Synthesis of ENI-systems Using Minimal Regions

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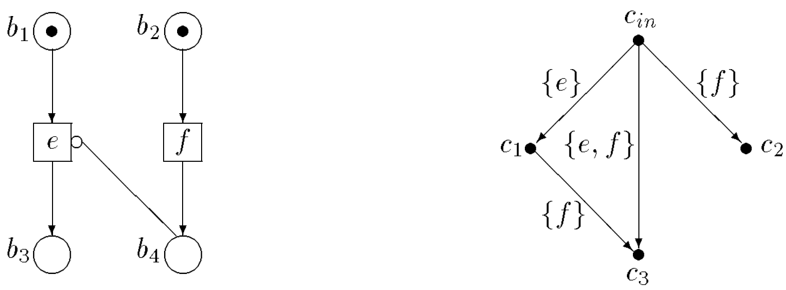
Abstract

We consider the synthesis problem for Elementary Net Systems with Inhibitor Arcs (ENI-systems) executed according to the *a-priori* semantics. The relationship between nets and transition systems generated by them (TSENI) is established via the notion of a *region*. The general synthesis problem for ENI-systems was solved in [20], and here we show how to optimise this solution using only minimal regions and selected inhibitor arcs. We also compare the proposed method of eliminating inhibitor arcs in ENI-systems with that introduced in [8] and show that they are similar.
Keywords: *Petri nets, concurrency, transition systems, regions, synthesis of nets.*

1 Introduction

The synthesis problem for Petri nets consists in constructing a net system for a given transition system in such a way that the net's behaviour is isomorphic to the transition system. This problem was solved for the class of Elementary Net Systems in [13] using the notion of a region which links nodes of transition systems (global states) with conditions in the corresponding nets (local states). The solution was later extended to the pure bounded Place Transition Nets ([7]), general Petri Nets ([18]), Safe Nets ([22]) and Elementary Nets Systems with Inhibitor Arcs ([20, 8]), by adopting the original definition of a region or using some extended notion of a generalised region. It also turned out that using all possible regions which can be found according to the general synthesis method leads to exponential algorithms. In [3], it was proved that the synthesis problem for the class of elementary nets is NP-complete. More efficient methods of synthesis were discussed in [12] and [6]. They were based on an idea that not all of the regions derived by the original method were actually needed. Practical algorithms for the synthesis problem were studied in [2] and [11].
In this paper, we consider the synthesis of Elementary Nets Systems with Inhibitor Arcs (ENI-systems) using minimal regions (w.r.t. set inclusion). The general problem of synthesis for these nets was solved in [20] where the related class of transition systems, called TSENI transition systems, were also axiomatised. Here we will show that minimal regions are sufficient to solve the synthesis problem for ENI-systems. We will show as well how to reduce the number of inhibitor arcs without changing the behaviour of a constructed net. It turns out that the redundancy in the number of regions and in the number of inhibitor arcs is linked and both can be tackled at the same time. The synthesis problem for Elementary Nets Systems with Inhibitor Arcs was studied in [8] but, unlike in this paper, only sequential behaviours were considered there. We will compare the method of elimination of inhibitor arcs presented in this paper with the one developed in [8]. As it turns out, the two methods are similar to each other.
The kind of Petri nets we are interested in is shown in Figure 1(a). The meaning of all the elements of $\mathcal{N}$ is standard except for the inhibitor arc between condition b_4 and event e (represented by an edge ending with a small circle) which indicates that e can only be fired if b_4 is empty. This has a clear interpretation if one considers purely interleaving net semantics: $\mathcal{N}$ can execute e or f or ef (i.e. e followed by f). However, when we consider a non-interleaving semantics based on step sequences, then one is faced with the problem whether or not the concurrent step $\{e, f\}$ should be allowed. Basically, both interpretations



$(a)\ \mathcal{N}$ $(b)\ TS_{\mathcal{N}}$

Figure 1: ENI-system $\mathcal{N}$ and its TSENI transition system $TS_{\mathcal{N}}$.

are possible, as discussed in [9]. The one in which it is possible to execute $\{e, f\}$ is called there the *a-priori* semantics, and that in which $\{e, f\}$ is disallowed is called the *a-posteriori* semantics. In this paper we will interpret all inhibitor arcs using the former semantics; examples of other work on nets with inhibitor arcs include [5, 10, 17]. TSENI transition systems are essentially a subset of general *step transition systems* of [18] as their arcs are labelled by sets of events rather than by single events (see Figure 1(b)); examples of other work on transition system models include [1, 4, 14, 16, 19].
The paper is organised as follows. In section 2, we recall from [20] and [21] the definitions and basic properties of ENI-systems and TSENI transition systems; in particular we recall the original construction of a net ($\mathcal{N}_{Sat}$) for a given TSENI transition system. Section 3 examines some properties of regions and minimal regions of TSENI transition systems. In section 4, we define for a given TSENI transition system a net which uses only minimal regions ($\mathcal{N}_{Min}$), and prove that it is an ENI-system. Section 5 examines the relationship between $\mathcal{N}_{Sat}$ and $\mathcal{N}_{Min}$ by defining a net morphism between the two nets. Using the results obtained in [21], it is proved that the transition systems generated by both nets are isomorphic. Section 6 looks at the possibility of a further minimisation of $\mathcal{N}_{Min}$ by removing some of its inhibitor arcs. We introduce a method for an elimination of redundant inhibitor arcs based on non-minimal regions of $\mathcal{N}_{Sat}$. The last section presents an example.

2 Preliminaries

2.1 Nets with inhibitor arcs

In this section we recall (with only notational adjustments) the definition of ENI-systems (see [15]).
Let $\mathcal{E}$ be a non-empty set of *events* fixed throughout this paper. A *net with inhibitor arcs* is a tuple $N = (B, E, F, I)$ such that B and $E \subseteq \mathcal{E}$ are finite disjoint sets, $F \subseteq (B \times E) \cup (E \times B)$ and $I \subseteq B \times E$. The meaning and graphical representation of B (conditions), E (events) and F (flow relation) is the same as in the standard net theory. An *inhibitor* arc $(s, e) \in I$ means that e can be enabled only if s is not marked (in the diagrams, it is represented by an edge ending with a small circle). We denote, for every $x \in B \cup E$, ${}^\bullet x = \{y \mid (y, x) \in F\}$ (pre-elements), $x^\bullet = \{y \mid (x, y) \in F\}$ (post-elements), $\overset{\blacksquare}{x} = \{y \mid (x, y) \in I \cup I^{-1}\}$ (I-elements). The dot-notation extends in the usual way to sets, for example, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$. It is assumed that for every $e \in E$,

$$e^\bullet \neq \emptyset \neq {}^\bullet e \quad \text{and} \quad e^\bullet \cap {}^\bullet e = \emptyset = (e^\bullet \cup {}^\bullet e) \cap \overset{\blacksquare}{e}\,. \tag{1}$$

An *elementary net system with inhibitor arcs* (ENI-system) is a tuple $\mathcal{N} = (B, E, F, I, c_{in})$ such that $N_{\mathcal{N}} = (B, E, F, I)$ is the (underlying) net with inhibitor arcs and $c_{in} \subseteq B$ is the *initial case* (in general, any subset of B is a *case*). We will assume that $\mathcal{N}$ is fixed until the end of this section.
The concurrency semantics of ENI-systems will be based on steps of simultaneously executed events. We first define valid steps:

$$V_{\mathcal{N}} = \Big\{ u \subseteq E \;\Big|\; u \neq \emptyset \;\wedge\; \forall e \neq f \in u : \; ({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset \Big\}. \tag{2}$$

The transition relation of $N_{\mathcal{N}}$, denoted by $\rightarrow_{N_{\mathcal{N}}}$, is given by:

$$\rightarrow_{N_{\mathcal{N}}} = \Big\{ (c, u, c') \in 2^B \times V_{\mathcal{N}} \times 2^B \;\Big|\; c \setminus c' = {}^\bullet u \;\wedge\; c' \setminus c = u^\bullet \;\wedge\; \overset{\blacksquare}{u} \cap c = \emptyset \Big\}. \tag{3}$$

The *state space* of $\mathcal{N}$, denoted by $C_{\mathcal{N}}$, is the least subset of 2^B containing c_{in} such that if $c \in C_{\mathcal{N}}$ and $(c, u, c') \in \rightarrow_{N_{\mathcal{N}}}$ then $c' \in C_{\mathcal{N}}$. The *transition relation* of $\mathcal{N}$, denoted by $\rightarrow_{\mathcal{N}}$, is defined as $\rightarrow_{N_{\mathcal{N}}}$ restricted to $C_{\mathcal{N}} \times V_{\mathcal{N}} \times C_{\mathcal{N}}$, and the set of *active steps* of $\mathcal{N}$ is given by $U_{\mathcal{N}} = \{u \mid (c, u, c') \in \rightarrow_{\mathcal{N}}\}$. We will use $c \xrightarrow{u}_{\mathcal{N}} c'$ to denote that $(c, u, c') \in \rightarrow_{\mathcal{N}}$. Also, $c \xrightarrow{u}_{\mathcal{N}}$ if $(c, u, c') \in \rightarrow_{\mathcal{N}}$, for some c'. A *step sequence* of $\mathcal{N}$ is a sequence $\varrho = u_1 \dots u_n$ of sets in $U_{\mathcal{N}}$ for which there are cases $c_1, \dots, c_n$ satisfying $c_{in} \xrightarrow{u_1}_{\mathcal{N}} c_1$, $c_1 \xrightarrow{u_2}_{\mathcal{N}} c_2$, $\dots$, $c_{n-1} \xrightarrow{u_n}_{\mathcal{N}} c_n$. We will denote this by $c_{in}[\varrho\rangle c_n$. The above defines the *a-priori* operational semantics of $\mathcal{N}$ (see [9]).

Proposition 2.1 [20] Let $c \in C_{\mathcal{N}}$ and $u \in V_{\mathcal{N}}$.

1. $c \xrightarrow{u}_{\mathcal{N}}$ if and only if ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$.
2. If $c \xrightarrow{u}_{\mathcal{N}} c'$ then $c' = (c \setminus {}^\bullet u) \cup u^\bullet$. □

2.2 Transition systems of nets with inhibitor arcs

In this section we recall the main definitions and results concerning the TSENI transition systems (see [20]). A *transition system* is a quadruple $TS = (S, U, T, s_{in})$ where:

TS1 S is a non-empty finite set of *states*.

TS2 $U \subseteq 2^{\mathcal{E}}$ is a set of *steps*; u is finite and non-empty, for every $u \in U$.

TS3 $T \subseteq S \times U \times S$ is the *transition relation*.

TS4 $s_{in} \in S$ is the *initial state*.

We will denote $s \xrightarrow{u} s'$ whenever $(s, u, s') \in T$; moreover $s \xrightarrow{u}$ if $s \xrightarrow{u} s'$, for some s'. By $E_{TS} = \bigcup_{u \in U} u$ we will denote all the events which can appear in steps labelling transitions in TS.
The notion of a region links the nodes of a transition system (global states) with the conditions in the corresponding net (local states). A set of states $r \subseteq S$ is a *region* if the following two conditions are satisfied:

R1 If $s \xrightarrow{u} s'$ and $s \in r$ and $s' \notin r$ then there is $e \in u$ such that:

- (a) If $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \in r$.
- (b) If $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.

R2 If $s \xrightarrow{u} s'$ and $s \notin r$ and $s' \in r$ then there is $e \in u$ such that:

- (a) If $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \notin r$.
- (b) If $q \xrightarrow{v} q'$ and $e \in v$ then $q \notin r$ and $q' \in r$.

The event $e \in u$ which satisfies the conditions in (R1) (or (R2)) is *unique*; it will be called r*-crossing* in u.

It can be shown that a complement of a region is also a region. The set of *non-trivial* regions (i.e. those different from S and $\emptyset$) will be denoted by R_{TS}. Moreover, for every state $s \in S$, we will denote by R_s the set of non-trivial regions containing s, $R_s = \{r \in R_{TS} \mid s \in r\}$. The sets of pre-regions, ${}^\circ u$, and post-regions, u°, of a step $u \in U$ are defined as:

$$\begin{array}{rcl} {}^\circ u & = & \{r \in R_{TS} \mid \exists (s,u,s') \in T : \; s \in r \; \wedge \; s' \notin r\} \\ \text{and} \quad u^\circ & = & \{r \in R_{TS} \mid \exists (s,u,s') \in T : \; s \notin r \; \wedge \; s' \in r\}. \end{array}$$

We will use ${}^\circ e$ and e° instead of respectively ${}^\circ\{e\}$ and $\{e\}^\circ$, for every $e \in E_{TS}$. The set which comprises sets of events which are potential steps in the transition system (they do not share pre- nor post-regions) is denoted V_{TS}, and defined by:

$$V_{TS} = \left\{ u \subseteq E_{TS} \;\middle|\; u \neq \emptyset \; \wedge \; \forall e \neq f \in u : \; ({}^\circ e \cup e^\circ) \cap ({}^\circ f \cup f^\circ) = \emptyset \right\}.$$

In the ENI-system constructed from a TSENI transition system, pre-regions will constitute pre-conditions and post-regions will constitute post-conditions of events. We also define inhibitor-regions, which in the constructed net will play the role of conditions connected with events by means of inhibitor arcs. We start with an auxiliary definition. Let $e \in E_{TS}$ be an event, and $r \in R_{TS}$ be a non-trivial region. Then

$$\mathcal{B}_r^e = \left\{ (s, \{e\}, s') \in T \;\middle|\; s \in r \; \wedge \; s' \in r \right\}$$

is the set of all the transitions labelled by $\{e\}$ which are totally included in r, and the set of *inhibitor-regions* (I-regions) of e is defined as follows:

$$\overset{\Box}{e} = \{r \in R_{TS} \mid \mathcal{B}_r^e = \emptyset \; \wedge \; \mathcal{B}_{S \setminus r}^e \neq \emptyset\}.$$

We can extend the last notion to a set of events $u \in U$, by $\overset{\Box}{u} = \bigcup_{e \in u} \overset{\Box}{e}$. We now can define the class of transition systems which will be the subject of our investigation throughout this paper. A transition system TS is a *TSENI transition system* if it satisfies the following six axioms:

A1 For every $(s,u,s') \in T$, $s \neq s'$.

A2 For every $u \in U$, there are $s, s' \in S$ such that $(s,u,s') \in T$.

A3 For every $s \in S \setminus \{s_{in}\}$, there are $(s_0,u_0,s_1),(s_1,u_1,s_2),\ldots,(s_{n-1},u_{n-1},s_n) \in T$ such that $s_0 = s_{in}$ and $s_n = s$.

A4 If $s \xrightarrow{u}$ and $e \in u$ then $s \xrightarrow{\{e\}}$.

A5 For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.

A6 Let $s \in S$ and $u \in V_{TS}$ be such that, for every $e \in u$, ${}^\circ e \subseteq R_s$ and $\overset{\Box}{e} \cap R_s = \emptyset$. Then $s \xrightarrow{u}$.

We now recall some facts proved in [20]. Assuming that $s \xrightarrow{u} s'$, the following hold:

$$r \in {}^\circ u \Rightarrow s \in r \; \wedge \; s' \notin r \qquad \text{and} \qquad r \in u^\circ \Rightarrow s \notin r \; \wedge \; s' \in r \tag{4}$$

$$u = \{e\} \; \wedge \; r \in \overset{\Box}{e} \Rightarrow s, s' \notin r. \tag{5}$$

Moreover,

$$\forall e \in E_{TS} : \{e\} \in U \tag{6}$$

$$\forall e \in E_{TS} : \; {}^\circ e \neq \emptyset \neq e^\circ \; \wedge \; {}^\circ e \cap e^\circ = {}^\circ e \cap \overset{\Box}{e} = e^\circ \cap \overset{\Box}{e} = \emptyset. \tag{7}$$

Finally, if $u \in U$ then u is a potential step in TS ($u \in V_{TS}$), and

$$u^\circ = \{S \setminus r \mid r \in {}^\circ u\} \tag{8}$$

$${}^\circ u = \bigcup_{e \in u} {}^\circ e \qquad \text{and} \qquad u^\circ = \bigcup_{e \in u} e^\circ. \tag{9}$$

2.3 Translations between ENI-systems and TSENI transition systems

We now recall how to construct a TSENI transition system from a given ENI-system, and vice versa (see [20]). The first construction is straightforward.
Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI-system. Then $TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \rightarrow_{\mathcal{N}}, c_{in})$ is the *transition system generated* by $\mathcal{N}$.

Theorem 2.2 [20] $TS_{\mathcal{N}}$ is a TSENI transition system. □

The reverse translation is based on the pre- post- and I-regions of events appearing in a transition system. Let $TS = (S, U, T, s_{in})$ be a TSENI transition system. The net system *associated* with TS is defined as $\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$ where F_{TS} and I_{TS} are defined thus:

$$\begin{array}{rrcl} & F_{TS} & = & \{(r,e) \in R_{TS} \times E_{TS} \mid r \in {}^\circ e\} \;\cup\; \{(e,r) \in E_{TS} \times R_{TS} \mid r \in e^\circ\} \\ \text{and} & I_{TS} & = & \{(r,e) \in R_{TS} \times E_{TS} \mid r \in \overset{\square}{e}\,\}. \end{array} \tag{10}$$

Theorem 2.3 [20] $\mathcal{N}_{TS}$ is an ENI-system. □

The next result states that the ENI-system associated with a TSENI transition system TS generates a transition system which is isomorphic to TS.

Theorem 2.4 [20] Let $\mathcal{N} = \mathcal{N}_{TS}$.

1. $C_{\mathcal{N}} = \{R_s \mid s \in S\}$.
2. $\rightarrow_{\mathcal{N}} = \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.
3. $TS_{\mathcal{N}}$ is isomorphic to TS with $s \mapsto R_s$ (for $s \in S$) being an isomorphism. □

2.4 Morphisms for inhibitor nets and related transition systems

We now recall morphisms between TSENI transition systems and between ENI-systems (see [21]).
Below, for any (partial or total) function $f : X \rightarrow Y$ we will denote by $dom(f)$ the domain of f, by $codom(f)$ the codomain of f, and by $\widehat{f}$ the lifting of f to a total function $\widehat{f} : 2^X \rightarrow 2^Y$ defined, for every $X' \subseteq X$, by $\widehat{f}(X') = f(X' \cap dom(f))$.

Definition 2.5 Let $TS_i = (S_i, U_i, T_i, s^i_{in})$ $(i = 1, 2)$ be TSENI transition systems. A *transition system morphism* from TS_1 to TS_2 is a pair of functions $(\sigma, \eta) : TS_1 \rightarrow TS_2$ such that:

MTS1 $\sigma : S_1 \rightarrow S_2$ is a total function satisfying $\sigma(s^1_{in}) = s^2_{in}$.

MTS2 $\eta : E_{TS_1} \rightarrow E_{TS_2}$ is a partial function, which is injective on every $u \in U_1$.

MTS3 For every $(s, u, s') \in T_1$, either $\widehat{\eta}(u) = \emptyset$ and $\sigma(s) = \sigma(s')$, or $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$. □

Definition 2.6 Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems. A *net morphism* from $\mathcal{N}_1$ to $\mathcal{N}_2$ is a pair of functions $(\alpha, \beta) : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ such that:

MENI1 $\alpha : B_2 \rightarrow B_1$ is a partial function.

MENI2 $\beta : E_1 \rightarrow E_2$ is a partial function.

MENI3 For every $b \in dom(\alpha)$, $\alpha(b) \in c^1_{in}$ if and only if $b \in c^2_{in}$.

MENI4 For every $e \in E_1 \setminus dom(\beta)$, $\alpha^{-1}({}^\bullet e) = \emptyset = \alpha^{-1}(e^\bullet)$.

MENI5 For every $e \in dom(\beta)$: $\alpha^{-1}({}^\bullet e) = {}^\bullet\beta(e)$, $\alpha^{-1}(e^\bullet) = \beta(e)^\bullet$ and $\overset{\blacksquare}{\beta(e)} \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$, where $\mathcal{M}_{(\alpha,\beta)} = \{b \in B_2 \mid b \in c^2_{in} \;\vee\; \exists e \in dom(\beta) : b \in \beta(e)^\bullet\}$. □

The following two propositions will be needed in section 5. They were proved for ENI-systems in [21] and they are similar to the results obtained for Elementary Nets Systems in [19].

Proposition 2.7 [21] Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems, and $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ be a net morphism. Moreover, let $f_\alpha : C_{\mathcal{N}_1} \to 2^{B_2}$ be a mapping such that, for every $c \in C_{\mathcal{N}_1}$, $f_\alpha(c) = \alpha^{-1}(c) \cup \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right)$.

1. For every $c \in C_{\mathcal{N}_1}$, $f_\alpha(c) \in C_{\mathcal{N}_2}$.
2. If $c \xrightarrow{u}_{\mathcal{N}_1} c'$ and $\widehat{\beta}(u) = \emptyset$ then $f_\alpha(c) = f_\alpha(c')$.
3. If $c \xrightarrow{u}_{\mathcal{N}_1} c'$ and $\widehat{\beta}(u) \neq \emptyset$ then $f_\alpha(c) \xrightarrow{\widehat{\beta}(u)}_{\mathcal{N}_2} f_\alpha(c')$. □

Proposition 2.8 [21] Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems, and $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ be a net morphism. Moreover, let $f_\alpha : C_{\mathcal{N}_1} \to C_{\mathcal{N}_2}$ be a total function defined by $f_\alpha(c) = \alpha^{-1}(c) \cup \left(c^2_{in} \setminus \alpha^{-1}(c^1_{in})\right)$, and $f_\beta : E_{TS_{\mathcal{N}_1}} \to E_{TS_{\mathcal{N}_2}}$ be a mapping defined by $f_\beta = \beta$. Then (f_α, f_β) is a transition system morphism from $TS_{\mathcal{N}_1}$ to $TS_{\mathcal{N}_2}$. □

3 Properties of (minimal) regions of TSENI transition systems

Let $TS = (S, U, T, s_{in})$ be a TSENI transition system fixed for the rest of this paper. The results in this section were formulated for transition systems describing sequential behaviour: Elementary Transition Systems in [6, 8, 11], and Condition Event Transition Systems in [6]. Here we show that they hold for TSENI transition systems, where non-sequential behaviour is represented explicitly.

Proposition 3.1 If r' and r are regions in R_{TS} such that $r' \subset r$ then $r_{diff} = r \setminus r' \in R_{TS}$.

Proof. First we prove that (R1) holds for r_{diff}. Let $s \xrightarrow{u} s'$, $s \in r_{diff} = r \setminus r'$ and $s' \notin r_{diff}$. We need to consider two cases (see figure 2).
Case 1: $s' \in r'$. Since r' is a region, there is $e \in u$ such that:

(i) If $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \notin r'$.

(ii) If $q \xrightarrow{v} q'$ and $e \in v$ then $q \notin r'$ and $q' \in r'$.

To show (R1) for r_{diff} it suffices to prove that $s'', q \in r$ in the formulae above.
Suppose that $s \xrightarrow{u'} s''$, $u' \subseteq u \setminus \{e\}$ and $s'' \in S \setminus r$ in (i). Then we have $s \in r$ (by $s \in r_{diff}$) and $s'' \notin r$ (by $s'' \in S \setminus r$). Since r is a region, there is $e' \in u'$ such that:

(iii) If $w \xrightarrow{h} w'$ and $e' \in h$ then $w \in r$ and $w' \notin r$.

From (iii) with $w = s$, $w' = s'$ and $h = u$ (notice that $e' \in u$) we obtain $s' \notin r$, which produces a contradiction with $s' \in r' \subset r$. Hence $s'' \in r$ in (i).
Suppose now that $q \xrightarrow{v} q'$, $e \in v$ and $q \in S \setminus r$ in (ii). Then we have $q \notin r$ and $q' \in r' \subset r$. Since r is a region, there exists $e'' \in v$ such that:

(iv) If $u'' \subseteq v \setminus \{e''\}$ and $q \xrightarrow{u''} s'''$ then $s''' \notin r$.

(v) If $p \xrightarrow{v'} p'$ and $e'' \in v'$ then $p \notin r$ and $p' \in r$.

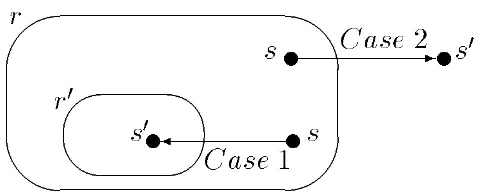


Figure 2: Illustration for proposition 3.1.

From (A4) and $q \xrightarrow{v} q'$ it follows that there exists q'' such that $q \xrightarrow{\{e\}} q''$. By (ii), $q'' \in r' \subset r$. If $e \neq e''$ then $q'' \notin r$, by (iv) with $u'' = \{e\}$ and $s''' = q''$, producing a contradiction. Suppose $e = e''$. Then (v) is satisfied with $p = s$, $p' = s'$ and $v' = u$. This implies $s \notin r$, contradicting $s \in r_{diff} \subset r$. Hence $q \in r$ in (ii).

Case 2: $s' \notin r$. Since r is a region, there is $e \in u$ such that:

(vi) If $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \in r$.

(vii) If $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.

Now, to show (R1) for r_{diff} it suffices to prove that $s'', q \notin r'$ in the formulae above.

Suppose that $s \xrightarrow{u'} s''$, $u' \subseteq u \setminus \{e\}$ and $s'' \in r'$ in (vi). Since r' is a region and $s \notin r'$, there exists $e' \in u'$ such that:

(viii) If $w \xrightarrow{h} w'$ and $e' \in h$ then $w \notin r'$ and $w' \in r'$.

From (viii) with $w = s$, $w' = s'$ and $h = u$ (notice that $e' \in u$) we obtain $s' \in r'$, which contradicts $s' \notin r$ (because $r' \subset r$). Hence $s'' \notin r'$ in (vi).

Suppose now that $q \xrightarrow{v} q'$, $e \in v$ and $q \in r'$ in (vii). Then $q \in r'$ and $q' \notin r'$ (because $q' \notin r$). Since r' is a region, there exists $e'' \in v$ such that:

(ix) If $u'' \subseteq v \setminus \{e''\}$ and $q \xrightarrow{u''} s'''$ then $s''' \in r'$.

(x) If $p \xrightarrow{v'} p'$ and $e'' \in v'$ then $p \in r'$ and $p' \notin r'$.

From (A4) and $q \xrightarrow{v} q'$ it follows that there exists q'' such that $q \xrightarrow{\{e\}} q''$. By (vii), $q'' \notin r$ and hence $q'' \notin r'$. If $e \neq e''$ then $q'' \in r'$, by (ix) with $u'' = \{e\}$ and $s''' = q''$, producing a contradiction. Suppose $e = e''$. Then (x) is satisfied with $p = s$, $p' = s'$ and $v' = u$. This implies $s \in r'$, contradicting $s \in r_{diff} = r \setminus r'$. Hence $q \notin r'$ in (vii).

That (R2) holds for r_{diff} can be proved in a similar way. Hence r_{diff} is a region. Moreover, as $r_{diff} \neq \emptyset$, $r_{diff} \in R_{TS}$. □

Proposition 3.2 If r' and r'' are disjoint regions in R_{TS} then $r' \cup r''$ is a (possibly trivial) region.

Proof. Define $r = r' \cup r''$. If $r = S$ then r is a trivial region in TS. Suppose that $r \neq S$ and r is not a region. From $r' \in R_{TS}$ it follows that $S \setminus r' \in R_{TS}$. Moreover, $r'' \subset S \setminus r'$ (because $r' \cap r'' = \emptyset$). Hence, by proposition 3.1, $(S \setminus r') \setminus r'' = S \setminus (r' \cup r'') \in R_{TS}$, a contradiction with $r' \cup r'' \notin R_{TS}$. □

Definition 3.3 A region $r \in R_{TS}$ is *minimal* if $r' \not\subset r$ for every $r' \in R_{TS}$.

The proof of the next result is similar to that of property 3.3 in [11].

Theorem 3.4 Every $r \in R_{TS}$ can be represented as a disjoint union of minimal regions.

Proof. If r is minimal then the result holds. If r is non-minimal then there exists a minimal region $r' \subset r$. From proposition 3.1 it follows that $r'' = r \setminus r'$ is a region in R_{TS}. If r'' is minimal we have $r = r' \cup r''$. Otherwise, we continue in the same way with r'' instead of r. In this way we will build a sequence of mutually disjoint, minimal regions which will be finite as S is finite, and whose union is equal to r. $\square$

Proposition 3.5 Let r be a non-minimal region in R_{TS}, $u \in U$, $e \in E_{TS}$ and $s \in S$.

1. If $r \in {}^\circ u$ then there exists a minimal region $r' \subset r$ such that $r' \in {}^\circ u$.
2. If $r \in u^\circ$ then there exists a minimal region $r' \subset r$ such that $r' \in u^\circ$.
3. If $r \in \overset{\square}{e}$ then for every minimal region $r' \subset r$, $r' \in \overset{\square}{e}$.
4. If $r \in R_s$ then there exists a minimal region $r' \subset r$ such that $r' \in R_s$.

Proof. (1) There exists $s \xrightarrow{u} s'$ such that $s \in r$ and $s' \notin r$. From theorem 3.4 it follows that r can be represented as a disjoint union of a set R of minimal regions. Let r' be a minimal region in R such that $s \in r'$. Since $s' \notin r$, $s' \notin r'$. Hence $r' \in {}^\circ u$.
(2) Can be proved similarly as (1).
(3) From the definition of an inhibitor region of e, it follows that for every non-trivial region $r' \subset r$, $r' \in \overset{\square}{e}$.
(4) Follows directly from theorem 3.4. $\square$

4 Minimisation of ENI-systems

Let $\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$ be an ENI-system associated with TS (see (10)). $\mathcal{N}_{TS}$ will be called *saturated* because it uses all the non-trivial regions as conditions; we will denote it by $\mathcal{N}_{Sat}$.
Let $\mathcal{R} \in 2^{R_{TS}}$ be a set of non-trivial regions of TS. Then $Min(\mathcal{R}) = \{r \in \mathcal{R} \mid r \text{ is minimal}\}$ will denote the set of minimal regions in $\mathcal{R}$.
We now define a net system $\mathcal{N}_{Min}$ (called *minimal*), which was obtained from $\mathcal{N}_{Sat}$ by deleting all the conditions associated with non-minimal regions and adjacent arcs:

$$\mathcal{N}_{Min} = (Min(R_{TS}), E_{TS}, \widehat{F}_{TS}, \widehat{I}_{TS}, Min(R_{s_{in}}))$$

where $\widehat{F}_{TS}$ and $\widehat{I}_{TS}$ are defined thus:

$$\begin{array}{lcl} \widehat{F}_{TS} & = & \{(r,e) \in R_{TS} \times E_{TS} \mid r \in Min({}^\circ e)\} \ \cup \ \{(e,r) \in E_{TS} \times R_{TS} \mid r \in Min(e^\circ)\} \\ \widehat{I}_{TS} & = & \{(r,e) \in R_{TS} \times E_{TS} \mid r \in Min(\overset{\square}{e})\}. \end{array} \tag{11}$$

Directly from the definition of $\mathcal{N}_{Sat}$, i.e. (10), we have that, for every $e \in E_{TS}$,

$${}^\bullet e = {}^\circ e,\ e^\bullet = e^\circ \text{ and } \overset{\blacksquare}{e} = \overset{\square}{e} \ \ (\text{in } \mathcal{N}_{Sat}). \tag{12}$$

Similarly, for $\mathcal{N}_{Min}$ we obtain from (11) that, for every $e \in E_{TS}$,

$${}^\bullet e = Min({}^\circ e),\ e^\bullet = Min(e^\circ) \text{ and } \overset{\blacksquare}{e} = Min(\overset{\square}{e}) \ (\text{in } \mathcal{N}_{Min}). \tag{13}$$

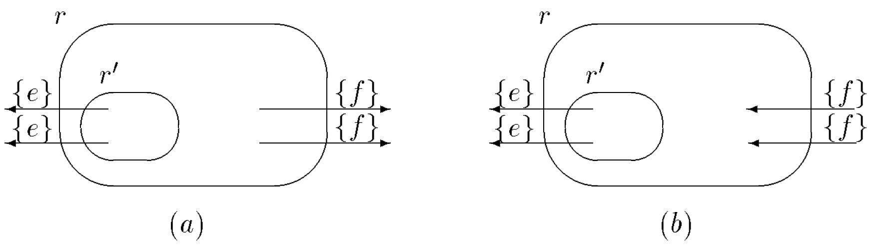


Figure 3: Illustration for proposition 4.2.

Proposition 4.1 $\mathcal{N}_{Min}$ is an ENI-system.

Proof. Since $\mathcal{N}_{Sat}$ is an ENI-system, it suffices to show that, for every $e \in E_{TS}$, $e^\bullet$ and ${}^\bullet e$ are both non-empty sets in $\mathcal{N}_{Min}$. Thus, by (13), it suffices to show that for all $e \in E_{TS}$, $Min(e^\circ) \neq \emptyset \neq Min({}^\circ e)$. From (7) it follows that $e^\circ \neq \emptyset \neq {}^\circ e$, for all $e \in E_{TS}$. And the former follows directly from (6) and proposition 3.5(1,2). $\square$

The following proposition shows that any active step of events from $\mathcal{N}_{Min}$ is a valid step in $\mathcal{N}_{Sat}$, although in the latter there are more conditions.

Proposition 4.2 $U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Sat}}$.

Proof. Let $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$. We need to show that $u \in V_{\mathcal{N}_{Sat}}$. From the definition of a valid step in ENI-system, (2), (12) and (13) we have:

$$V_{\mathcal{N}_{Sat}} = \left\{ u \subseteq E_{TS} \;\middle|\; u \neq \emptyset \;\wedge\; \forall e \neq f \in u : \; ({}^\circ e \cup e^\circ) \cap ({}^\circ f \cup f^\circ) = \emptyset \right\}$$

$$V_{\mathcal{N}_{Min}} = \left\{ u \subseteq E_{TS} \;\middle|\; u \neq \emptyset \;\wedge\; \forall e \neq f \in u : \; (Min({}^\circ e) \cup Min(e^\circ)) \cap (Min({}^\circ f) \cup Min(f^\circ)) = \emptyset \right\}.$$

Let $e, f \in u$ and $e \neq f$. We will prove that ${}^\circ e \cap {}^\circ f = \emptyset$. Suppose that there is $r \in {}^\circ e \cap {}^\circ f$. Then $r \in R_{TS}$ is non-minimal due to the definition of $V_{\mathcal{N}_{Min}}$ and $U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$. From (6) and proposition 3.5(1) it follows that there exists a minimal region $r' \subset r$ such that $r' \in {}^\circ e$. We consider two cases.
Case 1: $r' \in {}^\circ f$. Then $r' \in Min({}^\circ e) \cap Min({}^\circ f)$. Since $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$, we obtain a contradiction.
Case 2: $r' \notin {}^\circ f$. Then $r \setminus r' \in {}^\circ f$ (see figure 3(a)). (Notice that proposition 3.1 guarantees that $r \setminus r' \in R_{TS}$.) We observe that $r' \in \overset{\square}{f}$. From $u \in U_{\mathcal{N}_{Min}}$ we have that there exist $c, c' \in C_{\mathcal{N}_{Min}}$ such that $c \xrightarrow{u}_{\mathcal{N}_{Min}} c'$. From proposition 2.1(1) it follows that ${}^\bullet u \subseteq c$ and $\overset{\blacksquare}{u} \cap c = \emptyset$ (in $\mathcal{N}_{Min}$). Hence, ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{f} \cap c = \emptyset$ which, after applying (13), means that $Min({}^\circ e) \subseteq c$ and $Min(\overset{\square}{f}) \cap c = \emptyset$. But $r' \in {}^\circ e$, $r' \in \overset{\square}{f}$ and the fact that r' is minimal implies $r' \in Min({}^\circ e)$ and $r' \in Min(\overset{\square}{f})$, a contradiction.
Hence ${}^\circ e \cap {}^\circ f = \emptyset$. To prove $e^\circ \cap f^\circ = \emptyset$, suppose that there exists r in $e^\circ \cap f^\circ$. From (8) it follows that $S \setminus r \in {}^\circ e \cap {}^\circ f$, which contradicts the previously proven fact. What remains to be shown is ${}^\circ e \cap f^\circ = \emptyset$ (as the case $e^\circ \cap {}^\circ f = \emptyset$ is symmetric).
Suppose that there exists a non-minimal region $r \in {}^\circ e \cap f^\circ$. From (6) and proposition 3.5(1) it follows that there exists a minimal region $r' \subset r$ such that $r' \in {}^\circ e$. We again consider two cases.
Case 1: $r' \in f^\circ$. Then $r' \in Min({}^\circ e) \cap Min(f^\circ)$. Since $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$, we obtain a contradiction.
Case 2: $r' \notin f^\circ$. Then $r \setminus r' \in f^\circ$ (see figure 3(b)). We observe that $r' \in \overset{\square}{f}$. From $u \in U_{\mathcal{N}_{Min}}$ we have that there exist $c, c' \in C_{\mathcal{N}_{Min}}$ such that $c \xrightarrow{u}_{\mathcal{N}_{Min}} c'$. From proposition 2.1(1) it follows that ${}^\bullet u \subseteq c$ and $\overset{\blacksquare}{u} \cap c = \emptyset$ (in $\mathcal{N}_{Min}$). Hence, ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{f} \cap c = \emptyset$ which, after applying (13), means that $Min({}^\circ e) \subseteq c$ and $Min(\overset{\square}{f}) \cap c = \emptyset$. But $r' \in {}^\circ e$, $r' \in \overset{\square}{f}$ and the fact that r' is minimal implies $r' \in Min({}^\circ e)$ and $r' \in Min(\overset{\square}{f})$, a contradiction. Hence ${}^\circ e \cap f^\circ = \emptyset$, which completes the proof. $\square$

5 $TS_{\mathcal{N}_{Sat}}$ and $TS_{\mathcal{N}_{Min}}$ are isomorphic

In this section we examine the relationship between the behaviour of the saturated and minimal net constructed for a TSENI transition system $TS = (S, U, T, s_{in})$. First we define a mapping between ENI-systems $\mathcal{N}_{Sat}$ and $\mathcal{N}_{Min}$ as follows: $(\widetilde{\alpha}, \widetilde{\beta}) : \mathcal{N}_{Sat} \to \mathcal{N}_{Min}$, where $\widetilde{\alpha} : Min(R_{TS}) \to R_{TS}$ and $\widetilde{\beta} : E_{TS} \to E_{TS}$ are both total identity functions. Notice that,

$$\forall X \subseteq R_{TS} : \ \widetilde{\alpha}^{-1}(X) = Min(X). \tag{14}$$

Proposition 5.1 $(\widetilde{\alpha}, \widetilde{\beta})$ is a net morphism from $\mathcal{N}_{Sat}$ to $\mathcal{N}_{Min}$.

Proof. (MENI1) and (MENI2) are clearly satisfied. For (MENI3) we need to show that for every $r \in dom(\widetilde{\alpha})$, $\widetilde{\alpha}(r) \in R_{s_{in}} \Leftrightarrow r \in Min(R_{s_{in}})$. It follows easily from the fact that $Min(R_{s_{in}}) = \widetilde{\alpha}^{-1}(R_{s_{in}})$ (see (14)). (MENI4) holds since for all $e \in E_{TS}$, $e \in dom(\widetilde{\beta})$. Finally, we show that (MENI5) holds as follows. For every $e \in E_{TS}$,

$$\begin{array}{llllll}
r \in {}^{\bullet}\widetilde{\beta}(e) & (\text{in } \mathcal{N}_{Min}) & \Leftrightarrow \ \widetilde{\alpha}(r) \in {}^{\bullet}e & (\text{in } \mathcal{N}_{Sat}) & \Leftrightarrow \ r \in \widetilde{\alpha}^{-1}({}^{\bullet}e) & (\text{in } \mathcal{N}_{Min}) \\
r \in \widetilde{\beta}(e)^{\bullet} & (\text{in } \mathcal{N}_{Min}) & \Leftrightarrow \ \widetilde{\alpha}(r) \in e^{\bullet} & (\text{in } \mathcal{N}_{Sat}) & \Leftrightarrow \ r \in \widetilde{\alpha}^{-1}(e^{\bullet}) & (\text{in } \mathcal{N}_{Min}) \\
r \in \overset{\blacksquare}{\widetilde{\beta}(e)} & (\text{in } \mathcal{N}_{Min}) & \Leftrightarrow \ \widetilde{\alpha}(r) \in \overset{\blacksquare}{e} & (\text{in } \mathcal{N}_{Sat}) & \Leftrightarrow \ r \in \widetilde{\alpha}^{-1}(\overset{\blacksquare}{e}) & (\text{in } \mathcal{N}_{Min})
\end{array}$$

Hence $(\widetilde{\alpha}, \widetilde{\beta})$ is a well defined net morphism from $\mathcal{N}_{Sat}$ to $\mathcal{N}_{Min}$. □

Consider the mappings f_α and f_β defined in proposition 2.8 for a net morphism (α, β) between two ENI-systems $\mathcal{N}_1$ and $\mathcal{N}_2$. According to proposition 2.8, $(f_\alpha, f_\beta) : TS_{\mathcal{N}_1} \to TS_{\mathcal{N}_2}$ is a transition system morphism. We will show that for the specific $(\widetilde{\alpha}, \widetilde{\beta})$ defined above, $(f_{\widetilde{\alpha}}, f_{\widetilde{\beta}})$ is in fact an isomorphism. Before proving this we have the following result.

Proposition 5.2 Let $e \in E_{TS}$ and $s \in S$ in TS.

1. If $Min({}^{\circ}e) \subseteq Min(R_s)$ then ${}^{\circ}e \subseteq R_s$ and $e^{\circ} \cap R_s = \emptyset$.
2. If $Min(\overset{\square}{e}) \cap Min(R_s) = \emptyset$ then $\overset{\square}{e} \cap R_s = \emptyset$.

Proof. (1) Suppose that $r \in {}^{\circ}e$ is a non-minimal region such that $r \notin R_s$. From proposition 3.5(1) and (6) it follows that there exists a minimal region $r' \subset r$ such that $r' \in {}^{\circ}e$. Clearly, $r \notin R_s$ implies $r' \notin R_s$, a contradiction with $Min({}^{\circ}e) \subseteq Min(R_s)$. Hence ${}^{\circ}e \subseteq R_s$ holds.
Suppose now that there exists $r \in e^{\circ} \cap R_s$. Then $S \setminus r \in {}^{\circ}e$ and $S \setminus r \notin R_s$, and we proceed as before, obtaining a contradiction with $Min({}^{\circ}e) \subseteq Min(R_s)$. Hence $e^{\circ} \cap R_s = \emptyset$ is satisfied.
(2) Suppose that $r \in \overset{\square}{e} \cap R_s$ is a non-minimal region. From proposition 3.5(4) we have that there exists a minimal region $r' \subset r$ such that $r' \in R_s$. From proposition 3.5(3) we have $r' \in \overset{\square}{e}$, which contradicts $Min(\overset{\square}{e}) \cap Min(R_s) = \emptyset$. □

Proposition 5.3 $(f_{\widetilde{\alpha}}, f_{\widetilde{\beta}})$ is an isomorphism between $TS_{\mathcal{N}_{Sat}}$ and $TS_{\mathcal{N}_{Min}}$.

Proof. From theorem 2.4(1) it follows that for $\mathcal{N}_{Sat} = \mathcal{N}_{TS}$, $C_{\mathcal{N}_{Sat}} = \{R_s \mid s \in S\}$. As a result, $f_{\widetilde{\alpha}} : \{R_s \mid s \in S\} \to C_{\mathcal{N}_{Min}}$ and, for all $s \in S$,

$$f_{\widetilde{\alpha}}(R_s) \overset{(prop.2.8)}{=} \widetilde{\alpha}^{-1}(R_s) \cup \Big(Min(R_{s_{in}}) \setminus \widetilde{\alpha}^{-1}(R_{s_{in}})\Big) \overset{(14)}{=} \widetilde{\alpha}^{-1}(R_s) \cup \Big(\widetilde{\alpha}^{-1}(R_{s_{in}}) \setminus \widetilde{\alpha}^{-1}(R_{s_{in}})\Big) = \widetilde{\alpha}^{-1}(R_s).$$

Hence, for all $s \in S$, $f_{\widetilde{\alpha}}(R_s) = \widetilde{\alpha}^{-1}(R_s) \overset{(14)}{=} Min(R_s) \in C_{\mathcal{N}_{Min}}$. Thus $f_{\widetilde{\alpha}}$ maps the set of regions containing a specific state into its subset of minimal regions. We will prove that $f_{\widetilde{\alpha}}$ is a bijection.
First we show that $f_{\widetilde{\alpha}}$ is injective. Suppose $R_{s_1} \neq R_{s_2}$ and $Min(R_{s_1}) = Min(R_{s_2})$. Then, there exists a non-minimal region $r \in R_{s_1} \setminus R_{s_2}$ (the case $r \in R_{s_2} \setminus R_{s_1}$ is symmetric). From proposition 3.5(4) it follows that there exists a minimal region $r' \subset r$ such that $r' \in R_{s_1}$. Since $Min(R_{s_1}) = Min(R_{s_2})$ and r' is a minimal region, we obtain $r' \in R_{s_2}$. This implies that $s_2 \in r' \subset r$ and, as a result, that $r \in R_{s_2}$. Hence we obtained a contradiction, and so $f_{\widetilde{\alpha}}$ is injective.
We now show that $f_{\widetilde{\alpha}}$ is onto. For all $s \in S$, $f_{\widetilde{\alpha}}(R_s) \in C_{\mathcal{N}_{Min}}$. We need to prove that for every $c \in C_{\mathcal{N}_{Min}}$, there exists $s \in S$ such that $Min(R_s) = c$. To the contrary, suppose that this is not the case. We observe that $f_{\widetilde{\alpha}}(R_{s_{in}}) = Min(R_{s_{in}})$. Thus there exists a step sequence $\varrho = \varrho' u$ of sets of $U_{\mathcal{N}_{Min}}$ such that $Min(R_{s_{in}})[\varrho\rangle c'$ and $c' \neq Min(R_s)$, for all $s \in S$, and there exists $s' \in S$ such that $Min(R_{s_{in}})[\varrho'\rangle Min(R_{s'}) \xrightarrow{u}_{\mathcal{N}_{Min}} c'$. We will show that u is enabled at $R_{s'}$ in $\mathcal{N}_{Sat}$, i.e.

$$Min(R_{s'}) \xrightarrow{u}_{\mathcal{N}_{Min}} \quad \Rightarrow \quad R_{s'} \xrightarrow{u}_{\mathcal{N}_{Sat}} . \tag{15}$$

From proposition 2.1(1) we have ${}^\bullet u \subseteq Min(R_{s'})$, $u^\bullet \cap Min(R_{s'}) = \emptyset$ and $\overset{\blacksquare}{u} \cap Min(R_{s'}) = \emptyset$ (in $\mathcal{N}_{Min}$). Hence, ${}^\bullet e \subseteq Min(R_{s'})$ and $\overset{\blacksquare}{e} \cap Min(R_{s'}) = \emptyset$, for all $e \in u \subseteq E_{TS}$. By (13) we have $Min({}^\circ e) \subseteq Min(R_{s'})$ and $Min(\overset{\square}{e}) \cap Min(R_{s'}) = \emptyset$, for all $e \in u$. From this and proposition 5.2(1,2) it follows that ${}^\circ e \subseteq R_{s'}$, $e^\circ \cap R_{s'} = \emptyset$ and $\overset{\square}{e} \cap R_{s'} = \emptyset$, for all $e \in u$ which, after applying (12), means that ${}^\bullet e \subseteq R_{s'}$, $e^\bullet \cap R_{s'} = \emptyset$ and $\overset{\blacksquare}{e} \cap R_{s'} = \emptyset$, for all $e \in u$ (in $\mathcal{N}_{Sat}$). We recall that from proposition 4.2 we have $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Sat}}$, and $R_{s'} \in C_{\mathcal{N}_{Sat}}$ is satisfied as well. So, we can apply proposition 2.1(1) to obtain $R_{s'} \xrightarrow{u}_{\mathcal{N}_{Sat}}$ which proves (15). This implies that there exists $s'' \in S$ such that $R_{s'} \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s''}$ and then from proposition 2.1(2) and (12) we get $R_{s''} = (R_{s'} \setminus {}^\circ u) \cup u^\circ$. Notice that u is a step in TS as $u \in U_{\mathcal{N}_{Sat}} = U$ (see theorem 2.4(2)). From $Min(R_{s'}) \xrightarrow{u}_{\mathcal{N}_{Min}} c'$ and proposition 2.1(2) we have the following:

$$\begin{aligned} c' &= \Big(Min(R_{s'}) \setminus {}^\bullet u\Big) \cup u^\bullet & &\overset{(13)}{=} \Big(Min(R_{s'}) \setminus Min({}^\circ u)\Big) \cup Min(u^\circ) \\ &\overset{(14)}{=} \Big(\widetilde{\alpha}^{-1}(R_{s'}) \setminus \widetilde{\alpha}^{-1}({}^\circ u)\Big) \cup \widetilde{\alpha}^{-1}(u^\circ) & &= \widetilde{\alpha}^{-1}\Big((R_{s'} \setminus {}^\circ u) \cup u^\circ\Big) \\ &= \widetilde{\alpha}^{-1}(R_{s''}) & &\overset{(14)}{=} Min(R_{s''}). \end{aligned}$$

Hence we obtained a contradiction, and thus proved that $f_{\widetilde{\alpha}}$ is onto.
Thus $f_{\widetilde{\alpha}}$ is a bijection from $\{R_s \mid s \in S\}$ to $\{Min(R_s) \mid s \in S\}$, and $f_{\widetilde{\alpha}}(R_{s_{in}}) = Min(R_{s_{in}})$. The second mapping, $f_{\widetilde{\beta}} : E_{TS_{\mathcal{N}_{Sat}}} \to E_{TS_{\mathcal{N}_{Min}}}$, defined in proposition 2.8 by $f_{\widetilde{\beta}} = \widetilde{\beta}$ is a bijection as well, as $\widetilde{\beta}$ is a total identity function from E_{TS} to E_{TS}, and $E_{TS_{\mathcal{N}_{Sat}}} = E_{TS}$ (follows from theorem 2.4(2)). Finally, we need to prove that

$$R_s \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s'} \quad \Leftrightarrow \quad Min(R_s) \xrightarrow{u}_{\mathcal{N}_{Min}} Min(R_{s'}).$$

The "$\Rightarrow$" implication follows from proposition 2.7(3). We need to show that the reverse implication holds as well. Let $Min(R_s) \xrightarrow{u}_{\mathcal{N}_{Min}} Min(R_{s'})$. From the already proved (15) we have that $R_s \xrightarrow{u}_{\mathcal{N}_{Sat}}$. This implies that there exists $s'' \in S$ such that $R_s \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s''}$ and then from proposition 2.1(2) and (12) we get $R_{s''} = (R_s \setminus {}^\circ u) \cup u^\circ$. From this and (14) we obtain

$$\begin{aligned} Min(R_{s''}) &= \widetilde{\alpha}^{-1}\Big((R_s \setminus {}^\circ u) \cup u^\circ\Big) & &= \Big(\widetilde{\alpha}^{-1}(R_s) \setminus \widetilde{\alpha}^{-1}({}^\circ u)\Big) \cup \widetilde{\alpha}^{-1}(u^\circ) \\ &= \Big(Min(R_s) \setminus Min({}^\circ u)\Big) \cup Min(u^\circ) & &\overset{(prop.2.1(2),(13))}{=} Min(R_{s'}). \end{aligned}$$

Hence, $Min(R_{s''}) = Min(R_{s'})$. Since $f_{\widetilde{\alpha}}$ is an injective function, $R_{s''} = R_{s'}$. But TS is a TSENI transition system and, from axiom (A5) we get $s' = s''$. Consequently, $R_s \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s'}$. □

Theorem 5.4 TS is isomorphic to $TS_{\mathcal{N}_{Min}}$.

Proof. From theorem 2.4(3) we have that TS is isomorphic to $TS_{\mathcal{N}_{Sat}}$. Proposition 5.3 states, on the other hand, that $TS_{\mathcal{N}_{Sat}}$ is isomorphic to $TS_{\mathcal{N}_{Min}}$. Hence TS is isomorphic to $TS_{\mathcal{N}_{Min}}$. $\square$

6 Reduced ENI-systems

In this section we will further reduce $\mathcal{N}_{Min}$ without changing its behaviour, by removing some inhibitor arcs. Below we denote the disjoint union of sets by $\uplus$.

Proposition 6.1 Let $r' \subseteq r$ be regions in R_{TS} and $u \in U$.

1. If $r \in {}^\circ u$ then $r' \in {}^\circ u \cup \overset{\square}{u}$.
2. If $r \in u^\circ$ then $r' \in u^\circ \cup \overset{\square}{u}$.

Proof. (1) There exists $s \xrightarrow{u} s'$ such that $s \in r$ and $s' \notin r$. If $s \in r'$ then, because $s' \notin r'$ (by $s' \notin r$), we have $r' \in {}^\circ u$. Suppose that $s \notin r'$. From the definition of a region and (A4) it follows that there exist $e \in u$ and $s'' \in S$ such that $s \xrightarrow{\{e\}} s''$ and $s'' \notin r$. From (4) we obtain that for all $p \xrightarrow{\{e\}} p'$, $p \in r$ and $p' \notin r$. This means $p' \notin r'$, and therefore there is no arc labelled with e inside r' or coming into r'. There are no arcs labelled with e coming out of r' as well, because, by (4), this would mean that all such arcs would be coming out of r', contradicting $s \notin r'$. So, in this case $r' \in \overset{\square}{e} \subseteq \overset{\square}{u}$.
(2) Can be proven in a similar way as (1). $\square$

Proposition 6.2 Let r be a non-minimal region of R_{TS} and $u \in U$.

1. If $r \in {}^\circ u$ then there exist minimal regions r' and r_i $(i = 1, \ldots, n)$ such that $r' \in {}^\circ u$, $r_i \in \overset{\square}{u}$ (for $i = 1, \ldots, n$) and $r = r' \uplus \biguplus_{i=1}^{n} r_i$.
2. If $r \in u^\circ$ then there exist minimal regions r' and r_i $(i = 1, \ldots, n)$ such that $r' \in u^\circ$, $r_i \in \overset{\square}{u}$ (for $i = 1, \ldots, n$) and $r = r' \uplus \biguplus_{i=1}^{n} r_i$.

Proof. (1) From proposition 3.5(1) it follows that there exists a minimal region $r' \subset r$ such that $r' \in {}^\circ u$. Then $r'' = r \setminus r'$, which according to proposition 3.1 is a region in R_{TS}, does not belong to ${}^\circ u$ (see (4)). Hence from proposition 6.1 it follows that $r'' \in \overset{\square}{u}$. Thus there is $e \in u$ such that $r'' \in \overset{\square}{e}$. If r'' is minimal then $n = 1$ and $r_1 = r''$. If r'' is non-minimal, theorem 3.4 says that it can be represented as a disjoint union of minimal regions $r_1, \ldots, r_n$ $(n \geq 2)$, and from proposition 3.5(3) it follows that for all $i = 1, \ldots, n$, $r_i \in \overset{\square}{e}$. Consequently, in both cases, $r_i \in \overset{\square}{u}$ (for $i = 1, \ldots, n$).
(2) The proof of this part is similar to (1). $\square$

Note that the representation of a non-minimal region r, given in proposition 6.2, does not need to be unique (see the last paragraph of section 7).

Proposition 6.3 Let $e \in E_{TS}$ and r be a non-minimal region in R_{TS} such that $r \in {}^\circ e$.
Then there are minimal regions $r' \in {}^\circ e$ and $r_i \in \overset{\square}{e}$ $(i = 1, \ldots, n;\ n \geq 1)$ such that $r = r' \uplus \biguplus_{i=1}^{n} r_i$. Moreover, if one deletes the set of inhibitor arcs $\mathcal{I} = \{(e, r_1), \ldots, (e, r_n)\}$ from $\mathcal{N}_{Sat}$ or $\mathcal{N}_{Min}$ then the transition system of the resulting net remains the same (up to isomorphism).

Proof. From proposition 6.2(1) and (6) it follows that the above representation of r is possible. Recall that $C_{\mathcal{N}_{Sat}} = \{R_s \mid s \in S\}$ and $C_{\mathcal{N}_{Min}} = \{Min(R_s) \mid s \in S\}$. Suppose a condition corresponding to the region r' is marked at R_s. This means $r' \in R_s$ and so $s \in r'$. Consequently $s \notin r_i$ $(i = 1, \ldots, n)$ as the minimal regions in the representation are mutually disjoint. Hence $r_i \notin R_s$ $(i = 1, \ldots, n)$ which means they are not marked. In this case the inhibitor arcs (e, r_i) are not needed. If r' is not marked at R_s then e is not enabled and it does not matter whether the r_i's are marked or not. Thus in both cases the marking of the r_i's does not change the enabledness of e at any marking R_s. Hence the inhibitor arcs in $\mathcal{I}$ can be removed without changing the transition system generated by the net. □

We will denote by $\mathcal{I}_{TS}$ the union of all the sets $\mathcal{I}$ in proposition 6.3, after taking into account every $e \in E_{TS}$, every non-minimal pre-region r of e, and every possible representation of r described there. The net obtained from $\mathcal{N}_{Min}$ by deleting all the inhibitor arcs in $\mathcal{I}_{TS}$, will be called *reduced* and denoted by

$$\mathcal{N}_{Rcd} = (Min(R_{TS}), E_{TS}, \widehat{F}_{TS}, \widehat{I}_{TS} \setminus \mathcal{I}_{TS}, Min(R_{s_{in}})).$$

Theorem 6.4 *$TS_{\mathcal{N}_{Min}}$ is isomorphic to $TS_{\mathcal{N}_{Rcd}}$.*

Proof. Follows from proposition 6.3. □

Definition 6.5 An ENI-system $\mathcal{N}'$ is a *state machine* if its initial case is a singleton set and every event has exactly one pre-condition and one post-condition. A *state machine component* of an ENI-system $\mathcal{N} = (B, E, F, I, c_{in})$ is a state machine $\mathcal{N}' = (B', E', F', I', c'_{in})$ such that $B' \subseteq B$, $E' = \{e \in E \mid (e^\bullet \cup {}^\bullet e) \cap B' \neq \emptyset\}$, $F' = F \cap (B' \times E' \cup E' \times B')$, $I' = I \cap (B' \times E')$ and $c'_{in} = c_{in} \cap B'$. A *state machine decomposition* of $\mathcal{N}$ is a set of state machine components, $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, \ldots, n)$ such that $B = \bigcup_{i=1}^n B_i$, $E = \bigcup_{i=1}^n E_i$ and $F = \bigcup_{i=1}^n F_i$. □

In [6] it was shown that the states of an elementary transition system can be decomposed into disjoint minimal regions; moreover any such decomposition induces a state machine component. The set of all possible decompositions determines a set of state machine components which cover the minimal net associated with this elementary transition system. In this paper we have proved, in theorem 3.4, that any non-trivial region of a TSENI transition system can be represented as a disjoint union of minimal regions. The decomposability of minimal ENI-systems into state machines can then be proved in a similar way as it was done in [6] for Elementary Net Systems. For example $\mathcal{N}_{Min}$ ($\mathcal{N}_{Rcd}$) considered in section 7 has two state machine components: one induced by the decomposition $S = r_2 \uplus r_3 \uplus r_5$ and the other by $S = r_1 \uplus r_4 \uplus r_5$.
The ability of decomposing a net into state machine components can be useful for finding those inhibitor arcs which can be removed from the net without changing its behaviour. In [8], where the sequential behaviour of Elementary Net Systems with Inhibitor Arcs was investigated, it was shown that inhibitor arcs which are present within a state machine component are superfluous. We will show that the method of eliminating inhibitor arcs introduced in this section for ENI-systems is similar to the method described in [8].

Theorem 6.6 *Let $SM_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, \ldots, l)$ be the state machine components of $\mathcal{N}_{Min}$. Then $(e, r_{inh}) \in \mathcal{I}_{TS}$ if and only if there exists SM_k $(1 \leq k \leq l)$ such that $(e, r_{inh}) \in I_k$.*

Proof. Let $(e, r_{inh}) \in \mathcal{I}_{TS}$. Then there exists a non-minimal region $r \in R_{TS}$ such that $r \in {}^\circ e$ and r can be represented as $r = r' \uplus \biguplus_{i=1}^n r_i$ $(n \geq 1)$, where $r' \in {}^\circ e$ and $r_i \in \overset{\square}{e}$ (for $i = 1, \ldots, n$) are minimal regions. Let $1 \leq i_k \leq n$ be such that $r_{i_k} = r_{inh}$. We have $S \setminus r \in e^\circ$. Define r'' as $S \setminus r$, if it is minimal; otherwise define r'' as a minimal post-region of e appearing in the representation of $S \setminus r$ in proposition 6.2(2). Then $S = r' \uplus r'' \uplus \biguplus_{i=1}^n r_i \uplus \biguplus_{j=0}^m \overline{r}_j$, where $m \geq 0$ and $\overline{r}_j \in \overset{\square}{e}$ $(j = 1, \ldots, m)$ are minimal

regions. Define SM_k as a state machine component of $\mathcal{N}_{Min}$ induced by the decomposition of S given above. Clearly, $(e, r_{inh}) \in I_k$.
To prove the reverse implication we assume that $(e, r_{inh}) \in I_k \setminus \mathcal{I}_{TS}$ for some $1 \leq k \leq l$. Then there are r_{pred}, $r_{succ} \in B_k$ such that (r_{pred}, e), $(e, r_{succ}) \in F_k$ and r_{pred}, r_{succ} and r_{inh} are mutually disjoint non-empty sets (they are minimal regions from the decomposition associated with SM_k). Hence, by proposition 3.2, $r = r_{pred} \cup r_{inh}$ is a non-trivial region in TS and $r \in {}^\bullet e = {}^\circ e$ in $\mathcal{N}_{Sat}$. By proposition 6.3, $(e, r_{inh}) \in \mathcal{I}_{TS}$, a contradiction. $\square$

7 An example

Figure 4 shows the saturated ENI-system $\mathcal{N}_{Sat}$ associated with a TSENI transition system TS, and two stages of minimisation of $\mathcal{N}_{Sat}$. The regions in TS are:

$$
\begin{array}{llllllllllll}
r_1 & = & \{s_0, s_1\} & r_2 & = & \{s_0, s_2\} & r_3 & = & \{s_1, s_3\} & r_4 & = & \{s_2, s_3\} \\
r_5 & = & \{s_4\} & r_6 & = & \{s_0, s_1, s_2, s_3\} & r_7 & = & \{s_0, s_1, s_4\} & r_8 & = & \{s_0, s_2, s_4\} \\
r_9 & = & \{s_1, s_3, s_4\} & r_{10} & = & \{s_2, s_3, s_4\} & & & & & &
\end{array}
$$

and the pre-regions, post-regions and I-regions of events are:

$$
\begin{array}{lllllllll}
{}^\circ a & = & \{r_2, r_8\} & a^\circ & = & \{r_3, r_9\} & \overset{\square}{a} & = & \{r_4, r_5, r_{10}\} \\
{}^\circ b & = & \{r_1, r_7\} & b^\circ & = & \{r_4, r_{10}\} & \overset{\square}{b} & = & \{r_3, r_5, r_9\} \\
{}^\circ c & = & \{r_3, r_4, r_6\} & c^\circ & = & \{r_5, r_7, r_8\} & \overset{\square}{c} & = & \{r_1, r_2\}.
\end{array}
$$

The minimal regions of TS are: r_1, r_2, r_3, r_4 and r_5. To obtain $\mathcal{N}_{Min}$, we minimise $\mathcal{N}_{Sat}$ by removing conditions associated with non-minimal regions and the adjacent arcs. At this stage two inhibitor arcs are deleted: (a, r_{10}) and (b, r_9). The resulting $\mathcal{N}_{Min}$ has still redundant inhibitor arcs which can be identified by looking at non-minimal pre-regions of events in $\mathcal{N}_{Sat}$, and representing them as disjoint unions of minimal pre-regions and I-regions, as described in proposition 6.2. For event a we have: $r_8 = r_2 \uplus r_5$, for b: $r_7 = r_1 \uplus r_5$, for c: $r_6 = r_3 \uplus r_2$ and $r_6 = r_4 \uplus r_1$. Thus from proposition 6.3 it follows that the following inhibitor arcs are redundant: (a, r_5), (b, r_5), (c, r_2) and (c, r_1). Notice that the representation of a non-minimal pre-region, given in proposition 6.2, does not need to be unique; for example, as in the case of r_6. In such a situation we can eliminate more inhibitor arcs. At the end of this process we obtain $\mathcal{N}_{Rcd}$.

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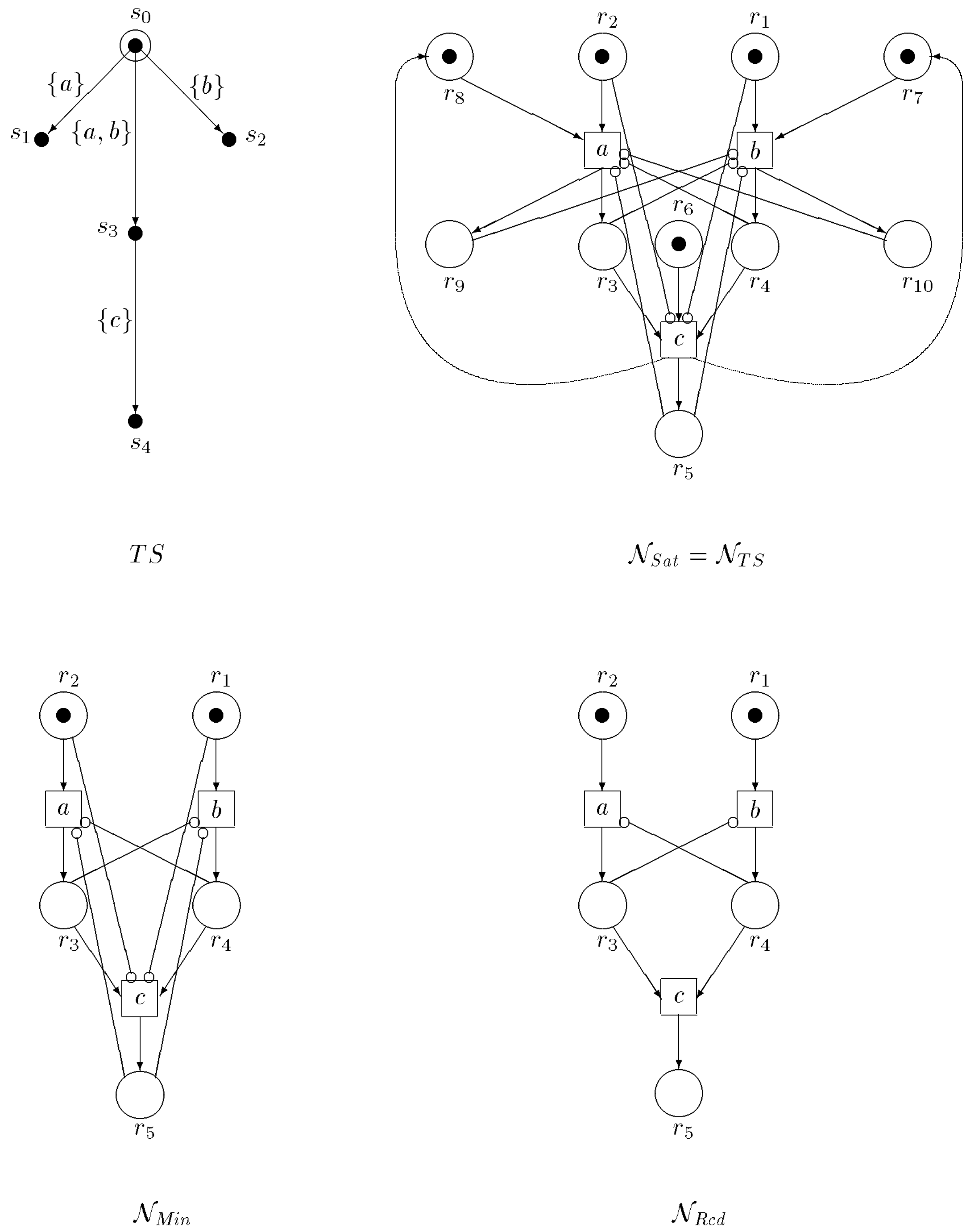

Figure 4: Minimisation of the ENI-system for a given TSENI transition system.

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