that $s, s' \in \sigma^{-1}(r)$ and, consequently, $\sigma(s), \sigma(s') \in r$. Moreover, from $(s, \{e\}, s') \in T_1$ and (MTS3) it follows that $(\sigma(s), \{d\}, \sigma(s')) \in T_2$. Hence $\mathcal{B}^d_r \neq \emptyset$, which contradicts $r \in \overset{\Box}{d}$. As a result, $\mathcal{B}^e_{\sigma^{-1}(r)} = \emptyset$, and so $\sigma^{-1}(r) \in \overset{\Box}{e}$. □

[1] Notice that the injectivity of η on every step $u \in U_1$ guarantees the uniqueness of the r-crossing element in u.

Example 3.5 The implication in the second part of proposition 3.4 cannot be reversed. Figure 2 shows two transitions systems such that for a suitable morphism and $e \in dom(\eta)$, $\sigma^{-1}(r) \in \overset{\square}{e}$ but $r \notin \overset{\square}{\eta(e)}$. The details of this counterexample are as follows. The morphism $\widetilde{f} = (\widetilde{\sigma}, \widetilde{\eta})$ is defined by:

$$\begin{array}{lllllllll} \widetilde{\sigma}(s_0) & = & s'_0 & \widetilde{\sigma}(s_1) & = & s'_1 & \widetilde{\sigma}(s_2) & = & s'_2 \\ \widetilde{\sigma}(s_3) & = & s'_3 & \widetilde{\eta}(a) & = & a' & \widetilde{\eta}(b) & = & b'. \end{array}$$

The regions in TS_1 are:

$$\begin{array}{llllll} r_1 & = & \{s_0, s_1\} & r_2 & = & \{s_0, s_2\} \\ r_3 & = & \{s_2, s_3\} & r_4 & = & \{s_1, s_3\} \end{array}$$

and the pre-regions, post-regions and I-regions of events are given by:

$$\begin{array}{lllllllll} {}^\circ a & = & \{r_2\} & a^\circ & = & \{r_4\} & \overset{\square}{a} & = & \{r_3\} \\ {}^\circ b & = & \{r_1\} & b^\circ & = & \{r_3\} & \overset{\square}{b} & = & \{r_4\}. \end{array}$$

The regions in TS_2 are:

$$\begin{array}{llllll} r'_1 & = & \{s'_0, s'_1\} & r'_2 & = & \{s'_0, s'_2\} \\ r'_3 & = & \{s'_2, s'_3\} & r'_4 & = & \{s'_1, s'_3\} \end{array}$$

and the pre-regions, post-regions and I-regions of events are given by:

$$\begin{array}{lllllllll} {}^\circ a' & = & \{r'_2\} & a'^\circ & = & \{r'_4\} & \overset{\square}{a'} & = & \emptyset \\ {}^\circ b' & = & \{r'_1\} & b'^\circ & = & \{r'_3\} & \overset{\square}{b'} & = & \emptyset. \end{array}$$

Since $\widetilde{\sigma}^{-1}(r'_i) = r_i$, for $i = 1, 2, 3, 4$, to produce a counterexample one can take $e = a$ and $r = r'_3$, or $e = b$ and $r = r'_4$. □

Example 3.6 Proposition 3.4 states that, if $r \subseteq S_2$ is a region in TS_2 then $\sigma^{-1}(r)$ is a region in TS_1. Notice that it does not assume that r nor $\sigma^{-1}(r)$ are non-trivial regions. It may happen that for a non-trivial r, $\sigma^{-1}(r)$ will be trivial. Consider, for example, the transition systems in Figure 2, and the transition system morphism $f = (\sigma, \eta)$ defined as follows:

$$\begin{array}{lllllllll} \sigma(s_0) & = & s'_0 & \sigma(s_1) & = & s'_1 & \sigma(s_2) & = & s'_0 \\ \sigma(s_3) & = & s'_1 & \eta(a) & = & a' & \eta(b) & - & \text{not defined.} \end{array}$$

In the above example σ is not injective. Observe that $\sigma^{-1}(r'_1) = S_1$ and $\sigma^{-1}(r'_3) = \emptyset$. □

4 Inhibitor nets and their morphisms

In this section we will introduce a class of morphisms for the ENI-systems.

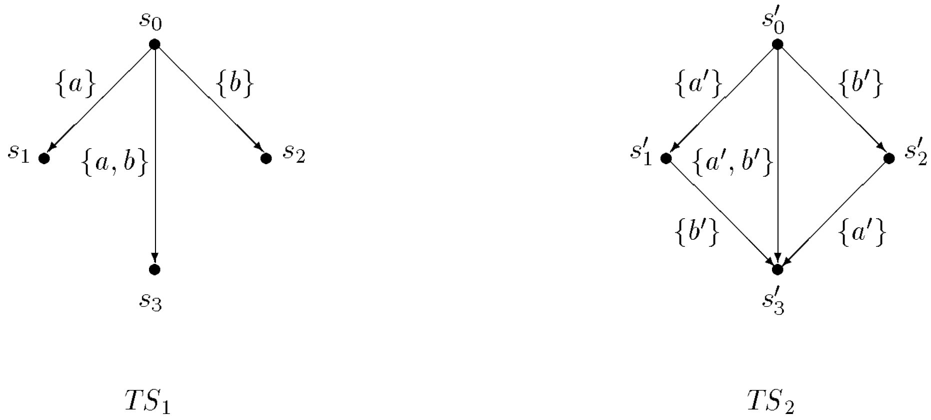


Figure 2: Two transition systems for Examples 3.5 and 3.6.

Definition 4.1 Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems. A net morphism from $\mathcal{N}_1$ to $\mathcal{N}_2$ is a pair $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ such that the following hold.

MENI1 $\alpha : B_2 \to B_1$ is a partial function.

MENI2 $\beta : E_1 \to E_2$ is a partial function.

MENI3 For every $b \in dom(\alpha)$, $\alpha(b) \in c^1_{in}$ if and only if $b \in c^2_{in}$.

MENI4 For every $e \in E_1 \setminus dom(\beta)$, $\alpha^{-1}({}^\bullet e) = \emptyset = \alpha^{-1}(e^\bullet)$.

MENI5 For every $e \in dom(\beta)$:

$$\begin{aligned} \alpha^{-1}({}^\bullet e) &= {}^\bullet\beta(e), \\ \alpha^{-1}(e^\bullet) &= \beta(e)^\bullet, \\ \overset{\blacksquare}{\beta(e)} \cap \mathcal{M}_{(\alpha,\beta)} &\subseteq \alpha^{-1}(\overset{\blacksquare}{e}), \end{aligned}$$

where $\mathcal{M}_{(\alpha,\beta)} = \{b \in B_2 \mid b \in c^2_{in} \vee \exists e \in dom(\beta) : b \in \beta(e)^\bullet\}$. □

In the above definition $\mathcal{M}_{(\alpha,\beta)}$ denotes a set of all the places of a net $\mathcal{N}_2$ which are potentially marked by at least one marking reachable from c^2_{in} when $\mathcal{N}_1$ is simulated. The net morphisms defined above are similar to the N-morphisms of [18]. They preserve initial markings, in the sense that $\alpha^{-1}(c^1_{in}) \subseteq c^2_{in}$, and the environments of events. The crucial difference is due to the presence of inhibitor arcs. The condition $\overset{\blacksquare}{\beta(e)} \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ means that, in the net $\mathcal{N}_2$, we can have more concurrency (less inhibition). Notice that we only take into account I-places of $\mathcal{N}_2$ which can potentially disable events, i.e. those which can potentially be marked when $\mathcal{N}_2$ simulates $\mathcal{N}_1$. The net version of proposition 3.3 is not true, i.e. two net morphisms $(\alpha_i, \beta_i) : \mathcal{N}_1 \to \mathcal{N}_2$, $(i = 1, 2)$ which satisfy $\beta_1 = \beta_2$ can be different. This is due to the fact that we do not require nets to be simple and allow isolated places.

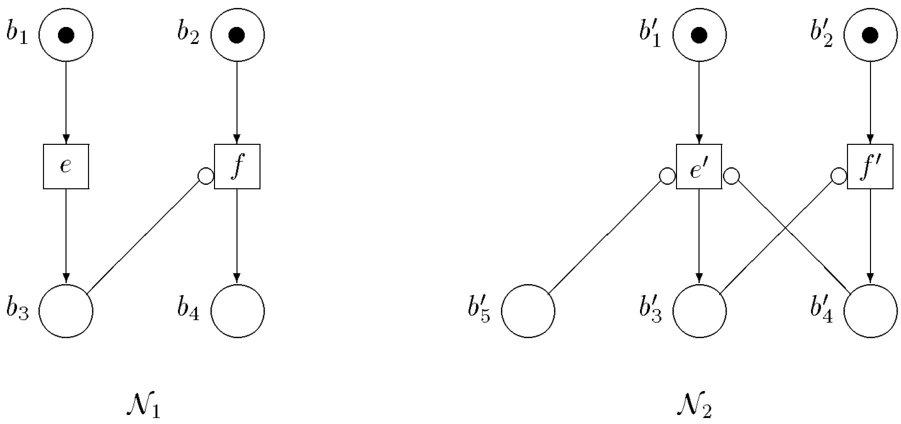


Figure 3: Two ENI-systems for Example 4.2.

Example 4.2 Figure 3 shows two ENI-system. We can define a net morphism $g = (\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ between them in the following way:

$$\alpha(b'_1) \;=\; b_1 \qquad\qquad \alpha(b'_3) \;=\; b_3 \qquad\qquad \beta(e) \;=\; e'.$$

Notice that $\alpha(b'_2)$, $\alpha(b'_4)$, $\alpha(b'_5)$ and $\beta(f)$ are not defined.
The pre-elements, post-elements and I-elements of events in $\mathcal{N}_1$ are given by:

$$\begin{array}{rclcrclcrcl} {}^{\bullet}e &=& \{b_1\} && e^{\bullet} &=& \{b_3\} && \overset{\blacksquare}{e} &=& \emptyset \\ {}^{\bullet}f &=& \{b_2\} && f^{\bullet} &=& \{b_4\} && \overset{\blacksquare}{f} &=& \{b_3\}. \end{array}$$

The pre-elements, post-elements and I-elements of events in $\mathcal{N}_2$ are given by:

$$\begin{array}{rclcrclcrcl} {}^{\bullet}e' &=& \{b'_1\} && e'^{\bullet} &=& \{b'_3\} && \overset{\blacksquare}{e'} &=& \{b'_4, b'_5\} \\ {}^{\bullet}f' &=& \{b'_2\} && f'^{\bullet} &=& \{b'_4\} && \overset{\blacksquare}{f'} &=& \{b'_3\}. \end{array}$$

It is straightforward to check that g is well defined net morphism. Observe that for e: $\alpha^{-1}(\overset{\blacksquare}{e}) = \emptyset$, $\beta(\overset{\blacksquare}{e}) = \{b'_4, b'_5\}$ and $\mathcal{M}_{(\alpha,\beta)} = \{b'_1, b'_2, b'_3\}$, so the inclusion in (MENI5) holds. Notice that although $\mathcal{N}_2$ has more inhibitor arcs than $\mathcal{N}_1$, it has no more *active inhibitor arcs* when it simulates $\mathcal{N}_1$. Both inhibitor places of e' in $\mathcal{N}_2$ give rise to *passive inhibitor arcs*: b'_5 is never marked in $\mathcal{N}_2$, and b'_4 is never marked in $\mathcal{N}_2$ when $\mathcal{N}_1$ is simulated (f' will be never executed as it is not an image of any event in $\mathcal{N}_1$). □

Proposition 4.3 Let $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ be a net morphism between ENI-systems $\mathcal{N}_1$ and $\mathcal{N}_2$. Then, for every $X \subseteq B_1$,

$$\alpha^{-1}(X) \cap \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right) = \emptyset.$$

Proof. Suppose that there is d such that: (i) $d \in \alpha^{-1}(X)$, and (ii) $d \in (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1))$. From (i) it follows that there is $b \in B_1$ such that $\alpha(d) = b \in X$. From (ii) it follows that $d \in c_{in}^2$ which means, by (MENI3), that $b = \alpha(d) \in c_{in}^1$. The latter in turn gives $d \in \alpha^{-1}(c_{in}^1)$ contradicting (ii). $\square$

Proposition 4.4 Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c_{in}^i)$ $(i = 1, 2)$ be ENI-systems and $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ be a net morphism. Moreover, let $f_\alpha : C_{\mathcal{N}_1} \to 2^{B_2}$ be a mapping such that, for every $c \in C_{\mathcal{N}_1}$,

$$f_\alpha(c) = \alpha^{-1}(c) \cup \left(c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)\right).$$

Then the following hold:

1. For every $c \in C_{\mathcal{N}_1}$, $f_\alpha(c) \in C_{\mathcal{N}_2}$.
2. If $(c, u, c') \in \to_{\mathcal{N}_1}$ and $\widehat{\beta}(u) = \emptyset$ then $f_\alpha(c) = f_\alpha(c')$.
3. If $(c, u, c') \in \to_{\mathcal{N}_1}$ and $\widehat{\beta}(u) \neq \emptyset$ then $(f_\alpha(c), \widehat{\beta}(u), f_\alpha(c')) \in \to_{\mathcal{N}_2}$.

Proof. (1) Let $c \in C_{\mathcal{N}_1}$ and ϱ be a shortest step sequence of $\mathcal{N}_1$ such that $c_{in}^1[\varrho\rangle c$. The proof proceeds by induction on the length k of ϱ.
The base case is $k = 0$. Then $c = c_{in}^1$ and, by (MENI3), we have $\alpha^{-1}(c_{in}^1) \subseteq c_{in}^2$. Hence

$$f_\alpha(c_{in}^1) = \alpha^{-1}(c_{in}^1) \cup \left(c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)\right) = c_{in}^2 \in C_{\mathcal{N}_2}.$$

In the induction step, $k > 0$. Let $\varrho = \varrho' u$ and $c_{in}^1[\varrho'\rangle c'$ in $\mathcal{N}_1$. By the induction hypothesis, $c'' = f_\alpha(c') \in C_{\mathcal{N}_2}$. We have $(c', u, c) \in \to_{\mathcal{N}_1}$. Hence, by proposition 2.1(2), $c = (c' \setminus {}^\bullet u) \cup u^\bullet$ in $\mathcal{N}_1$, and we consider two cases.
Case 1: $\widehat{\beta}(u) = \emptyset$. Then

$$f_\alpha(c) = \alpha^{-1}(c) \cup \left(c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)\right) = \alpha^{-1}\left((c' \setminus {}^\bullet u) \cup u^\bullet\right) \cup \left(c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)\right) \overset{(MENI4)}{=} f_\alpha(c') \in C_{\mathcal{N}_2}.$$

Case 2: $\widehat{\beta}(u) \neq \emptyset$. Denote $w = \widehat{\beta}(u)$. We will show that w is enabled at $c'' \in C_{\mathcal{N}_2}$. First, knowing that $u \in V_{\mathcal{N}_1}$, we will prove that $w \in V_{\mathcal{N}_2}$. Clearly, for every pair $e_2 \neq f_2 \in w$ we can find a pair $e_1 \neq f_1 \in u$ such that $\beta(e_1) = e_2$ and $\beta(f_1) = f_2$. Then

$$\begin{aligned}({}^\bullet e_2 \cup e_2^\bullet) \cap ({}^\bullet f_2 \cup f_2^\bullet) &\overset{(MENI5)}{=} \left(\alpha^{-1}({}^\bullet e_1) \cup \alpha^{-1}(e_1^\bullet)\right) \cap \left(\alpha^{-1}({}^\bullet f_1) \cup \alpha^{-1}(f_1^\bullet)\right)\\ &= \alpha^{-1}\left(({}^\bullet e_1 \cup e_1^\bullet) \cap ({}^\bullet f_1 \cup f_1^\bullet)\right)\\ &\overset{(2)}{=} \emptyset.\end{aligned}$$

Hence $w \in V_{\mathcal{N}_2}$. Since $c'' \in C_{\mathcal{N}_2}$, we can use proposition 2.1(1) to show that w is enabled at c''. We need to prove that ${}^\bullet w \subseteq c''$, $w^\bullet \cap c'' = \emptyset$ and $\overset{\blacksquare}{w} \cap c'' = \emptyset$. By $w \in V_{\mathcal{N}_2}$, it suffices to prove that if $f \in w$ then ${}^\bullet f \subseteq c''$, $f^\bullet \cap c'' = \emptyset$ and $\overset{\blacksquare}{f} \cap c'' = \emptyset$. Let $e \in u$ be such that $\beta(e) = f$. We first show that

$${}^\bullet f \subseteq c'' = f_\alpha(c') = \alpha^{-1}(c') \cup \left(c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)\right).$$

From (MENI5) it follows that $\alpha^{-1}({}^{\bullet}e) = {}^{\bullet}f$. Hence what we need to show is that

$$\alpha^{-1}({}^{\bullet}e) \subseteq \alpha^{-1}(c') \cup \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right).$$

Now, from $(c', u, c) \in \rightarrow_{\mathcal{N}_1}$ it follows that ${}^{\bullet}e \subseteq c'$. Hence $\alpha^{-1}({}^{\bullet}e) \subseteq \alpha^{-1}(c')$ and so ${}^{\bullet}f \subseteq c''$. We next show that $f^{\bullet} \cap c'' = \emptyset$. By (MENI5), $\alpha^{-1}(e^{\bullet}) = f^{\bullet}$. From $(c', u, c) \in \rightarrow_{\mathcal{N}_1}$ it follows that $e^{\bullet} \cap c' = \emptyset$, so $\alpha^{-1}(e^{\bullet}) \cap \alpha^{-1}(c') = \emptyset$. Consequently, $f^{\bullet} \cap \alpha^{-1}(c') = \emptyset$. Moreover, $f^{\bullet} \cap (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) = \emptyset$ which follows from $f^{\bullet} = \alpha^{-1}(e^{\bullet})$ and proposition 4.3. Hence $f^{\bullet} \cap c'' = \emptyset$.

We now show that $\overset{\blacksquare}{f} \cap c'' = \emptyset$. From (MENI5) we have $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$, and from $(c', u, c) \in \rightarrow_{\mathcal{N}_1}$ it follows that $\overset{\blacksquare}{e} \cap c' = \emptyset$. Hence

$$\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap \alpha^{-1}(c') \subseteq \alpha^{-1}(\overset{\blacksquare}{e}) \cap \alpha^{-1}(c') = \emptyset.$$

Consequently, $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap \alpha^{-1}(c') = \emptyset$. Moreover, $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) = \emptyset$ which follows from $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ and proposition 4.3. Hence

$$\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap \left(\alpha^{-1}(c') \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))\right) = \emptyset.$$

We now show that $c'' = \alpha^{-1}(c') \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) \subseteq \mathcal{M}_{(\alpha,\beta)}$. Suppose $b \in c''$. It implies that $b \in c^2_{in} \setminus \alpha^{-1}(c^1_{in})$ from which it immediately follows that $b \in \mathcal{M}_{(\alpha,\beta)}$, or $b \in \alpha^{-1}(c')$ which means $\alpha(b) \in c'$. But $c' \in C_{\mathcal{N}_1}$, so $\alpha(b) \in c^1_{in}$ which by (MENI3) means $b \in c^2_{in}$, or there exists $g \in E_1$ such that $\alpha(b) \in g^{\bullet}$ (so $b \in \alpha^{-1}(g^{\bullet})$ and $\alpha^{-1}(g^{\bullet}) \neq \emptyset$). From (MENI4) and (MENI5) it follows that $g \in dom(\beta)$ and $\alpha^{-1}(g^{\bullet}) = \beta(g)^{\bullet}$. So $b \in \beta(g)^{\bullet}$ for some $g \in dom(\beta)$ and, consequently, $b \in \mathcal{M}_{(\alpha,\beta)}$. So $\overset{\blacksquare}{f} \cap c'' = \emptyset$. Hence we have shown that $c'' \xrightarrow{w}_{\mathcal{N}_2}$.

Let $\widetilde{c} = (c'' \setminus {}^{\bullet}w) \cup w^{\bullet}$ in $\mathcal{N}_2$. We have $\widetilde{c} \in C_{\mathcal{N}_2}$ and it suffices to show that $f_\alpha(c) = \widetilde{c}$ in order to prove that $f_\alpha(c) \in C_{\mathcal{N}_2}$. We proceed as follow:

$$\begin{aligned}
f_\alpha(c) &= f_\alpha((c' \setminus {}^{\bullet}u) \cup u^{\bullet}) \\
&= \alpha^{-1}\left((c' \setminus {}^{\bullet}u) \cup u^{\bullet}\right) \cup \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right) \\
&= \alpha^{-1}(c' \setminus {}^{\bullet}u) \cup \alpha^{-1}(u^{\bullet}) \cup \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right) \\
&= \left(\alpha^{-1}(c') \setminus \alpha^{-1}({}^{\bullet}u)\right) \cup \alpha^{-1}(u^{\bullet}) \cup \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right) \\
&\overset{(prop.\ 4.3)}{=} \left(\left(\alpha^{-1}(c') \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))\right) \setminus \alpha^{-1}({}^{\bullet}u)\right) \cup \alpha^{-1}(u^{\bullet}) \\
&= \left(f_\alpha(c') \setminus \alpha^{-1}({}^{\bullet}u)\right) \cup \alpha^{-1}(u^{\bullet}) \\
&= \left(c'' \setminus \alpha^{-1}({}^{\bullet}u)\right) \cup \alpha^{-1}(u^{\bullet}) \\
&\overset{(MENI5)}{=} (c'' \setminus {}^{\bullet}w) \cup w^{\bullet} \\
&= \widetilde{c}.
\end{aligned}$$

(2,3) Both were proved while showing (1). □

5 Categories $\mathcal{CAT}_{ENI}$ and $\mathcal{CAT}_{TSENI}$

We start by recalling some basic definitions concerning categories and functors from [1] and [4].

Definition 5.1 A category $\mathcal{K}$ comprises a collection of objects of $\mathcal{K}$, called $\mathcal{K}_0$, together with, for each pair A, B of objects of $\mathcal{K}$, a distinct (possibly empty) collection of morphisms from A to B, called $\mathcal{K}_1$, subject to the conditions (C1) and (C2) below. We write $f : A \to B$ to indicate that f is a morphism from A to B, and then refer to A as the source of f and to B as the target of f. For two morphisms, f and g, such that the target of f is the source of g, there is the composite morphism denoted by $g \circ f$. The source of $g \circ f$ is the source of f, and the target of $g \circ f$ is the target of g. For every object A of $\mathcal{K}$, we will denote by id_A a distinguished morphism from A to A, called the identity of the object A.

C1 $(h \circ g) \circ f = h \circ (g \circ f)$ whenever either side of the equality is defined.

C2 If $f : A \to B$, then $f \circ id_A = id_B \circ f = f$. □

Definition 5.2 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is a pair of functions $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that the following hold.

F1 If $f : A \to B$ in $\mathcal{C}$, then $F_1(f) : F_0(A) \to F_0(B)$ in $\mathcal{D}$.

F2 For every object A of $\mathcal{C}$, $F_1(id_A) = id_{F_0(A)}$.

F3 If $g \circ f$ is defined in $\mathcal{C}$, then $F_1(g) \circ F_1(f)$ is defined in $\mathcal{D}$ and $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

Whenever it does not lead to ambiguity, we will denote F_0 and F_1 by F. □

We now define two categories: the category of TSENI transition systems with morphisms defined as in section 3, and the category of ENI-systems with morphisms defined as in section 4. To define these categories, we need to say what are the identity morphisms, and define the compositions of two morphisms. Let $TS = (S, U, T, s_{in})$ be a TSENI transition system, and $\sigma_{id} : S \to S$ and $\eta_{id} : E_{TS} \to E_{TS}$ be total identity functions. Then $id_{TS} = (\sigma_{id}, \eta_{id})$ will denote an identity morphism $id_{TS} : TS \to TS$.
Let $TS_i = (S_i, U_i, T_i, s^i_{in})$ (for $i = 1, 2, 3$) be TSENI transition systems, and $f = (\sigma_f, \eta_f) : TS_1 \to TS_2$ and $g = (\sigma_g, \eta_g) : TS_2 \to TS_3$ be two transition system morphisms. Then the composition of the morphisms is defined by $g \circ f = (\sigma_g \circ \sigma_f, \eta_g \circ \eta_f) : TS_1 \to TS_3$, where $\sigma_g \circ \sigma_f$ is a total function composition and $\eta_g \circ \eta_f$ is a partial function composition. It is straightforward to prove that the TSENI transition systems with transition system morphisms form a category. We will denote it by $\mathcal{CAT}_{TSENI}$.
Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI-system, and $\alpha_{id} : B \to B$ and $\beta_{id} : E \to E$ be two total identity functions. Then $id_{\mathcal{N}} = (\alpha_{id}, \beta_{id})$ will denote an identity morphism $id_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$.
Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ (for $i = 1, 2, 3$) be ENI-systems, and $f = (\alpha_f, \beta_f) : \mathcal{N}_1 \to \mathcal{N}_2$ and $g = (\alpha_g, \beta_g) : \mathcal{N}_2 \to \mathcal{N}_3$ be two net morphisms. Then the composition of the morphisms is defined as follows: $g \circ f = (\alpha_f \circ \alpha_g, \beta_g \circ \beta_f) : \mathcal{N}_1 \to \mathcal{N}_3$, where both $\alpha_f \circ \alpha_g$ and $\beta_g \circ \beta_f$ are partial function compositions. Notice that α's and β's are composed in different order.

We now show that $g \circ f$ is a net morphism. It is clear that $\alpha_f \circ \alpha_g : B_3 \to B_1$ and $\beta_g \circ \beta_f : E_1 \to E_3$ are partial functions. (MENI3), (MENI4) and the first parts of (MENI5) are straightforward to show. We prove the last part of (MENI5), i.e., $\beta_g \circ \overset{\blacksquare}{\beta_f}(e) \cap \mathcal{M}_{(\alpha_f \circ \alpha_g, \beta_g \circ \beta_f)} \subseteq (\alpha_f \circ \alpha_g)^{-1}(\overset{\blacksquare}{e})$, for every $e \in dom(\beta_g \circ \beta_f)$. From the fact that f and g are net morphisms we have:

$$\forall e \in dom(\beta_g) : \ \overset{\blacksquare}{\beta_g}(e) \cap \mathcal{M}_{(\alpha_g, \beta_g)} \subseteq \alpha_g^{-1}(\overset{\blacksquare}{e}), \tag{15}$$

$$\forall e \in dom(\beta_f) : \ \overset{\blacksquare}{\beta_f}(e) \cap \mathcal{M}_{(\alpha_f, \beta_f)} \subseteq \alpha_f^{-1}(\overset{\blacksquare}{e}). \tag{16}$$

Consider $b \in \beta_g \circ \overset{\blacksquare}{\beta_f}(e) \cap \mathcal{M}_{(\alpha_f \circ \alpha_g, \beta_g \circ \beta_f)}$. This means $b \in \beta_g(\overset{\blacksquare}{\beta_f}(e))$, moreover, $b \in c_{in}^3$ or there exists $e' \in dom(\beta_g \circ \beta_f)$ such that $b \in \beta_g \circ \beta_f(e')^\bullet$. Thus $b \in \beta_g(\overset{\blacksquare}{\beta_f}(e))$ and, moreover, $b \in c_{in}^3$ or there exists $e'' = \beta_f(e') \in dom(\beta_g)$ such that $b \in \beta_g(e'')^\bullet$. From (15) we have $b \in \alpha_g^{-1}(\overset{\blacksquare}{\beta_f}(e))$ and then $\alpha_g(b) \in \overset{\blacksquare}{\beta_f}(e)$. So $b \in dom(\alpha_g)$ and, if $b \in c_{in}^3$ we have from the fact that g is a net morphism (MENI3), $\alpha_g(b) \in c_{in}^2$. If, on the other hand, there exists $e' \in dom(\beta_g \circ \beta_f)$ such that $b \in \beta_g(\beta_f(e'))^\bullet$, then from the fact that g is a net morphism (MENI5), there exists $e' \in dom(\beta_f)$ such that $b \in \alpha_g^{-1}(\beta_f(e')^\bullet)$. Consequently, there exists $e' \in dom(\beta_f)$ such that $\alpha_g(b) \in \beta_f(e')^\bullet$. We have proved that $\alpha_g(b) \in \overset{\blacksquare}{\beta_f}(e)$ and, moreover, $\alpha_g(b) \in c_{in}^2$ or there exists $e' \in dom(\beta_f)$ such that $\alpha_g(b) \in \beta_f(e')^\bullet$. Since f is a net morphism we have, by (16), $\alpha_g(b) \in \alpha_f^{-1}(\overset{\blacksquare}{e})$, which means $\alpha_f \circ \alpha_g(b) \in \overset{\blacksquare}{e}$ and, finally, $b \in (\alpha_f \circ \alpha_g)^{-1}(\overset{\blacksquare}{e})$. It is now easy to see that the ENI-systems with net morphisms form a category. We will denote it by $\mathcal{CAT}_{ENI}$.

6 A functor from $\mathcal{CAT}_{ENI}$ to $\mathcal{CAT}_{TSENI}$

To define a functor we need to show how the objects and morphisms of one category are mapped into objects and morphisms (respectively) of another category. In section 2.3, we have recalled the construction of the TSENI transition system, $TS_{\mathcal{N}}$, for a given ENI-system, $\mathcal{N}$. The next proposition defines the mapping between morphisms of $\mathcal{CAT}_{ENI}$ and morphisms of $\mathcal{CAT}_{TSENI}$.

Proposition 6.1 Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c_{in}^i)$ $(i = 1, 2)$ be ENI-systems and (α, β) be a net morphism from $\mathcal{N}_1$ to $\mathcal{N}_2$. Moreover, let $f_\alpha : C_{\mathcal{N}_1} \to C_{\mathcal{N}_2}$ be a total function defined as follows:

$$f_\alpha(c) = \alpha^{-1}(c) \cup \left(c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)\right)$$

and $f_\beta : E_{TS_{\mathcal{N}_1}} \to E_{TS_{\mathcal{N}_2}}$ be a mapping defined by $f_\beta = \beta$. Then (f_α, f_β) is a transition system morphism from $TS_{\mathcal{N}_1}$ to $TS_{\mathcal{N}_2}$.

Proof. (MTS1) and (MTS3) follow immediately from proposition 4.4. Note that $E_{TS_{\mathcal{N}_1}} \subseteq E_1$ and $\beta : E_1 \to E_2$ is a partial function. Moreover, from proposition 4.4(3) we have that, if $e \in E_{TS_{\mathcal{N}_1}}$ and $\beta(e)$ is defined, then $e \in E_{TS_{\mathcal{N}_2}}$. Hence f_β is a well defined partial function. What is left to show is the injectivity of $f_\beta = \beta$ on steps. Suppose that $u \in U_{\mathcal{N}_1}$. We need to show that for all different $e, f \in u \cap dom(\beta)$, $\beta(e) \neq \beta(f)$. Let us assume that $\beta(e) = \beta(f)$. Then, by (MENI5), $\alpha^{-1}({}^\bullet e) = \alpha^{-1}({}^\bullet f) \neq \emptyset$. Thus there is $x \in \alpha^{-1}({}^\bullet e) \cap \alpha^{-1}({}^\bullet f) = \alpha^{-1}({}^\bullet e \cap {}^\bullet f)$. Hence $\alpha(x) \in {}^\bullet e \cap {}^\bullet f$ which contradicts $u \in U_{\mathcal{N}_1}$ (since $u \in V_{\mathcal{N}_1}$). $\square$

Now we are ready to define a functor from $\mathcal{CAT}_{ENI}$ to $\mathcal{CAT}_{TSENI}$.

Theorem 6.2 Let $H : \mathcal{CAT}_{ENI} \to \mathcal{CAT}_{TSENI}$ be a mapping defined as follows, for every ENI-system $\mathcal{N}$ and net morphism (α, β),

$$\begin{aligned} H(\mathcal{N}) &= TS_{\mathcal{N}}, \\ H(\alpha, \beta) &= (f_\alpha, f_\beta). \end{aligned}$$

Then H is a functor.

Proof. Let $\mathcal{N} = (B, E, F, I, c_{in})$ and $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2, 3)$ be ENI-systems. Let $id_{\mathcal{N}} = (\alpha_{id}, \beta_{id}) : \mathcal{N} \to \mathcal{N}$ be an identity morphism and $f = (\alpha_f, \beta_f) : \mathcal{N}_1 \to \mathcal{N}_2$ and $g = (\alpha_g, \beta_g) : \mathcal{N}_2 \to \mathcal{N}_3$ be two net morphisms. Then (F1) follows from proposition 6.1. We will show that $H(id_{\mathcal{N}}) = id_{TS_{\mathcal{N}}}$ (i.e. (F2)) and $H(g \circ f) = H(g) \circ H(f)$ (i.e. (F3)). To prove the former we observe that $H(\alpha_{id}, \beta_{id}) = (f_{\alpha_{id}}, f_{\beta_{id}})$ where, for every $c \in C_{\mathcal{N}}$ and $e \in E_{TS_{\mathcal{N}}}$,

$$\begin{aligned} f_{\alpha_{id}}(c) &= \alpha^{-1}_{id}(c) \cup (c_{in} \setminus \alpha^{-1}_{id}(c_{in})) &= c \cup (c_{in} \setminus c_{in}) &= c \\ f_{\beta_{id}}(e) &= \beta_{id}(e) & &= e. \end{aligned}$$

The latter is proved in the following way. We have:

$$\begin{aligned} H(g \circ f) &= H((\alpha_g, \beta_g) \circ (\alpha_f, \beta_f)) \\ &= H(\alpha_f \circ \alpha_g, \beta_g \circ \beta_f) \\ &= (f_{\alpha_f \circ \alpha_g}, f_{\beta_g \circ \beta_f}) \end{aligned}$$

and

$$\begin{aligned} H(g) \circ H(f) &= H(\alpha_g, \beta_g) \circ H(\alpha_f, \beta_f) \\ &= (f_{\alpha_g}, f_{\beta_g}) \circ (f_{\alpha_f}, f_{\beta_f}) \\ &= (f_{\alpha_g} \circ f_{\alpha_f}, f_{\beta_g} \circ f_{\beta_f}). \end{aligned}$$

Since it is clear that $f_{\beta_g \circ \beta_f} = f_{\beta_g} \circ f_{\beta_f}$, it suffices to show that $f_{\alpha_f \circ \alpha_g} = f_{\alpha_g} \circ f_{\alpha_f}$[2]. We have, for every $c \in C_{\mathcal{N}_1}$,

$$\begin{aligned} f_{\alpha_f \circ \alpha_g}(c) &= (\alpha_f \circ \alpha_g)^{-1}(c) \cup \left(c^3_{in} \setminus (\alpha_f \circ \alpha_g)^{-1}(c^1_{in})\right) \\ &= \alpha_g^{-1} \circ \alpha_f^{-1}(c) \cup \left(c^3_{in} \setminus \alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})\right). \end{aligned}$$

From $\alpha_f^{-1}(c^1_{in}) \subseteq c^2_{in}$ and $\alpha_g^{-1}(c^2_{in}) \subseteq c^3_{in}$ it follows that $\alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in}) \subseteq \alpha_g^{-1}(c^2_{in}) \subseteq c^3_{in}$. Hence, for every $c \in C_{\mathcal{N}_1}$,

$$\begin{aligned} f_{\alpha_g} \circ f_{\alpha_f}(c) &= f_{\alpha_g}(f_{\alpha_f}(c)) \\ &= f_{\alpha_g}\left(\alpha_f^{-1}(c) \cup (c^2_{in} \setminus \alpha_f^{-1}(c^1_{in}))\right) \\ &= \alpha_g^{-1}\left(\alpha_f^{-1}(c) \cup (c^2_{in} \setminus \alpha_f^{-1}(c^1_{in}))\right) \cup \left(c^3_{in} \setminus \alpha_g^{-1}(c^2_{in})\right) \\ &= \alpha_g^{-1}\left(\alpha_f^{-1}(c)\right) \cup \alpha_g^{-1}\left(c^2_{in} \setminus \alpha_f^{-1}(c^1_{in})\right) \cup \left(c^3_{in} \setminus \alpha_g^{-1}(c^2_{in})\right) \\ &= \alpha_g^{-1} \circ \alpha_f^{-1}(c) \cup \left(\alpha_g^{-1}(c^2_{in}) \setminus \alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})\right) \cup \left(c^3_{in} \setminus \alpha_g^{-1}(c^2_{in})\right) \\ &= \alpha_g^{-1} \circ \alpha_f^{-1}(c) \cup (c^3_{in} \setminus \alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})). \end{aligned}$$

[2]We will prove this as an exercise, as $f_{\alpha_f \circ \alpha_g} = f_{\alpha_g} \circ f_{\alpha_f}$ follows from proposition 3.3 and $f_{\beta_g \circ \beta_f} = f_{\beta_g} \circ f_{\beta_f}$.

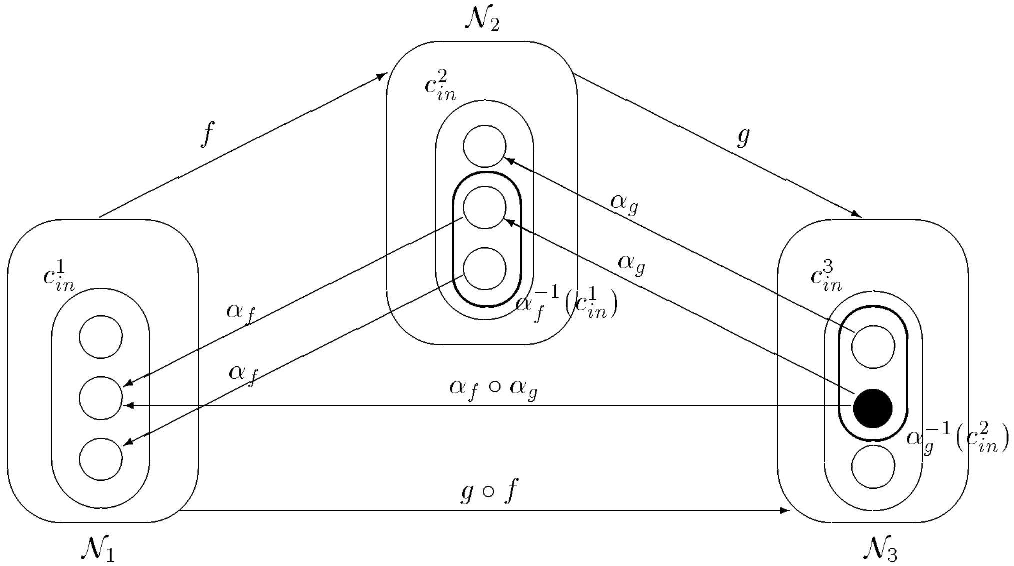


Figure 4: Illustration for theorem 6.2 where the black circle denotes $\alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})$.

Thus, for every $c \in C_{\mathcal{N}_1}$, $f_{\alpha_g} \circ f_{\alpha_f}(c) = f_{\alpha_f \circ \alpha_g}(c)$ which completes the proof. $\square$

7 A functor from $\mathcal{CAT}_{TSENI}$ to $\mathcal{CAT}_{ENI}$

We have shown how to construct a functor H from $\mathcal{CAT}_{ENI}$ to $\mathcal{CAT}_{TSENI}$. In this section, we will define a functor in the opposite direction. To build the ENI-system, $\mathcal{N}_{TS}$, for a given TSENI transition system, TS, we will use the construction recalled in section 2.3. The next proposition defines a mapping between morphisms of $\mathcal{CAT}_{TSENI}$ and morphisms of $\mathcal{CAT}_{ENI}$.

Proposition 7.1 Let $TS_i = (S_i, U_i, T_i, s^i_{in})$ $(i = 1, 2)$ be TSENI transition systems, and $(\sigma, \eta) : TS_1 \to TS_2$ be a transition system morphism. Moreover, let $f_\sigma : R_{TS_2} \to R_{TS_1}$ be a mapping such that $f_\sigma(r) = \sigma^{-1}(r)$, for every $r \in R_{TS_2}$ such that $\emptyset \neq \sigma^{-1}(r) \neq S_1$, and $f_\eta : E_{TS_1} \to E_{TS_2}$ be a mapping defined by $f_\eta = \eta$. Then (f_σ, f_η) is a net morphism from $\mathcal{N}_{TS_1}$ to $\mathcal{N}_{TS_2}$.

Proof. Let $\mathcal{N}_{TS_i} = (R_{TS_i}, E_{TS_i}, F_{TS_i}, I_{TS_i}, R_{s^i_{in}})$, for $i = 1, 2$. We observe that (MENI1) holds since f_σ is a partial function from R_{TS_2} to R_{TS_1} (follows from proposition 3.4), and (MENI2) holds since f_η is a partial function from E_{TS_1} to E_{TS_2}. To show (MENI3), for every $r \in dom(f_\sigma)$ we need to demonstrate that $f_\sigma(r) \in R_{s^1_{in}} \Leftrightarrow r \in R_{s^2_{in}}$. This is equivalent to showing $s^1_{in} \in \sigma^{-1}(r) \Leftrightarrow s^2_{in} \in r$ which clearly holds since $\sigma(s^1_{in}) = s^2_{in}$. To prove (MENI4), for every $e \in E_{TS_1} \setminus dom(\eta)$ we need to show that $f_\sigma^{-1}({}^\bullet e) = \emptyset = f_\sigma^{-1}(e^\bullet)$. Let us assume that $r \in f_\sigma^{-1}({}^\bullet e) \neq \emptyset$. Then $f_\sigma(r) \in {}^\bullet e$ and so $\sigma^{-1}(r) \in {}^\bullet e$ (in $\mathcal{N}_{TS_1}$) which means $\sigma^{-1}(r) \in {}^\circ e$ (in TS_1). From (9) and proposition 3.4(1) we have that $\eta(e)$ is defined, a contradiction. Hence

$f_\sigma^{-1}({}^\bullet e) = \emptyset$. The same can be shown for $f_\sigma^{-1}(e^\bullet)$. Finally, to show (MENI5), we need to prove that, for every $e \in dom(\eta)$,

$$\begin{aligned} f_\sigma^{-1}({}^\bullet e) &= {}^\bullet\eta(e), \\ f_\sigma^{-1}(e^\bullet) &= \eta(e)^\bullet, \\ \overset{\blacksquare}{\eta(e)} \cap \mathcal{M}_{(f_\sigma, f_\eta)} &\subseteq f_\sigma^{-1}(\overset{\blacksquare}{e}). \end{aligned}$$

The first equality can be proved as follows (note that by (9), $\{e\} \in U_1$).

$$\begin{aligned} r \in f_\sigma^{-1}({}^\bullet e) \quad &\Leftrightarrow \quad f_\sigma(r) \in {}^\bullet e = {}^\circ e \\ &\Leftrightarrow \quad \sigma^{-1}(r) \in {}^\circ e \\ &\overset{(prop.\ 3.4(1))}{\Leftrightarrow} \quad r \in {}^\circ\eta(e) = {}^\bullet\eta(e). \end{aligned}$$

The second equality can be proved in a similar way.
To prove the last part of (MENI5) notice that, $r \in \mathcal{M}_{(f_\sigma, f_\eta)}$ means that $r \in R_{TS_2}$ and, moreover, $r \in R_{s^2_{in}}$ or there is $e \in dom(\eta)$ such that $r \in \eta(e)^\bullet$. If $r \in R_{s^2_{in}}$, then $s^2_{in} \in r$ and together with $\sigma(s^1_{in}) = s^2_{in}$ we have $s^1_{in} \in \sigma^{-1}(r)$. So $\sigma^{-1}(r) \neq \emptyset$. If $r \in \eta(e)^\bullet$, for some $e \in dom(\eta)$, then $\sigma^{-1}(r) \neq \emptyset$ follows from (9) and proposition 3.4(1). The inclusion follows then from the fact that, for $r \in \overset{\blacksquare}{\eta(e)} = \overset{\square}{\eta(e)}$ and $\sigma^{-1}(r) \neq \emptyset$, proposition 3.4(2) states that $\sigma^{-1}(r) \in \overset{\square}{e} = \overset{\blacksquare}{e}$. Hence $f_\sigma(r) \in \overset{\blacksquare}{e}$, and so $r \in f_\sigma^{-1}(\overset{\blacksquare}{e})$. □

The next theorem defines a functor from $\mathcal{CAT}_{TSENI}$ to $\mathcal{CAT}_{ENI}$.

Theorem 7.2 Let $J : \mathcal{CAT}_{TSENI} \to \mathcal{CAT}_{ENI}$ be a mapping defined, for every TSENI transition system TS and transition system morphism (σ, η), by:

$$\begin{aligned} J(TS) &= \mathcal{N}_{TS}, \\ J(\sigma, \eta) &= (f_\sigma, f_\eta). \end{aligned}$$

Then J is a functor.

Proof. Let $TS = (S, U, T, s_{in})$ and $TS_i = (S_i, U_i, T_i, s^i_{in})$ (for $i = 1, 2, 3$) be TSENI transition systems. Let $id_{TS} = (\sigma_{id}, \eta_{id}) : TS \to TS$ be an identity morphism and $f = (\sigma_f, \eta_f) : TS_1 \to TS_2$ and $g = (\sigma_g, \eta_g) : TS_2 \to TS_3$ be two transition system morphisms. Then (F1) follows from proposition 7.1. We now show that $J(id_{TS}) = id_{\mathcal{N}_{TS}}$ (i.e. (F2)) and $J(g \circ f) = J(g) \circ J(f)$ (i.e. (F3)) also hold.
The former follows from $f_{\sigma_{id}}(r) = \sigma_{id}^{-1}(r) = r$ and $f_{\eta_{id}}(e) = \eta_{id}(e) = e$, for $r \in R_{TS}$ and $e \in E_{TS}$. The latter can be shown as follows. We have:

$$\begin{aligned} J(g \circ f) &= J((\sigma_g, \eta_g) \circ (\sigma_f, \eta_f)) \\ &= J(\sigma_g \circ \sigma_f, \eta_g \circ \eta_f) \\ &= (f_{\sigma_g \circ \sigma_f}, f_{\eta_g \circ \eta_f}) \end{aligned}$$

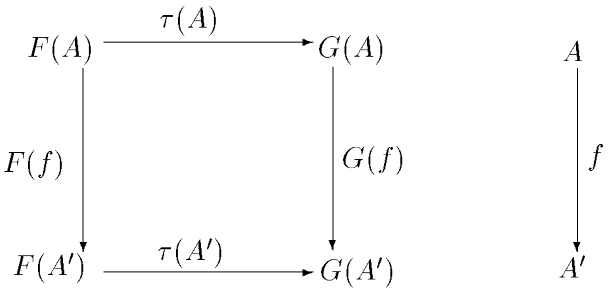


Figure 5: Natural transformation $\tau : F \to G$.

and

$$\begin{aligned} J(g) \circ J(f) &= (f_{\sigma_g}, f_{\eta_g}) \circ (f_{\sigma_f} \circ f_{\eta_f}) \\ &= (f_{\sigma_f} \circ f_{\sigma_g}, f_{\eta_g} \circ f_{\eta_f}). \end{aligned}$$

We then observe that $f_{\sigma_g \circ \sigma_f} = (\sigma_g \circ \sigma_f)^{-1} = \sigma_f^{-1} \circ \sigma_g^{-1} = f_{\sigma_f} \circ f_{\sigma_g}$ and $f_{\eta_g \circ \eta_f} = \eta_g \circ \eta_f = f_{\eta_g} \circ f_{\eta_f}$ which completes the proof. □

8 An adjunction between functors H and J

The two functors $J : \mathcal{CAT}_{TSENI} \to \mathcal{CAT}_{ENI}$ and $H : \mathcal{CAT}_{ENI} \to \mathcal{CAT}_{TSENI}$ are closely related since, in categorical terms, they form an adjunction. We recall some definitions from [4] and [15].

Definition 8.1 Given two functors $F, G : \mathcal{A} \to \mathcal{B}$, between categories $\mathcal{A}$ and $\mathcal{B}$, a *natural transformation* $\tau : F \to G$ is a function which assigns to each object A of $\mathcal{A}$ a morphism $\tau(A) : F(A) \to G(A)$ of $\mathcal{B}$ in such a way that every morphism $f : A \to A'$ in $\mathcal{A}$ yields a diagram as in Figure 5 which is commutative. The morphism $\tau(A)$ for an object A is called the *component* of the natural transformation τ at A. □

In the above definition, the commutativity of the diagram in Figure 5 means that the equality $G(f) \circ \tau(A) = \tau(A') \circ F(f)$ holds.

Definition 8.2 Let $\mathcal{A}$ and $\mathcal{B}$ be categories. If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors, we say that F is a *left adjoint* to G and G is a *right adjoint* to F provided there is natural transformation $\tau : id_{\mathcal{A}} \to G \circ F$[3] such that for any objects A of $\mathcal{A}$ and B of $\mathcal{B}$ and any morphism $f : A \to G(B)$, there is a unique morphism $g : F(A) \to B$ such that $f = G(g) \circ \tau(A)$. The triple (F, G, τ) constitutes an *adjunction.* The natural transformation τ is called the *unit* of the adjunction. □

Before proving the last theorem, we consider an ENI-system $\mathcal{N} = (B, E, F, I, c_{in})$ and the related transition system $H(\mathcal{N}) = TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \to_{\mathcal{N}}, c_{in})$. It was stated in theorem 2.2 that, for every $b \in B$, $r_b = \{c \in C_{\mathcal{N}} \mid b \in c\}$ is a (possibly trivial) region in $TS_{\mathcal{N}}$. We need to prove one more property of the regions in $H(\mathcal{N}) = TS_{\mathcal{N}}$ before presenting the main result of this paper.

[3] $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is an identity functor which maps objects and morphisms of $\mathcal{A}$ onto themselves.

Proposition 8.3 Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI-system and $H(\mathcal{N}) = TS_{\mathcal{N}}$ be the transition system generated by $\mathcal{N}$. Then, for all $b \in B$ and $e \in E_{TS_{\mathcal{N}}}$, the following hold:

1. $r_b \in {}^\circ e$ (in $H(\mathcal{N})$) $\Leftrightarrow b \in {}^\bullet e$ (in $\mathcal{N}$).
2. $r_b \in e^\circ$ (in $H(\mathcal{N})$) $\Leftrightarrow b \in e^\bullet$ (in $\mathcal{N}$).

Proof. (1) To show the ($\Rightarrow$) implication we proceed as follows. $r_b \in {}^\circ e$ (in $H(\mathcal{N})$) implies that there exist $c \xrightarrow{\{e\}}_{\mathcal{N}} d$ such that $c \in r_b$ and $d \notin r_b$. Hence $b \in c$ and $b \notin d$. From (3) it follows that $c \setminus d = {}^\bullet e$ and so $b \in {}^\bullet e$ (in $\mathcal{N}$).
To show the reverse implication, assume that $b \in {}^\bullet e$ (in $\mathcal{N}$). From (A2), (9) and $e \in E_{TS_{\mathcal{N}}}$ it follows that there exist $d, d' \in C_{\mathcal{N}}$ such that $d \xrightarrow{\{e\}}_{\mathcal{N}} d'$. From $b \in {}^\bullet e$ and $d \setminus d' = {}^\bullet e$ we have that $b \in d$ and $b \notin d'$. Hence $d \in r_b$ and $d' \notin r_b$. So r_b is a non-trivial region in $H(\mathcal{N}) = TS_{\mathcal{N}}$ and $r_b \in {}^\circ e$.
(2) Can be proved in the similar way. $\square$

Theorem 8.4 Let $\tau : id_{\mathcal{CAT}_{TSENI}} \to H \circ J$ be a natural transformation and $\tau(TS) : TS \to H \circ J(TS)$ be the component of τ at TS defined as follows. $\tau(TS) = ([\tau(TS)]_0, [\tau(TS)]_1)$ where $[\tau(TS)]_0 : S \to C_{\mathcal{N}_{TS}}$ and $[\tau(TS)]_1 : E_{TS} \to E_{TS_{\mathcal{N}_{TS}}}$ are total functions defined below.

$$\begin{array}{lrcl} \forall\, s \in S: & [\tau(TS)]_0(s) & = & R_s, \\ \forall\, e \in E_{TS}: & [\tau(TS)]_1(e) & = & e. \end{array} \tag{17}$$

Then, $J : \mathcal{CAT}_{TSENI} \to \mathcal{CAT}_{ENI}$ and $H : \mathcal{CAT}_{ENI} \to \mathcal{CAT}_{TSENI}$ form an adjunction with J as left adjoint and τ as a unit (see Figure 6).

Proof. We need to show that, for every TSENI transition system TS_1 in $\mathcal{CAT}_{TSENI}$ and every ENI-system $\mathcal{N}_2$ in $\mathcal{CAT}_{ENI}$, if there is a transition system morphism $f : TS_1 \to H(\mathcal{N}_2)$ then there is a unique net morphism $g : J(TS_1) \to \mathcal{N}_2$ such that

$$f = H(g) \circ \tau(TS_1). \tag{18}$$

Let $TS_1 = (S_1, U_1, T_1, s^1_{in})$ and $\mathcal{N}_2 = (B_2, E_2, F_2, I_2, c^2_{in})$. From the definitions of the functors, $J(TS_1) = \mathcal{N}_{TS_1} = (R_{TS_1}, E_{TS_1}, F_{TS_1}, I_{TS_1}, R_{s^1_{in}})$ and $H(\mathcal{N}_2) = TS_{\mathcal{N}_2} = (C_{\mathcal{N}_2}, U_{\mathcal{N}_2}, \rightarrow_{\mathcal{N}_2}, c^2_{in})$. Let $\tau(TS_1) = \psi$. It follows from theorem 2.4 that ψ is an isomorphism and a well defined transition system morphism from TS_1 to $TS_{\mathcal{N}_{TS_1}}$. For $f = (\sigma, \eta)$, we define $g = (\alpha, \beta)$ in the following way. $\alpha : B_2 \to R_{TS_1}$ is a mapping such that, for $b \in B_2$,

$$\alpha(b) = \begin{cases} \sigma^{-1}(r_b) & \text{if } S_1 \neq \sigma^{-1}(r_b) \neq \emptyset \\ \text{undefined} & \text{otherwise.} \end{cases}$$

and $\beta : E_{TS_1} \to E_2$ is defined by $\beta(e) = \eta(e)$. Notice that $\sigma^{-1}(r_b) \neq S_1$ and $\sigma^{-1}(r_b) \neq \emptyset$ implies $r_b \neq C_{\mathcal{N}_2}$ and $r_b \neq \emptyset$.
We will prove that $g = (\alpha, \beta)$ is a net morphism from $J(TS_1)$ to $\mathcal{N}_2$. We observe that (MENI1) and (MENI2) hold since α is a partial function and β is a partial function (as $\eta : E_{TS_1} \to$

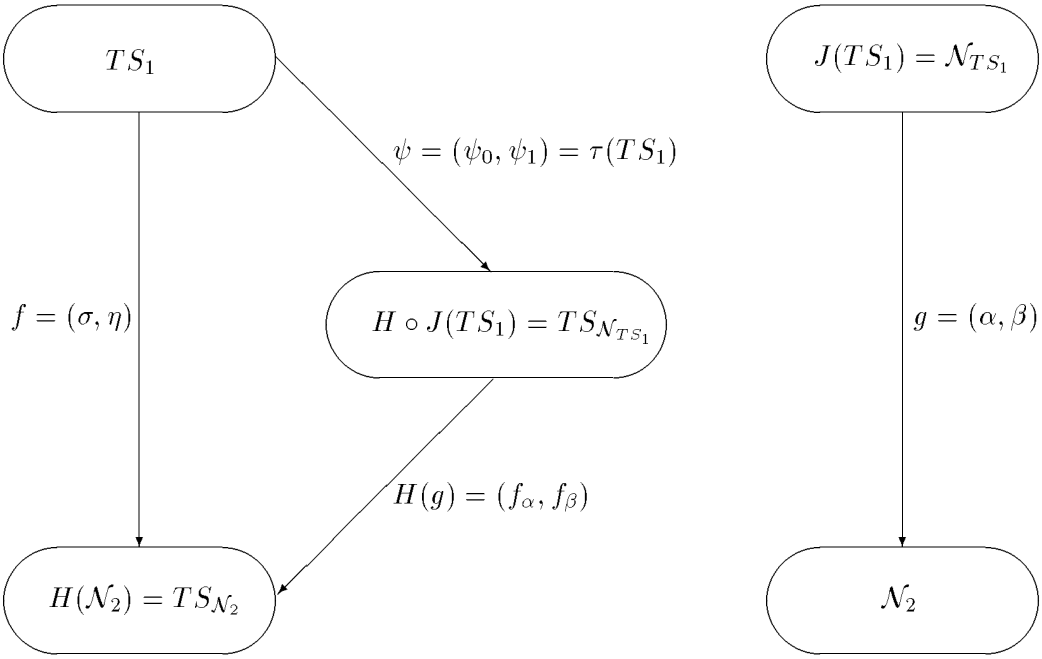


Figure 6: Illustration for theorem 8.4.

$E_{TS_{\mathcal{N}_2}} \subseteq E_2$ is a partial function). To show (MENI3), for all $b \in dom(\alpha)$, we need to demonstrate that $\alpha(b) \in R_{s^1_{in}} \Leftrightarrow b \in c^2_{in}$. This holds, since

$$\alpha(b) = \sigma^{-1}(r_b) \in R_{s^1_{in}} \Leftrightarrow s^1_{in} \in \sigma^{-1}(r_b) \Leftrightarrow \sigma(s^1_{in}) \in r_b \overset{(MTS1)}{\Leftrightarrow} c^2_{in} \in r_b \Leftrightarrow b \in c^2_{in}.$$

To prove (MENI4), for every $e \in E_{TS_1} \setminus dom(\beta)$, we need to show that $\alpha^{-1}({}^\bullet e) = \emptyset = \alpha^{-1}(e^\bullet)$. Note that $\eta(e)$ is not defined. Let us assume that $\alpha^{-1}({}^\bullet e) \neq \emptyset$. Then there is $b \in B_2$ such that $b \in \alpha^{-1}({}^\bullet e)$ which means $\alpha(b) \in {}^\bullet e$. From the definition of α and the fact that (in $\mathcal{N}_{TS_1}$) ${}^\bullet e = {}^\circ e$, we have $\sigma^{-1}(r_b) \in {}^\circ e$. Hence, from proposition 3.4(1) we obtain that $\eta(e)$ is defined, a contradiction. The same way of reasoning applies to $\alpha^{-1}(e^\bullet)$.
Finally, to show (MENI5), we need to prove that for all $e \in dom(\beta) = dom(\eta)$,

$$\begin{array}{rcl} \alpha^{-1}({}^\bullet e) & = & {}^\bullet\beta(e), \\ \alpha^{-1}(e^\bullet) & = & \beta(e)^\bullet, \\ \beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} & \subseteq & \alpha^{-1}(\overset{\blacksquare}{e}). \end{array}$$

First we prove that $\alpha^{-1}({}^\bullet e) = {}^\bullet\beta(e)$.

$$\begin{array}{lcl} b \in \alpha^{-1}({}^\bullet e) & \Leftrightarrow & \alpha(b) \in {}^\bullet e = {}^\circ e \text{ (in } \mathcal{N}_{TS_1}) \\ & \Leftrightarrow & \sigma^{-1}(r_b) \in {}^\circ e \text{ (a place in } \mathcal{N}_{TS_1} \text{ is a region in } TS_1) \\ & \overset{(prop.\ 3.4(1))}{\Leftrightarrow} & r_b \in {}^\circ\eta(e) \text{ (in } H(\mathcal{N}_2)) \\ & \overset{(prop.\ 8.3(1))}{\Leftrightarrow} & b \in {}^\bullet\eta(e) \text{ (in } \mathcal{N}_2) \\ & \Leftrightarrow & b \in {}^\bullet\beta(e). \end{array}$$

That $\alpha^{-1}(e^\bullet) = \beta(e)^\bullet$ can be proved in a similar way. We now prove that $\beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ holds (in $\mathcal{N}_2$). $b \in \beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)}$ implies $b \in \eta(\overset{\blacksquare}{e})$. From (A2) and (9) it follows that there exist $c, c' \in C_{\mathcal{N}_2}$ such that $c \overset{\{\eta(e)\}}{\longrightarrow}_{\mathcal{N}_2} c'$. From (3) we have $c \setminus c' = {}^\bullet\eta(e)$, $c' \setminus c = \eta(e)^\bullet$ and $\eta(\overset{\blacksquare}{e}) \cap c = \emptyset$. Since $b \in \eta(\overset{\blacksquare}{e})$, we have that $b \notin c$. By (1), we have $\eta(e)^\bullet \cap \eta(\overset{\blacksquare}{e}) = \emptyset$. Since $b \in \eta(\overset{\blacksquare}{e})$, we have $b \notin \eta(e)^\bullet = c' \setminus c$, which together with the fact that $b \notin c$ means $b \notin c'$. Hence $b \notin c$ and $b \notin c'$, and so $c, c' \notin r_b$.

Recall that $b \in \mathcal{M}_{(\alpha,\beta)} = \{b \in B_2 \mid b \in c_{in}^2 \vee \exists g \in dom(\beta) : \ b \in \beta(g)^\bullet\}$. If $b \in \beta(g)^\bullet$ for some $g \in dom(\beta)$, then from the already proved part of (MENI5) we have $b \in \alpha^{-1}(g^\bullet)$, so $b \in dom(\alpha)$. If $b \in \alpha^{-1}(R_{s_{in}^1})$ we have again $b \in dom(\alpha)$. If $b \in c_{in}^2 \setminus \alpha^{-1}(R_{s_{in}^1})$, then b belongs to every case reachable in $\mathcal{N}_2$ when $\mathcal{N}_{TS_1}$ is simulated (this follows from proposition 4.3 and the fact that for $g \in dom(\beta)$, $\alpha^{-1}({}^\bullet g) = {}^\bullet\beta(g)$ and $\alpha^{-1}(g^\bullet) = \beta(g)^\bullet$ which was already proved). But this contradicts $b \notin c, c'$. So $b \in \beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)}$ implies $b \in dom(\alpha)$, and thus that $\sigma^{-1}(r_b)$ is not trivial. Consequently, r_b is non-trivial and hence $\mathcal{B}^{\eta(e)}_{C_{\mathcal{N}_2} \setminus r_b} \neq \emptyset$. Suppose now that $f \overset{\{\eta(e)\}}{\longrightarrow} f'$ belongs to $\mathcal{B}^{\eta(e)}_{r_b}$. Then $f, f' \in r_b$ and we have $b \in f$ and $b \in f'$. But this and (3) contradicts $b \in \eta(\overset{\blacksquare}{e})$. Hence $\mathcal{B}^{\eta(e)}_{r_b} = \emptyset$ and, as a result, $r_b \in \eta(\overset{\square}{e})$ (in $H(\mathcal{N}_2)$). From proposition 3.4(2) and $\sigma^{-1}(r_b) \neq \emptyset$, $\sigma^{-1}(r_b) \in \overset{\square}{e}$. Since $b \in dom(\alpha)$, $\alpha(b) \in \overset{\square}{e}$. Hence, in $\mathcal{N}_{TS_1}$, $\alpha(b) \in \overset{\blacksquare}{e}$ and so $b \in \alpha^{-1}(\overset{\blacksquare}{e})$ in $\mathcal{N}_2$. This means that the inclusion $\beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ holds. Thus we have shown that $g = (\alpha, \beta)$ is a net morphism from $J(TS_1)$ to $\mathcal{N}_2$.

We now want to show that $H((\alpha,\beta)) \circ \tau(TS_1) = f$ where $f = (\sigma, \eta)$, $\tau(TS_1) = (\psi_0, \psi_1)$ and $H((\alpha,\beta)) = (f_\alpha, f_\beta)$. What we need to show is that $(f_\alpha, f_\beta) \circ (\psi_0, \psi_1) = (\sigma, \eta)$, i.e. $(f_\alpha \circ \psi_0, f_\beta \circ \psi_1) = (\sigma, \eta)$. It is enough to prove that $f_\beta \circ \psi_1 = \eta$, and then $f_\alpha \circ \psi_0 = \sigma$ follows from proposition 3.3.

It is easy to show that $f_\beta \circ \psi_1 = \eta$ holds. The first of the functions involved $\psi_1 : E_{TS_1} \to E_{TS_{\mathcal{N}_{TS_1}}}$ (see (17)) is a total identity function (notice that from theorem 2.4 we have $E_{TS_1} = E_{TS_{\mathcal{N}_{TS_1}}}$). The second function $f_\beta : E_{TS_{\mathcal{N}_{TS_1}}} \to E_{TS_{\mathcal{N}_2}}$ is defined as follows: $f_\beta = \beta = \eta$, where $\eta : E_{TS_1} \to E_{TS_{\mathcal{N}_2}}$. So $f_\beta \circ \psi_1(e) = f_\beta(\psi_1(e)) = f_\beta(e) = \eta(e)$ for all $e \in dom(\eta)$.

We now prove the uniqueness of $g = (\alpha, \beta)$. Let us assume that there is another net morphism $g' = (\alpha', \beta')$ satisfying (18). Then $H((\alpha,\beta)) \circ (\psi_0, \psi_1) = (\sigma, \eta)$ and $H((\alpha', \beta')) \circ (\psi_0, \psi_1) = (\sigma, \eta)$. From the above it follows that $(f_\alpha, f_\beta) \circ (\psi_0, \psi_1) = (f_{\alpha'}, f_{\beta'}) \circ (\psi_0, \psi_1)$ which means that:

$$f_\alpha \circ \psi_0 = f_{\alpha'} \circ \psi_0, \tag{19}$$

$$f_\beta \circ \psi_1 = f_{\beta'} \circ \psi_1. \tag{20}$$

From (20) we obtain $f_\beta = f_{\beta'}$ and consequently $\beta = \beta'$ [4]. From (19) we have $f_\alpha(R_s) = f_{\alpha'}(R_s)$, for all $s \in S_1$, which means

$$\forall s \in S_1 : \ \alpha^{-1}(R_s) \cup \left(c^2_{in} \setminus \alpha^{-1}(R_{s^1_{in}})\right) = \alpha'^{-1}(R_s) \cup \left(c^2_{in} \setminus \alpha'^{-1}(R_{s^1_{in}})\right).$$

Observe that the sets $\alpha^{-1}(R_{s^1_{in}})$ and $\alpha'^{-1}(R_{s^1_{in}})$ are equal. From theorem 2.3 we know that every place in $\mathcal{N}_{TS_1}$ is a pre- or post-place of some event. This and (MENI3), (MENI4) and (MENI5) gives us

$$\begin{aligned}
\alpha^{-1}(R_{s^1_{in}}) &= \Big(\bigcup_{e \in dom(\beta)} {}^\bullet\beta(e) \ \cup \bigcup_{e \in dom(\beta)} \beta(e)^\bullet \Big) \ \cap \ c^2_{in} \\
&\overset{(\beta=\beta')}{=} \Big(\bigcup_{e \in dom(\beta')} {}^\bullet\beta'(e) \ \cup \bigcup_{e \in dom(\beta')} \beta'(e)^\bullet \Big) \ \cap \ c^2_{in} \ = \ \alpha'^{-1}(R_{s^1_{in}}).
\end{aligned}$$

Hence we obtain

$$\begin{array}{llrcl}
\forall s \in S_1 : & & \alpha^{-1}(R_s) & = & \alpha'^{-1}(R_s) \\
\forall s \in S_1 & \forall b \in B_2 : & b \in \alpha^{-1}(R_s) & \Leftrightarrow & b \in \alpha'^{-1}(R_s) \\
\forall s \in S_1 & \forall b \in B_2 : & \alpha(b) \in R_s & \Leftrightarrow & \alpha'(b) \in R_s \\
\forall s \in S_1 & \forall b \in B_2 : & s \in \alpha(b) & \Leftrightarrow & s \in \alpha'(b) \\
\forall b \in B_2 & \forall s \in S_1 : & s \in \alpha(b) & \Leftrightarrow & s \in \alpha'(b) \\
\forall b \in B_2 : & & \alpha(b) & = & \alpha'(b)
\end{array}$$

Thus $\alpha = \alpha'$, which gives $g = g'$, a contradiction. □

Example 8.5 We now illustrate the last theorem with the following example (see Figure 7). Consider the TSENI transition system TS_1 shown in Figure 7. It has four non-trivial regions: $r_1 = \{s^1_{in}, s_2\}$, $r_2 = \{s^1_{in}, s_1\}$, $r_3 = \{s_1, s_3\}$ and $r_4 = \{s_2, s_3\}$. Moreover, the pre-, post- and I-regions of a and b are: ${}^\circ a = \{r_1\}$, ${}^\circ b = \{r_2\}$, $a^\circ = \{r_3\}$, $b^\circ = \{r_4\}$, $\overset{\Box}{a} = \{r_4\}$ and $\overset{\Box}{b} = \{r_3\}$. We can build $\mathcal{N}_{TS_1}$ and then $TS_{\mathcal{N}_{TS_1}}$ using constructions recalled in section 2.3. Note that, according to theorem 2.4, $TS_{\mathcal{N}_{TS_1}}$ is isomorphic to TS_1. Consider now ENI-system $\mathcal{N}_2$ and the TSENI transition system generated by it, $TS_{\mathcal{N}_2}$. The reachable cases of $\mathcal{N}_2$ are:

$$\begin{array}{rclcrcl}
c^2_{in} & = & \{b_1, b_2, b_5\} & \quad & c_1 & = & \{b_3, b_2, b_5\} \\
c_2 & = & \{b_1, b_4, b_5\} & & c_3 & = & \{b_3, b_4, b_5\}.
\end{array}$$

We have the following regions of $TS_{\mathcal{N}_2}$ associated with every $b \in B_2$:

$$\begin{array}{rclcrclcrcl}
r_{b_1} & = & \{c^2_{in}, c_2\} & \quad & r_{b_2} & = & \{c^2_{in}, c_1\} & \quad & r_{b_3} & = & \{c_1, c_3\} \\
r_{b_4} & = & \{c_2, c_3\} & & r_{b_5} & = & \{c^2_{in}, c_1, c_2, c_3\} & & r_{b_6} & = & \emptyset.
\end{array}$$

[4] It is essential to prove that $\alpha = \alpha'$ as well, as we do not have the net version of proposition 3.3. Notice that proving $\beta = \beta'$ first was important.

Observe that $r_{b_5} = C_{\mathcal{N}_2}$ and $r_{b_6} = \emptyset$ are trivial regions. Let us define a transition system morphism $f = (\sigma, \eta)$ from TS_1 to $TS_{\mathcal{N}_2}$ in a following way.

$$\begin{array}{lllllllll} \sigma(s^1_{in}) & = & c^2_{in} & \sigma(s_1) & = & c_1 & \sigma(s_2) & = & c^2_{in} \\ \sigma(s_3) & = & c_1 & \eta(a) & = & e & \eta(b) & - & \text{not defined.} \end{array}$$

According to the construction in theorem 8.4 the net morphism $g = (\alpha, \beta)$ from $\mathcal{N}_{TS_1}$ to $\mathcal{N}_2$ is defined by:

$$\begin{array}{lll} \alpha(b_1) & = & \sigma^{-1}(r_{b_1}) = r_1 \\ \alpha(b_2) & - & \text{not defined, because } \sigma^{-1}(r_{b_2}) = S_1 \\ \alpha(b_3) & = & \sigma^{-1}(r_{b_3}) = r_3 \\ \alpha(b_4) & - & \text{not defined, because } \sigma^{-1}(r_{b_4}) = \emptyset \\ \alpha(b_5) & - & \text{not defined, because } r_{b_5} = C_{\mathcal{N}_2} \\ \alpha(b_6) & - & \text{not defined, because } r_{b_6} = \emptyset \\ \beta(a) & = & e \\ \beta(b) & - & \text{not defined.} \end{array}$$

Recall that $f_\alpha(\psi_0(s)) = f_\alpha(R_s) = \alpha^{-1}(R_s) \cup \left(c^2_{in} \setminus \alpha^{-1}(R_{s^1_{in}})\right)$. In our example $c^2_{in} \setminus \alpha^{-1}(R_{s^1_{in}}) = \{b_2, b_5\}$. We can now verify that $f_\alpha(\psi_0(s)) = \sigma(s)$ for all $s \in S_1$.

$$\begin{array}{lllllllll} R_{s^1_{in}} & = & \{r_1, r_2\} & \alpha^{-1}(R_{s^1_{in}}) & = & \{b_1\} & f_\alpha(R_{s^1_{in}}) & = & \{b_1\} \cup \{b_2, b_5\} = c^2_{in} \\ R_{s_1} & = & \{r_2, r_3\} & \alpha^{-1}(R_{s_1}) & = & \{b_3\} & f_\alpha(R_{s_1}) & = & \{b_3\} \cup \{b_2, b_5\} = c_1 \\ R_{s_2} & = & \{r_1, r_4\} & \alpha^{-1}(R_{s_2}) & = & \{b_1\} & f_\alpha(R_{s_2}) & = & \{b_1\} \cup \{b_2, b_5\} = c^2_{in} \\ R_{s_3} & = & \{r_3, r_4\} & \alpha^{-1}(R_{s_3}) & = & \{b_3\} & f_\alpha(R_{s_3}) & = & \{b_3\} \cup \{b_2, b_5\} = c_1 \end{array}$$

We observe that $f_\alpha(C_{\mathcal{N}_{TS_1}}) = \{c^2_{in}, c_1\}$. □

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References

[1] Arbib M.A., Manes E.G.: *Arrows, structures, and functors.* Academic Press, Inc. (1975).

[2] Arnold A.: *Finite Transition Systems.* Prentice Hall International (1994).

[3] Badouel E., Darondeau Ph.: *Theory of regions.* Third Advanced Course on Petri Nets, Dagstuhl Castle. To appear in the Lecture Notes in Computer Science (1997).

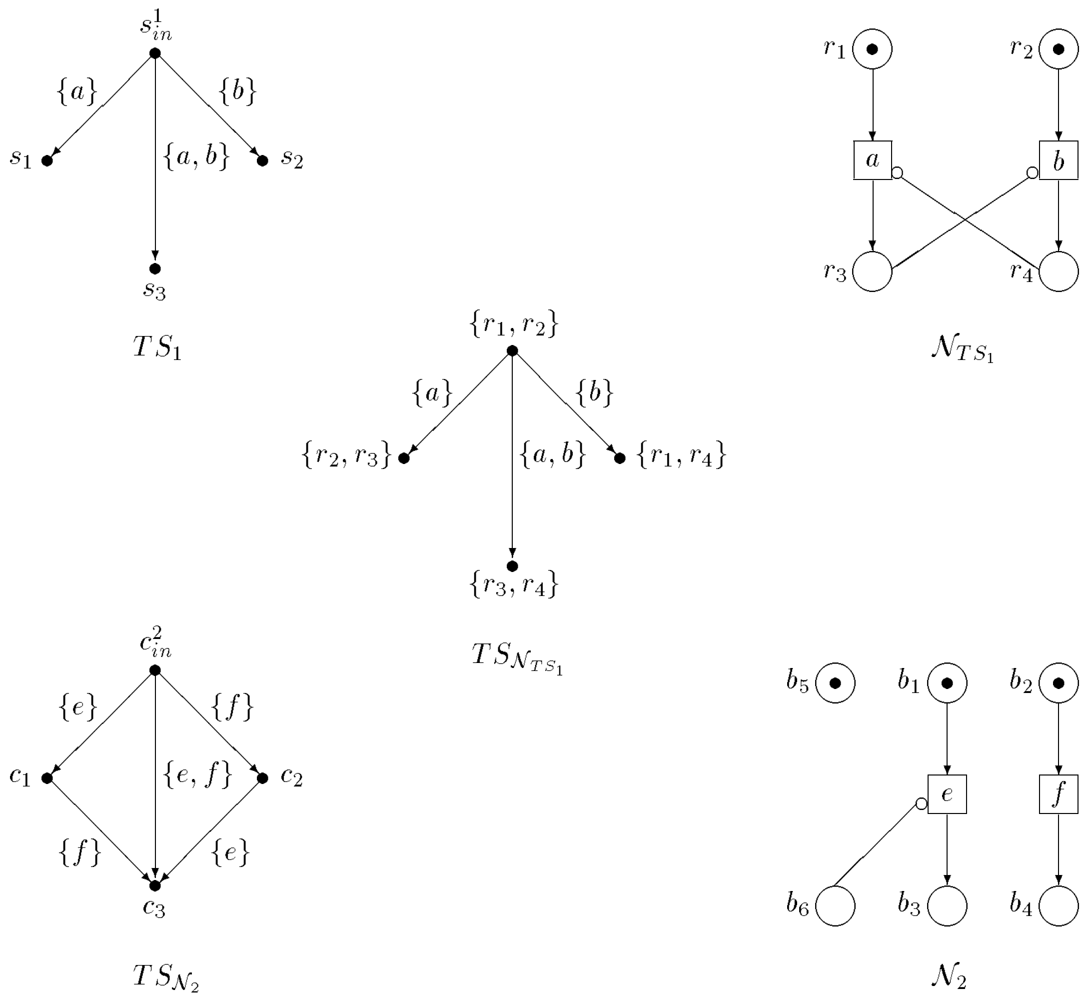

Figure 7: Example 8.5 (an illustration for theorem 8.4).

[4] Barr M., Wells C.: *Category theory for computing science.* Prentice Hall International (1990).

[5] Bednarczyk M.A.: *Categories of asynchronous systems.* Ph.D. Thesis, University of Sussex (1988).

[6] Billington J.: *Extensions to coloured nets.* Proc. of 3rd Int. Workshop on Petri Nets and Performance Models, Kyoto, Japan (1989), 61-70.

[7] Bernardinello L., De Michelis G., Petruni K., Vigna S.: *On the synchronic structure of transition systems.* In: J.Desel (Ed.) Structures in Concurrency Theory, Berlin 1995, Workshops in Computing, Springer (1995), 69-84.

[8] Chiola G., Donatelli S., Francheschinis G.: *Priorities, inhibitor arcs and concurrency in P/T nets.* Proc. of 12th Intern. Conf. on Appl. and Theory of Petri Nets, Gjern, Denmark (1991), 182-205.

[9] Christiansen S., Hansen N.D.: *Coloured Petri nets extended with place capacities, test arcs and inhibitor arcs.* Proc. of Application and Theory of Petri Nets'93, Lecture Notes in Computer Science 651, Springer (1993), 186-205.

[10] Cortadella J., Kishinevsky M., Lavagno L., Yakovlev A.: *Synthesizing petri nets from state-based models.* Proceedings of ICCAD'95 (1995), 164-171.

[11] Desel J., Reisig W.: *The synthesis problem of petri nets.* Acta Informatica, Vol. 33 (1996), 297-315.

[12] Hoogeboom H.J., Rozenberg G.: *Diamond properties of elementary net systems.* Fundamenta Informaticae XIV (1991), 287-300.

[13] Janicki R., Koutny M.: *Semantics of inhibitor nets.* Information and Computation, Vol. 123, No. 1 (1995), 1-16.

[14] Keller R.M.: *Formal verification of parallel programs.* CACM, Vol. 19, No. 7 (1976), 371-389.

[15] MacLane S.: *Categories for the working mathematician.* Springer-Verlag, (1971).

[16] Montanari U., Rossi F.: *Contextual nets.* Acta Informatica 32 (1995), 545-596.

[17] Mukund M.: *Petri nets and step transition systems.* International Journal of Foundations of Computer Science, Vol. 3, No. 4 (1992), 443-478.

[18] Nielsen M., Rozenberg G., Thiagarajan P.S.: *Elementary transition systems.* Theoretical Computer Science 96 (1992), 3-33.

[19] Pietkiewicz-Koutny M.: *Transition systems of elementary net systems with inhibitor arcs.* Technical Report No 547, 1996 and Proc. of 18th International Conference, ICATPN'97 Toulouse, France, Lecture Notes in Computer Science 1248 (June 1997), 310-327.

[20] Winskel G., Nielsen M.: *Models for concurrency.* In: S.Abramsky, Dov M.Gabbay and T.S.E.Maibaum (Eds.), Handbook of Logic in Computer Science, Vol. 4 (1995), 1-148.