An $O\left(T^2\right)$ Discrete-Time Adaptive Regulator for Uncertain MIMO Systems with Bounded Input Delays

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***Abstract*— This paper proposes a discrete-time adaptive regulator for a MIMO linear time-invariant system with unknown, constant input time delays that may differ across the input channels. It is assumed the delay has a known upper-bound. In addition, the plant is subject to an unmeasurable exogeneous disturbance. To mitigate the effect of the disturbance, a second-order delay disturbance observer is used. A stability analysis shows that the proposed regulator drives the plant state to zero asymptotically with an $O(T^2)$ bound on the regulation error and simulation results are shown to verify the approach.**

I. INTRODUCTION

This paper addresses the problem of regulating the state of a continuous-time MIMO linear system with uncertain parameters and uncertain input time delay, using a discrete-time controller. Input delays arise in many different contexts, for example, communications delays in multi-agent systems [1], and as a result of model reduction of PDEs [2]. Moreover, in a multi-input system it is also possible that different input channels have different time delays.

The efficacy of model-based control depends on the availability of accurate models, which is compounded by the predictor-based nature of most time-delay compensation techniques [3], but in practical situations some error is inevitable. Thus, a controller must be capable of handling uncertainty in the plant parameters *and* the time delay(s). Exogeneous disturbances are another source of uncertainty and the problem of reducing this and quantifying the residual disturbance is also studied here.

The key ideas of the present work are outlined as follows:

The continuous-time plant is sampled, and the controller is formulated in a discrete-time setting. This simplifies the analysis and greatly facilitates implementation on sampled-data systems. Furthermore, the integrals found in continuous-time time-delay compensators can, if improperly discretised, lead to instability [4].

To compensate for the time delay, a discrete-time application of Artstein's model reduction [5] is employed. This involves finding a suitable substitution to transform the original dynamics with time-delay into a delay-free dynamics. Various tasks such as controller design and controllability analysis can be performed on the latter, with results that transfer back to the former. Artstein's reduction can be extended to cope with uncertain plant parameters and time delay, by taking a robust approach (as is done in [6], which notably considers a time-varying delay), or by an adaptive approach that is pursued here. This is based on previous work by the authors [7], [8], which was restricted to scalar systems without disturbance.

Two measures are employed to mitigate the effect of disturbances. The first is a standard use of a deadzone in the adaptation law, to ensure the adaptation process does not suffer from parameter drift instability [9, Ch. 8]. The second is to estimate and compensate for the disturbance using a second-order delay disturbance observer. To illustrate the idea, consider a linear system with disturbance ω_k:

$$x_{k+1} = Ax_k + \omega_k$$

The disturbance in the previous time step ω_{k-1} is

$$\omega_{k-1} = x_k - Ax_{k-1}$$

If ω_{k-1} is used to estimate the disturbance in the current time step k, then ω_k is just the sum of the estimate and its error, given by

$$\omega_k = \hat{\omega}_k + \tilde{\omega}_k = \omega_{k-1} + \omega_k - \omega_{k-1} = \omega_{k-1} + \Delta\omega_k$$

Notice the error is a first-order difference of ω_k, denoted $\Delta\omega_k$. Assuming that the disturbance is smooth, this first-order observer incurs an error of $\|\Delta\omega_k\| \in O(T^2)$ [10]. Substituting this back into the original dynamics yields a higher order model with a smaller disturbance:

$$x_{k+1} = (A+I)x_k - Ax_{k-1} + \Delta\omega_k$$

Designing a controller around this model thus achieves partial compensation of the disturbance, as estimated by the disturbance observer. The estimate can be further refined by estimating $\Delta\omega_k$ itself using $\Delta\omega_{k-1}$, which leaves an error equal to the second-order difference of the disturbance, *viz.* $\Delta^2\omega_k$. Continuing further, it can be shown that with an n-th order disturbance observer,

$$\omega_k = \sum_{i=0}^{n-1} \Delta^i \omega_{k-1} + \Delta^n \omega_k$$

Moreover, it can be shown that substituting an n-th order disturbance estimate into the original dynamics is equivalent to taking the n-th order difference of the original dynamics and making x_{k+1} the subject.

The paper is organised as follows: The problem is defined in Section II. The adaptive plant model and adaptation laws are developed in IIIA. In IIIB, the state substitutes are defined and the resulting delay-free dynamics is obtained. After

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proving its controllability in Lemma 1, the control law is proposed. In IIIC the closed loop system is shown to be stable, via Lemmas 2–4 and Theorem 1. Section IV illustrates the method with simulation results.

II. PROBLEM DEFINITION

Consider the n^{th} order MIMO system in continuous-time with input delay given as

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\boldsymbol{\xi}(\mathbf{u}) + \boldsymbol{\delta}(t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{\xi}(\mathbf{u}) \in \mathbb{R}^m$ are the state vector and the delayed inputs vector, respectively. The delayed inputs vector is given as $\boldsymbol{\xi}(\mathbf{u}) = \left[u_1(t-\tau_1) \,\vdots\, u_2(t-\tau_2) \,\vdots\, \cdots \,\vdots\, u_m(t-\tau_m)\right]$ where $u_j \in \mathbb{R}$ for $j = 1,\ldots,m$ are the control inputs and $\tau_j \in \mathbb{R}^+$ for $j = 1,\ldots,m$ are uncertain input time-delays. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant uncertain matrices and $\boldsymbol{\delta}(t) \in \mathbb{R}^n$ is an unmeasurable bounded exogenous disturbance vector. Assume, without loss of generality, that the system (1) is written in such a way that $\tau_1 < \tau_2 < \cdots < \tau_l = \cdots = \tau_q < \cdots < \tau_m$ and $\tau_l = \cdots = \tau_q = \tau_{lq}$.

Now consider that the system (1) is sampled at a uniform time interval T (where in general the time delay $\tau_j \; \forall j \in [1,m]$ may not be an integer multiple of T) such that it is described by the sampled-data model given as,

$$\begin{aligned}\mathbf{x}_{k+1} = F\mathbf{x}_k &+ \sum_{j=1, j\notin[l,q]}^{m} \left(G_{1,j}\mathbf{u}_{k-d_j} + G_{2,j}\mathbf{u}_{k-d_j-1}\right)\\ &+ \left(G_{1,lq}\mathbf{u}_{k-d_{lq}} + G_{2,lq}\mathbf{u}_{k-d_{lq}-1}\right) + \boldsymbol{\omega}_k \end{aligned} \tag{2}$$

where $k \in \mathbb{Z}^+$ corresponds to the k^{th} time-step and $d_{lq}, d_j \in \mathbb{Z}^+$ for $j = 1,\ldots,l,q\ldots,m$ are the uncertain constant delays in time-steps that satisfy $d_{lq}T \le \tau_{lq} \le (d_{lq}+1)T$, $d_jT \le \tau_j \le (d_j+1)T$ and $d_1 < d_2 < \cdots < d_{lq} < \cdots < d_m$. The matrices $F \in \mathbb{R}^{n\times n}$, $G_{1,j} \in \mathbb{R}^{n\times m}$, $G_{2,j} \in \mathbb{R}^{n\times m}$, $G_{1,lq} \in \mathbb{R}^{n\times m}$ and $G_{2,lq} \in \mathbb{R}^{n\times m}$ are computed using the relations

$$F = e^{AT}, \; G_{1,j} = \int_0^{(d_j+1)T-\tau_j} e^{A\sigma}d\sigma B_j,$$

$$G_{2,j} = \int_{(d_j+1)T-\tau_j}^{T} e^{A\sigma}d\sigma B_j, \; G_{1,lq} = \int_0^{(d_{lq}+1)T-\tau_{lq}} e^{A\sigma}d\sigma B_{lq}$$

and

$$G_{2,lq} = \int_{(d_{lq}+1)T-\tau_{lq}}^{T} e^{A\sigma}d\sigma B_{lq}$$

where $B_j \in \mathbb{R}^{n\times m}$ is a matrix with the entries of the j^{th} column being the j^{th} column of the matrix B and all other entries being zero while $B_{lq} \in \mathbb{R}^{n\times m}$ is a matrix with the entries of the l^{th} to q^{th} columns being the l^{th} to q^{th} columns of the matrix B and all other entries being zero. Finally, the disturbance signal $\boldsymbol{\omega}_k$ is computed as

$$\boldsymbol{\omega}_k = \int_0^T e^{A\sigma}B\boldsymbol{\delta}\left((k+1)T-\sigma\right)d\sigma$$

The system (1) and the sampled-data system (2) satisfy the following assumptions:

Assumption 1: The disturbance $\boldsymbol{\delta}(t)$ is smooth and bounded and as a result $\|\boldsymbol{\omega}_k\| \in O(T)$ and $\|\boldsymbol{\omega}_k - 2\boldsymbol{\omega}_{k-1} + \boldsymbol{\omega}_{k-2}\| = \|\boldsymbol{\nu}_k\| \le \nu_{\max} \in O\left(T^3\right)$, [10].

Assumption 2: The delay $\tau_{\max} = \sup\{\tau_1,\ldots,\tau_q\}$ is bounded as $\tau_{\max} \le \tau_{\text{p}}$ and τ_{p} satisfies $pT \le \tau_{\text{p}} \le (p+1)T$ where p is the upper-bound on the delay in time-steps.

The control problem is to find a bounded control input vector $\mathbf{u}_k$ in sampled-time which will drive the state vector, $\mathbf{x}(t)$, to zero asymptotically, while keeping all system signals bounded.

III. MAIN RESULT

In this section an adaptive estimator and the adaptive law design is presented followed by the control law design. The control law is computed from a delay free dynamics derived from the adaptive estimator using a reduction approach inspired by [5]. Finally, the section concludes with a rigorous stability analysis of the system to verify the validity of the approach.

A. Adaptive Estimator Design

Consider the 2^{nd}-order difference, applied to the system (2), expressed in the form

$$\begin{aligned}\mathbf{x}_{k+1} = 2\mathbf{x}_k - \mathbf{x}_{k-1} &+ F\left(\mathbf{x}_k - 2\mathbf{x}_{k-1} + \mathbf{x}_{k-2}\right)\\ &+ \sum_{j=1, j\neq(l,q)}^{m} G_{1,j}\left(\mathbf{u}_{k-d_j} - 2\mathbf{u}_{k-d_j-1} + \mathbf{u}_{k-d_j-2}\right)\\ &+ \sum_{j=1, j\neq(l,q)}^{m} G_{2,j}\left(\mathbf{u}_{k-d_j-1} - 2\mathbf{u}_{k-d_j-2} + \mathbf{u}_{k-d_j-3}\right)\\ &+ G_{1,lq}\left(\mathbf{u}_{k-d_{lq}} - 2\mathbf{u}_{k-d_{lq}-1} + \mathbf{u}_{k-d_{lq}-2}\right)\\ &+ G_{2,lq}\left(\mathbf{u}_{k-d_{lq}-1} - 2\mathbf{u}_{k-d_{lq}-2} + \mathbf{u}_{k-d_{lq}-3}\right)\\ &+ \boldsymbol{\omega}_k - 2\boldsymbol{\omega}_{k-1} + \boldsymbol{\omega}_{k-2}\end{aligned} \tag{3}$$

which is re-written in the form

$$\begin{aligned}\mathbf{x}_{k+1} &= (F+2I)\,\mathbf{x}_k - (2F+I)\,\mathbf{x}_{k-1} + F\mathbf{x}_{k-2}\\ &\quad+ \sum_{j=1, j\neq(l,q)}^{m}\Big(G_{1,j}\mathbf{u}_{k-d_j} + (G_{2,j} - 2G_{1,j})\,\mathbf{u}_{k-d_j-1}\\ &\quad+ (G_{1,j} - 2G_{2,j})\,\mathbf{u}_{k-d_j-2} + G_{2,j}\mathbf{u}_{k-d_j-3}\Big)\\ &\quad+ G_{1,lq}\mathbf{u}_{k-d_{lq}} + (G_{2,lq} - 2G_{1,lq})\,\mathbf{u}_{k-d_{lq}-1}\\ &\quad+ (G_{1,lq} - 2G_{2,lq})\,\mathbf{u}_{k-d_{lq}-2} + G_{2,lq}\mathbf{u}_{k-d_{lq}-3}\\ &\quad+ \boldsymbol{\omega}_k - 2\boldsymbol{\omega}_{k-1} + \boldsymbol{\omega}_{k-2}\\ &= \sum_{i=0}^{2}\Phi_{i+1}\mathbf{x}_{k-i} + \sum_{i=d_1}^{d_m+3}\Gamma_i\mathbf{u}_{k-i} + \boldsymbol{\nu}_k\end{aligned} \tag{4}$$

where $\Phi_1 = (F+2I)$, $\Phi_2 = -(2F+I)$ and $\Phi_3 = F$, respectively. The input matrices Γ_{d_1} and Γ_{d_m+3} are given as $\Gamma_{d_1} = G_{1,1}$ and $\Gamma_{d_m+3} = G_{2,m}$, respectively, while the matrices $\Gamma_i \; \forall i \in (d_1, d_m+3)$ contain a combination of the matrices $G_{1,j}$, $G_{2,j}$, $G_{1,lq}$ and $G_{2,lq}$.

As the system (4), is written in terms of the uncertain delays $d_1,\ldots,d_m$, it will be useful to define the matrix $\Theta^\top \triangleq \left[\,\Phi_1 \,\vdots\, \cdots \,\vdots\, \Phi_3 \,\vdots\, \Psi_0 \,\vdots\, \cdots \,\vdots\, \Psi_{p_{\text{m}}}\,\right] \in \mathbb{R}^{n\times[m(p_{\text{m}}+1)+3n]}$

which is the augmented matrix of uncertain matrices and the vector $\boldsymbol{\zeta}_k^\top \triangleq \left[\, \mathbf{x}_k^\top \;\vdots\; \cdots \;\vdots\; \mathbf{x}_{k-2}^\top \;\vdots\; \mathbf{u}_k^\top \;\vdots\; \cdots \;\vdots\; \mathbf{u}_{k-p_\mathrm{m}}^\top \,\right] \in \mathbb{R}^{[m(p_\mathrm{m}+1)+3n]}$ which is the augmented signal vector such that the system (4) is written in the form

$$\mathbf{x}_{k+1} = \Theta^\top \boldsymbol{\zeta}_k + \boldsymbol{\nu}_k \tag{5}$$

where $p_\mathrm{m} = p + 3$ and the matrices $\Psi_i \in \mathbb{R}^{n\times m}$ are defined as

$$\Psi_i = \begin{cases} \Gamma_i & d_1 \le i \le d_m + 3 \\ [0] & \text{otherwise} \end{cases} \qquad i \in [0, p_\mathrm{m}]. \tag{6}$$

Now consider the adaptive estimator given as

$$\hat{\mathbf{x}}_{k+1} = \sum_{j=0}^{2} \hat{\Phi}_{j,k}\mathbf{x}_{k-i} + \sum_{i=0}^{p_\mathrm{m}} \hat{\Psi}_{i,k}\mathbf{u}_{k-i} = \hat{\Theta}_k^\top \boldsymbol{\zeta}_k \tag{7}$$

where $\hat{\mathbf{x}}_k$ is the estimate of the state vector $\mathbf{x}_k$ and $\hat{\Theta}_k^\top \triangleq \left[\, \hat{\Phi}_{1,k} \;\vdots\; \cdots \;\vdots\; \hat{\Phi}_{3,k} \;\vdots\; \hat{\Psi}_{0,k} \;\vdots\; \cdots \;\vdots\; \hat{\Psi}_{p_\mathrm{m},k} \,\right] \in \mathbb{R}^{n\times[m(p_\mathrm{m}+1)+3n]}$ is the estimate of the parameter matrix Θ respectively. The purpose of the adaptive estimator (7) is to facilitate in the computation of the control law which would otherwise be difficult due to the uncertain parameters in the system (5).

Proceeding with the adaptive law design, if the output estimation error is defined as $\tilde{\mathbf{x}}_k \triangleq \mathbf{x}_k - \hat{\mathbf{x}}_k$ then the output estimation dynamics is obtained as

$$\tilde{\mathbf{x}}_{k+1} = \tilde{\Theta}_k^\top \boldsymbol{\zeta}_k + \boldsymbol{\nu}_k \tag{8}$$

where $\tilde{\Theta}_k \triangleq \Theta - \hat{\Theta}_k$ is the augmented parameter estimation error matrix. From (8) the adaptive law is derived as

$$\hat{\Theta}_{k+1} = \begin{cases} \hat{\Theta}_k + \alpha_k\rho_k P_{k+1}\boldsymbol{\zeta}_k\tilde{\mathbf{x}}_{k+1}^\top & \forall k \in [k_0, \infty) \\ \hat{\Theta}_{k_0} & \forall k \in [0, k_0) \end{cases} \tag{9}$$

$$P_{k+1} = \begin{cases} P_k - \dfrac{\alpha_k\rho_k P_{\mathrm{x},k+1}}{1+\alpha_k\rho_k\mu_k} & \forall k \in [k_0, \infty) \\ P_{k_0} & \forall k \in [0, k_0) \end{cases} \tag{10}$$

$$\rho_k = \begin{cases} 1 - \dfrac{\nu_{\max}}{\|\tilde{\mathbf{x}}_{k+1}\|} & \text{if } \|\tilde{\mathbf{x}}_{k+1}\| \ge \nu_{\max} \\ 0 & \text{if } \|\tilde{\mathbf{x}}_{k+1}\| < \nu_{\max} \end{cases} \tag{11}$$

where $k_0 \ge 0$ is the initial time-step, $\alpha_k > 0$ is a positive coefficient that guarantees that $|\hat{\Phi}_{3,k}| \ne 0$ and that the system is controllable, $P_k \in \mathbb{R}^{[m(p_\mathrm{m}+1)+3n]\times[m(p_\mathrm{m}+1)+3n]}$ is the symmetric positive-definite covariance matrix, $P_{\mathrm{x},k} \triangleq P_k\boldsymbol{\zeta}_k\boldsymbol{\zeta}_k^\top P_k$ and $\mu_k \triangleq \boldsymbol{\zeta}_k^\top P_k \boldsymbol{\zeta}_k$.

B. Control Law Design

Consider once more the adaptive estimator (7). As was previously stated, a reduction approach will be utilized to derive a delay free dynamics from the adaptive estimator (7) that will simplify the computation of the control law.

To proceed with the reduction approach, consider the adaptive estimator (7) written in an augmented form as

$$\begin{aligned} &\left[\, \hat{\mathbf{x}}_{k+1} \;\vdots\; \mathbf{x}_k \;\vdots\; \mathbf{x}_{k-1} \,\right]^\top \\ &\quad = \bar{\Phi}_k \left[\, \mathbf{x}_k \;\vdots\; \mathbf{x}_{k-1} \;\vdots\; \mathbf{x}_{k-2} \,\right]^\top + \sum_{i=0}^{p_\mathrm{m}} \bar{\Psi}_{i,k}^\top \mathbf{u}_{k-i} \end{aligned} \tag{12}$$

where the matrices $\bar{\Phi}_k \in \mathbb{R}^{3n\times 3n}$ and $\bar{\Psi}_{i,k} \in \mathbb{R}^{3n\times m}$ are given as

$$\bar{\Phi}_k = \begin{bmatrix} \hat{\Phi}_{1,k} & \hat{\Phi}_{2,k} & \hat{\Phi}_{3,k} \\ I & [0] & [0] \\ [0] & I & [0] \end{bmatrix} \quad \text{and} \quad \bar{\Psi}_{i,k} = \begin{bmatrix} \hat{\Psi}_{i,k} \\ [0] \\ [0] \end{bmatrix}$$

repectively. Next, the vectors $\boldsymbol{\eta}_k, \hat{\boldsymbol{\eta}}_k \in \mathbb{R}^{3n}$ are introduced and defined as

$$\hat{\boldsymbol{\eta}}_{k+1} \triangleq \left[\, \hat{\mathbf{x}}_{k+1}^\top \;\vdots\; \mathbf{x}_k^\top \;\vdots\; \mathbf{x}_{k-1}^\top \,\right]^\top + \sum_{i=1}^{p_\mathrm{m}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i+1} \tag{13}$$

and

$$\boldsymbol{\eta}_k \triangleq \left[\, \mathbf{x}_k^\top \;\vdots\; \mathbf{x}_{k-1}^\top \;\vdots\; \mathbf{x}_{k-2}^\top \,\right]^\top + \sum_{i=1}^{p_\mathrm{m}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i} \tag{14}$$

where $\hat{\Omega}_{i,k} \in \mathbb{R}^{3n\times m}$. The vectors (13) and (14) are basically the sum of the output and the weighted control input history. The terms in (13) and (14) are rearranged such that the augmented output vectors are expressed as

$$\left[\, \hat{\mathbf{x}}_{k+1}^\top \;\vdots\; \mathbf{x}_k^\top \;\vdots\; \mathbf{x}_{k-1}^\top \,\right]^\top = \hat{\boldsymbol{\eta}}_{k+1} - \sum_{i=1}^{p_\mathrm{m}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i+1} \tag{15}$$

and

$$\left[\, \mathbf{x}_k^\top \;\vdots\; \mathbf{x}_{k-1}^\top \;\vdots\; \mathbf{x}_{k-2}^\top \,\right]^\top = \boldsymbol{\eta}_k - \sum_{i=1}^{p_\mathrm{m}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i}. \tag{16}$$

Substitution of (15) and (16) in (12) as well as adding and subtracting the term $\hat{\Omega}_{0,k}\mathbf{u}_k$ on the right-hand-side of the resulting expression give a system of the form

$$\begin{aligned} \hat{\boldsymbol{\eta}}_{k+1} &= \bar{\Phi}_k\boldsymbol{\eta}_k + \hat{\Omega}_{0,k}\mathbf{u}_k - \bar{\Phi}_k\sum_{i=1}^{p_\mathrm{m}}\hat{\Omega}_{i,k}\mathbf{u}_{k-i} - \hat{\Omega}_{0,k}\mathbf{u}_k \\ &\quad + \sum_{i=0}^{p_\mathrm{m}}\bar{\Psi}_{i,k}\mathbf{u}_{k-i} + \sum_{i=1}^{p_\mathrm{m}}\hat{\Omega}_{i,k}\mathbf{u}_{k-i+1} \\ &= \bar{\Phi}_k\boldsymbol{\eta}_k + \hat{\Omega}_{0,k}\mathbf{u}_k - \left(\hat{\Omega}_{0,k} - \hat{\Omega}_{1,k}\right)\mathbf{u}_k - \sum_{i=1}^{p_\mathrm{m}-1}\Big(\hat{\Phi}_k\hat{\Omega}_{i,k} \\ &\quad - \hat{\Omega}_{i+1,k}\Big)\mathbf{u}_{k-i} - \hat{\Phi}_k\hat{\Omega}_{p_\mathrm{m},k}\mathbf{u}_{k-p_\mathrm{m}} + \sum_{i=0}^{p_\mathrm{m}}\hat{\boldsymbol{\psi}}_{i,k}\mathbf{u}_{k-i}. \end{aligned} \tag{17}$$

The parameters $\hat{\Omega}_{i,k}\forall i \in [0, p_\mathrm{m}]$ are computed from the matrices $\bar{\Phi}_k$ and $\bar{\Psi}_{i,k}\forall i \in [0, p_\mathrm{m}]$ as

$$\hat{\Omega}_{i,k} = \begin{cases} \displaystyle\sum_{j=0}^{p_\mathrm{m}} \bar{\Phi}_k^{-j}\bar{\Psi}_{j,k} & i = 0 \\ \displaystyle\sum_{j=i}^{p_\mathrm{m}} \bar{\Phi}_k^{i-j-1}\bar{\Psi}_{j,k} & i \in [1, p_\mathrm{m}] \end{cases} \tag{18}$$

and that results in the simplification of the system (17) into the delay free dynamics of the form

$$\hat{\boldsymbol{\eta}}_{k+1} = \bar{\Phi}_k \boldsymbol{\eta}_k + \hat{\Omega}_{0,k}\mathbf{u}_k. \tag{19}$$

The control law can now be designed using the system (19) with the condition that $\bar{\Phi}_k, \hat{\Omega}_{0,k}$ is a controllable pair. The controllability of the system (19) is addressed in *Lemma 1*.

Remark 1: In (18), the inverse of the matrix $\bar{\Phi}_k$ is required and, therefore, $\bar{\Phi}_k$ must be a non-singular matrix. From the definition of $\bar{\Phi}_k$, the determinant $|\bar{\Phi}_k| = -|\hat{\Phi}_{3,k}|$ and, since, $|\hat{\Phi}_{3,k}| \neq 0$ the matrix $\bar{\Phi}_k$ is non-singular.

Remark 2: Note that in the system (19), $\hat{\boldsymbol{\eta}}_{k+1}$ is a function of $\boldsymbol{\eta}_k$. It will be shown in **Lemma 4** that if a feedback gain $L_{\mathrm{x},k} \in \mathbb{R}^{3n\times m}$ is selected such that the matrix $\bar{\Phi}_k - \hat{\Omega}_{0,k}L_{\mathrm{x},k}^\top$ is Hurwitz, then $\boldsymbol{\eta}_k$ is uniformly bounded and, consequently, $\hat{\boldsymbol{\eta}}_k$ is uniformly bounded.

Lemma 1: It is possible to select the initial adaptive law parameters and the coefficient α_k such that:

(a) The matrix $\hat{\Phi}_{3,k}$ is non-singular, i.e., $|\hat{\Phi}_{3,k}| \neq 0$.

(b) The pair $\bar{\Phi}_k, \hat{\Omega}_{0,k}$ is controllable.

Proof: To prove part (a) of **Lemma 1**, consider the adaptive law (9) for $k \in [k_0, \infty)$ and define a matrix $S^\top = \left[\,[0] \;\vdots\; [0] \;\vdots\; I \;\vdots\; [0] \;\vdots\; \cdots \;\vdots\; [0]\,\right] \in \mathbb{R}^{n\times[m(p_\mathrm{m}+1)+3n]}$ such that $S^\top \hat{\Theta}_k = \hat{\Phi}_{3,k}$ and

$$\begin{aligned} S^\top \hat{\Theta}_{k+1} &= \hat{\Phi}_{3,k+1} \\ &= \hat{\Phi}_{3,k} + \alpha_k \rho_k S^\top P_{k+1}\boldsymbol{\zeta}_k \tilde{\mathbf{x}}_{k+1}^\top \\ &= \hat{\Phi}_{3,k}\left[I + \alpha_k\rho_k \hat{\Phi}_{3,k}^{-1} S^\top P_{k+1}\boldsymbol{\zeta}_k\tilde{\mathbf{x}}_{k+1}^\top\right]. \end{aligned} \tag{20}$$

From (20), if the initial value $|\hat{\Phi}_{3,k_0}| \neq 0$ and $\alpha_{k_0}^{-1}$ is not an eigenvalue of the matrix $-\hat{\Phi}_{3,k_0}^{-1} S^\top P_{k_0+1}\boldsymbol{\zeta}_{k_0}\tilde{\mathbf{x}}_{k_0+1}^\top$ then $\hat{\Phi}_{3,k_0+1}$ is non-singular. This can then be generalized for all $k \in [k_0, \infty)$ as a requirement that α_k^{-1} not be an eigenvalue of the matrix $-\hat{\Phi}_{3,k}^{-1} S^\top P_{k+1}\boldsymbol{\zeta}_k\tilde{\mathbf{x}}_{k+1}^\top$. This completes the proof of part (a) of **Lemma 1**.

To prove part (b) of **Lemma 1**, consider the pair $\bar{\Phi}_k, \hat{\Omega}_{0,k}$ and the fact that controllability requires that the controllability matrix $W_{\mathrm{c},k} \triangleq \left[\,\hat{\Omega}_{0,k} \;\vdots\; \bar{\Phi}_k\hat{\Omega}_{0,k} \;\vdots\; \cdots \;\vdots\; \bar{\Phi}_k^{n-1}\hat{\Omega}_{0,k}\,\right] \in \mathbb{R}^{3n\times 3n\cdot m}$ be of rank $3n$. To express $W_{\mathrm{c},k}$ explicitly in terms of the adaptive parameters, $\hat{\Omega}_{0,k}$ in (18) is given as

$$\hat{\Omega}_{0,k} = \bar{\Psi}_{0,k} + \bar{\Phi}_k^{-1}\bar{\Psi}_{1,k} + \cdots + \bar{\Phi}_k^{-p_\mathrm{m}}\bar{\Psi}_{p_\mathrm{m},k}. \tag{21}$$

Substitution of (21) in the definition of the controllability matrix $W_{\mathrm{c},k}$, it is obtained that

$$W_{\mathrm{c},k} = \sum_{i=0}^{p_\mathrm{m}} \bar{\Phi}_k^{-i}\left[\,\bar{\Psi}_{i,k} \;\vdots\; \bar{\Phi}_k\bar{\Psi}_{i,k} \;\vdots\; \cdots \;\vdots\; \bar{\Phi}_k^{3n-1}\bar{\Psi}_{i,k}\,\right] \tag{22}$$

which is now explicitly in terms of the adaptive parameters. Since (22) relies on the inverse of $\bar{\Phi}_k$ it is convenient to define $W_{\Phi,k} \triangleq \bar{\Phi}_k^{p_\mathrm{m}} W_{\mathrm{c},k}$ such that the premultiplication of both sides of (22) with $\hat{\Phi}_k^{p_\mathrm{m}}$ results in

$$W_{\Phi,k} = \sum_{i=0}^{p_\mathrm{m}} \bar{\Phi}_k^{p_\mathrm{m}-i}\left[\,\bar{\Psi}_{i,k} \;\vdots\; \bar{\Phi}_k\bar{\Psi}_{i,k} \;\vdots\; \cdots \;\vdots\; \bar{\Phi}_k^{3n-1}\bar{\Psi}_{i,k}\,\right]. \tag{23}$$

Consider now the adaptive law (9) when $|\hat{\Phi}_{3,k}| \neq 0$. The adaptive law for each parameter can be written as

$$\begin{aligned} \hat{\Phi}_{1,k+1} &= \hat{\Phi}_{1,k} + \alpha_k\Lambda_{1,k} \\ &\;\;\vdots \\ \hat{\Phi}_{3,k+1} &= \hat{\Phi}_{3,k} + \alpha_k\Lambda_{3,k} \\ \hat{\Psi}_{0,k+1} &= \hat{\Psi}_{0,k} + \alpha_k H_{0,k} \\ &\;\;\vdots \\ \hat{\Psi}_{p_\mathrm{m},k+1} &= \hat{\Psi}_{p_\mathrm{m},k} + \alpha_k H_{p_\mathrm{m},k} \end{aligned} \tag{24}$$

where $\Lambda_{i,k} = \rho_k\tilde{\mathbf{x}}_{k+1}\boldsymbol{\zeta}_k^\top P_{k+1}C_{\phi,i}^\top\ \forall i \in [1,3]$ and $H_{i,k} = \rho_k\tilde{\mathbf{x}}_{k+1}\boldsymbol{\zeta}_k^\top P_{k+1}C_{\psi,i}^\top\ \forall i \in [0,p_\mathrm{m}]$ with $C_{\phi,i}$ being the $(i-1)n+1$ to $i\cdot n$ rows of an identity matrix of size $m(p_\mathrm{m}+1)+3n$ and $C_{\psi,i}$ being the $i\cdot m+3n+1$ to $(i+1)m+3n$ rows of an identity matrix of size $m(p_\mathrm{m}+1)+3n$. Then $\bar{\Phi}_k$ is written as

$$\begin{aligned} \bar{\Phi}_{k+1} &= \bar{\Phi}_k + \alpha_k \begin{bmatrix} \Lambda_{1,k} & \cdots & \Lambda_{3,k} \\ [0] & \cdots & [0] \\ \vdots & & \vdots \\ [0] & \cdots & [0] \end{bmatrix} \\ &= \bar{\Phi}_k + \alpha_k\bar{\Lambda}_k \end{aligned} \tag{25}$$

and $\bar{\Psi}_{i,k}\ \forall i \in [0,p_\mathrm{m}]$ is similarly written as

$$\begin{aligned} \bar{\Psi}_{i,k+1} &= \bar{\Psi}_{i,k} + \alpha_k \begin{bmatrix} H_{i,k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \bar{\Psi}_{i,k} + \alpha_k\bar{H}_{i,k}. \end{aligned} \tag{26}$$

Substitution of (25) and (26) in (23), results in an expression for $W_{\Phi,k}$ given as

$$\begin{aligned} W_{\Phi,k+1} &= \sum_{i=0}^{p_\mathrm{m}}\left(\bar{\Phi}_k^{p_\mathrm{m}-i} + \alpha_k Q_{p_\mathrm{m}-i,k}\right)\Big[\left(\bar{\Psi}_{i,k} + \alpha_k\bar{H}_{i,k}\right) \;\vdots \\ &\quad \left(\bar{\Phi}_k + \alpha_k Q_{1,k}\right)\left(\bar{\Psi}_{i,k} + \alpha_k\bar{H}_{i,k}\right) \;\vdots\; \cdots \;\vdots\; \Big(\bar{\Phi}_k^{3n-1} \\ &\quad + \alpha_k Q_{3n-1,k}\Big)\left(\bar{\Psi}_{i,k-1} + \alpha_k\bar{H}_{i,k}\right)\Big] \end{aligned} \tag{27}$$

$$\begin{aligned} &= \sum_{i=0}^{p_\mathrm{m}} \bar{\Phi}_k^{p_\mathrm{m}-i}\left[\bar{\Psi}_{i,k} \;\vdots\; \bar{\Phi}_k\bar{\Psi}_{i,k} \;\vdots\; \cdots \;\vdots\; \bar{\Phi}_k^{3n-1}\bar{\Psi}_{i,k}\right] \\ &\quad + \alpha_k \sum_{i=0}^{p_\mathrm{m}}\Big[\bar{\Phi}_k^{p_\mathrm{m}-i}\bar{H}_{i,k} + Q_{p_\mathrm{m}-i,k}\bar{\Psi}_{i,k+1} \;\vdots\; \bar{\Phi}_k\bar{H}_{i,k} \\ &\quad + Q_{1,k}\bar{\Psi}_{i,k+1} + Q_{p_\mathrm{m}-i,k}\bar{\Phi}_{k+1}\bar{\Psi}_{i,k+1} \;\vdots\; \cdots \;\vdots\; \bar{\Phi}_k^{3n-1} \\ &\quad \times \bar{H}_{i,k} + Q_{3n-1,k}\bar{\Psi}_{i,k+1} + Q_{p_\mathrm{m}-i,k}\bar{\Phi}_k^{3n-1}\bar{\Psi}_{i,k+1}\Big] \end{aligned}$$

where $Q_{i,k} \triangleq \alpha_k^{-1}\left(\bar{\Phi}_{k+1}^i - \bar{\Phi}_k^i\right)$. Note that the first term on the right-hand-side of (27) is a single time-step delayed (23). Therefore, (27) is simplified as

$$W_{\Phi,k+1} = W_{\Phi,k} + \alpha_k M_{k+1} \tag{28}$$

where

$$M_{k+1} = \sum_{i=0}^{p_{\mathrm{m}}} \Big[\bar{\Phi}_k^{p_{\mathrm{m}}-i} \bar{H}_{i,k} + Q_{p_{\mathrm{m}}-i,k} \bar{\Psi}_{i,k+1} \vdots \bar{\Phi}_k \bar{H}_{i,k} + Q_{1,k} \times \bar{\Psi}_{i,k+1} + Q_{p_{\mathrm{m}}-i,k} \bar{\Phi}_{k+1} \bar{\Psi}_{i,k+1} \vdots \cdots \vdots \bar{\Phi}_k^{3n-1} \bar{H}_{i,k} + Q_{3n-1,k} \bar{\Psi}_{i,k+1} + Q_{p_{\mathrm{m}}-i,k} \bar{\Phi}_k^{3n-1} \bar{\Psi}_{i,k+1} \Big]. \tag{29}$$

Consider now the expression (28) when $k = k_0 + 1$ and suppose that the initial adaptive parameters are selected such that W_{Φ,k_0} has a rank of $3n$ which implies $W_{\Phi,k_0} W_{\Phi,k_0}^\top$ is a non-singular matrix then it is obtained that

$$W_{\Phi,k_0+1} W_{\Phi,k_0+1}^\top = W_{\Phi,k_0} W_{\Phi,k_0}^\top \Big(I + \alpha_{k_0} \left(W_{\Phi,k_0} W_{\Phi,k_0}^\top \right)^{-1} \Big(W_{\Phi,k_0} \times M_{k_0+1}^\top + W_{\Phi,k_0}^\top M_{k_0+1} + \alpha_k M_{k_0+1} M_{k_0+1}^\top \Big) \Big) \tag{30}$$

where W_{Φ,k_0+1} has a rank of $3n$ if and only if $\alpha_{k_0}^{-1} \neq \lambda \Big[- \left(W_{\Phi,k_0} W_{\Phi,k_0}^\top \right)^{-1} \Big(W_{\Phi,k_0} M_{k_0+1}^\top + W_{\Phi,k_0}^\top M_{k_0+1} + \alpha_{k_0} M_{k_0+1} M_{k_0+1}^\top \Big) \Big]$, where $\lambda[\cdot]$ is the set of eigenvalues. Then, in general, $W_{\Phi,k}$ has a rank of $3n$ if the initial value W_{Φ,k_0} is also has a rank of $3n$ and $\alpha_k^{-1} \neq \lambda \Big[- \left(W_{\Phi,k} W_{\Phi,k}^\top \right)^{-1} \Big(W_{\Phi,k} M_{k+1}^\top + W_{\Phi,k}^\top M_{k+1} + \alpha_k M_{k+1} M_{k+1}^\top \Big) \Big]$.

Furthermore, since $\bar{\Phi}_k$ is a non-singular matrix, if $W_{\Phi,k}$ has a rank of $3n$ then $W_{\mathrm{c},k}$ also has a rank of $3n$ and the pair $\bar{\Phi}_k, \hat{\Omega}_{0,k}$ is controllable. ■

Remark 3: Similar to in [11], the coefficient $\alpha_k > 0$ can be selected from a set of pre-defined values to ensure that $W_{\mathrm{c},k}$ has a rank of $3n$.

Considering that the controllability of the system (19) is established in **Lemma 1**, the control law is proposed as

$$\mathbf{u}_k = -L_{\mathrm{x},k}^\top \boldsymbol{\eta}_k \tag{31}$$

where the feedback gain vector $L_{\mathrm{x},k}$ can be computed using a Pole Placement or any optimal control approaches.

C. Stability Analysis

In this section, it is shown that the parameter adaptation produces bounded and convergent parameter estimates (**Lemma 2** and **Lemma 3**), that the adaptive system model converges in input-output behaviour to the true system (**Lemma 4**), and that the proposed adaptive control law drives the system state to zero asymptotically (**Theorem 1**).

Lemma 2: For the system (8) with the adaptive laws (9) and (10) it is true that

$$\lim_{k\to\infty} \frac{\alpha_k \rho_k}{1 + \alpha_k \rho_k \mu_k} \tilde{\mathbf{x}}_k^\top \tilde{\mathbf{x}}_k = 0 \tag{32}$$

Furthermore, it is also true that the parameter estimate $\hat{\boldsymbol{\theta}}_k$ is bounded, hence, the parameter estimation error $\tilde{\boldsymbol{\theta}}_k$ is also bounded.

Proof: To proceed with the proof, let $\tilde{\mathbf{x}}_k^\top = \left[\tilde{x}_{1,k} \vdots \cdots \vdots \tilde{x}_{n,k} \right]^\top$, $\tilde{\Theta}_k^\top = \left[\tilde{\boldsymbol{\theta}}_{1,k} \vdots \cdots \vdots \tilde{\boldsymbol{\theta}}_{n,k} \right]^\top$ and consider the following positive function

$$V_k = \sum_{j=1}^{n} \tilde{\boldsymbol{\theta}}_{j,k}^\top P_k^{-1} \tilde{\boldsymbol{\theta}}_{j,k}. \tag{33}$$

The forward difference of (33) is given by

$$\Delta V_k = V_{k+1} - V_k = \sum_{j=1}^{m} \left[\tilde{\boldsymbol{\theta}}_{j,k+1}^\top P_{k+1}^{-1} \tilde{\boldsymbol{\theta}}_{j,k+1} - \tilde{\boldsymbol{\theta}}_{j,k}^\top P_k^{-1} \tilde{\boldsymbol{\theta}}_{j,k} \right]. \tag{34}$$

Following an approach similar to in [11] it is obtained that

$$\lim_{k\to\infty} \Delta V_k = \lim_{k\to\infty} \frac{\alpha_k \rho_k}{1 + \alpha_k \rho_k \mu_k} \tilde{\mathbf{x}}_k^\top \tilde{\mathbf{x}}_k = 0 \tag{35}$$

which is true for $\|\tilde{\mathbf{x}}_k\| \geq \nu_{\max}$, [11]. The result (35) implies that $\lim_{k\to\infty} \|\tilde{\Theta}_{k+1} - \tilde{\Theta}_k\| = 0$. Consequently, $\lim_{k\to\infty} \|\hat{\Theta}_{k+1} - \hat{\Theta}_k\| = 0$, [11]. ■

Lemma 3: Using the results in **Lemma 2**, the vector $\boldsymbol{\eta}_k$ defined in (14) is bounded as

$$\|\boldsymbol{\eta}_k\| \leq c_0 + c_1 \max_{i\in[0,k]} \|\tilde{\mathbf{x}}_{k-i}\| \tag{36}$$

for some positive constants c_0,c_1 and, consequently, the model estimation error $\tilde{\mathbf{x}}_k$ converges to a bound of $\nu_{\max}$ asymptotically, i.e.

$$\lim_{k\to\infty} \|\tilde{\mathbf{x}}_k\| \leq \nu_{\max}. \tag{37}$$

Proof: Consider (13) and (14), the difference of the two vectors results in the expression

$$\begin{aligned} \boldsymbol{\eta}_k &= \hat{\boldsymbol{\eta}}_k + \left[\tilde{\mathbf{x}}_k^\top \vdots [0] \vdots [0] \right]^\top + \sum_{i=1}^{p_{\mathrm{m}}} \left(\hat{\Omega}_{i,k} - \hat{\Omega}_{i,k-1} \right) \mathbf{u}_{k-i} \\ &= \hat{\boldsymbol{\eta}}_k + \left[\tilde{\mathbf{x}}_k^\top \vdots [0] \vdots [0] \right]^\top + \sum_{i=1}^{p_{\mathrm{m}}} \Delta\hat{\Omega}_{i,k} \mathbf{u}_{k-i} \end{aligned} \tag{38}$$

where $\Delta\hat{\Omega}_{i,k} \triangleq \hat{\Omega}_{i,k} - \hat{\Omega}_{i,k-1}$. Substitution of (19) and (31) in a one time-step forward (38)

$$\begin{aligned} \boldsymbol{\eta}_{k+1} &= \bar{\Phi}_{\mathrm{m},k} \boldsymbol{\eta}_k + \sum_{i=1}^{p_{\mathrm{m}}} \Delta\hat{\Omega}_{i,k} \mathbf{u}_{k-i} + \left[\tilde{\mathbf{x}}_{k+1}^\top \vdots [0] \vdots [0] \right]^\top \\ &= \bar{\Phi}_{\mathrm{m},k} \boldsymbol{\eta}_k - \sum_{i=1}^{p_{\mathrm{m}}} \Delta\hat{\Omega}_{i,k} L_{\mathrm{x},k-i}^\top \boldsymbol{\eta}_{k-i} + \left[\tilde{\mathbf{x}}_{k+1}^\top \vdots [0] \vdots [0] \right]^\top \end{aligned} \tag{39}$$

where $\bar{\Phi}_{\mathrm{m},k} = \bar{\Phi}_k - \hat{\Omega}_{0,k} L_{\mathrm{x},k}^\top$. Expressing (39) in augmented form and defining $N_{i,k} \triangleq \Delta\hat{\Omega}_{i,k} L_{\mathrm{x},k-i}^\top \in \mathbb{R}^{3n\times 3n}$ such that,

$$\bar{\boldsymbol{\eta}}_{k+1} = \begin{bmatrix} \bar{\Phi}_{\mathrm{m},k} - N_{1,k} & -N_{2,k} & \cdots & -N_{p_{\mathrm{m}},k} \\ I & [0] & \cdots & [0] \\ [0] & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & [0] \end{bmatrix} \bar{\boldsymbol{\eta}}_k + \left[\tilde{\mathbf{x}}_{k+1}^\top \vdots 0 \vdots \cdots \vdots 0 \right]^\top \tag{40}$$

where $\bar{\boldsymbol{\eta}}_{k-1}^{\top} \triangleq \left[\boldsymbol{\eta}_{k-1}^{\top} \,\vdots\, \boldsymbol{\eta}_{k-2}^{\top} \,\vdots\, \cdots \,\vdots\, \boldsymbol{\eta}_{k-p_{\mathrm{m}}}^{\top}\right] \in \mathbb{R}^{3n \cdot p_{\mathrm{m}}}$. Using the results in **Lemma 2** and [8], $\lim_{k\to\infty} \|N_{i,k-1}\| = 0$ and that implies that the augmented system, (40), can be reduced to the form

$$\bar{\boldsymbol{\eta}}_{k+1} = \begin{bmatrix} \bar{\Phi}_{\mathrm{m},k} & [0] & \cdots & [0] \\ I & [0] & \cdots & [0] \\ [0] & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & [0] \end{bmatrix} \bar{\boldsymbol{\eta}}_k + \left[\tilde{\mathbf{x}}_{k+1}^{\top} \,\vdots\, 0 \,\vdots \cdots \vdots\, 0\right]^{\top} \tag{41}$$

which is stable and has $3n$ eigenvalues of the matrix $\bar{\Phi}_{\mathrm{m},k}$ while the remaining $3n \cdot p_{\mathrm{m}} - 3n$ eigenvalues are 0. Therefore, the system (41) is stable and a bound on $\boldsymbol{\eta}_k$ exists such that

$$\|\boldsymbol{\eta}_k\| \leq c_0 + c_1 \max_{i\in[0,k]} \|\tilde{\mathbf{x}}_{k-i}\| \tag{42}$$

for some positive constants c_0 and c_1. This establishes the bound on $\boldsymbol{\eta}_k$.

Consider now the control law (31). From (42) and the fact that $L_{\mathrm{x},k}$, is bounded then the control input is bounded as

$$\|\mathbf{u}_k\| \leq c_2 + c_3 \max_{i\in[0,k]} \|\tilde{\mathbf{x}}_{k-i}\| \tag{43}$$

for some positive constants c_2 and c_3. Using (38) and the fact that $\mathbf{x}_k = \hat{\mathbf{x}}_k + \tilde{\mathbf{x}}_k$ a bound on $\mathbf{x}_k$ is obtained as

$$\|\mathbf{x}_k\| \leq c_4 + c_5 \max_{i\in[0,k]} \|\tilde{\mathbf{x}}_{k-i}\| \tag{44}$$

for some positive constants c_4 and c_5. From the definition of $\boldsymbol{\zeta}_k$ and using (43), (44) there exists positive constants c_0^0 and c_1^0 such that

$$\|\boldsymbol{\zeta}_k\| \leq c_0^0 + c_1^0 \max_{i\in[0,k]} \|\tilde{\mathbf{x}}_{k-i}\|. \tag{45}$$

Consequently, from (45) and the **Key Technical Lemma**, it is obtained that

$$\lim_{k\to\infty} \|\tilde{\mathbf{x}}_k\| \leq \nu_{\max}. \tag{46}$$

■

Remark 4: Since $\|\tilde{\mathbf{x}}_k\|$ is uniformly bounded then, from (42), $\|\boldsymbol{\eta}_k\|$ is uniformly bounded. Furthermore, from (19), (31) and the fact that the adaptive parameters are bounded then $\|\hat{\boldsymbol{\eta}}_k\|$ is also uniformly bounded.

Theorem 1: The states of the closed-loop system approaches a bound of $\epsilon \in O\left(T^2\right)$ around zero, i.e. $\lim_{k\to\infty} \|\mathbf{x}_k\| \leq \epsilon$.

Proof: Consider **Lemma 3** and the stable dynamics given by (41), which is reduced to the form

$$\boldsymbol{\eta}_{k+1} = \bar{\Phi}_{\mathrm{m},k}\boldsymbol{\eta}_k + \left[\tilde{\mathbf{x}}_k^{\top} \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^{\top}. \tag{47}$$

From **Lemma 2**, it is shown that the adaptive parameters are bounded and converge at steady state. Therefore, there exists $\bar{\Phi}_{\mathrm{m,ss}}$ and $\hat{\Omega}_{0,\mathrm{ss}}$ such that $\bar{\Phi}_{\mathrm{m,ss}} = \lim_{k\to\infty} \bar{\Phi}_{\mathrm{m},k}$ and $\hat{\Omega}_{0,\mathrm{ss}} = \lim_{k\to\infty} \hat{\Omega}_{0,k}$. Then the dynamics (47) is written as

$$\boldsymbol{\eta}_{k+1} = \bar{\Phi}_{\mathrm{m,ss}}\boldsymbol{\eta}_k + \boldsymbol{\gamma}_k \tag{48}$$

where

$$\boldsymbol{\gamma}_k = \left(\bar{\Phi}_{\mathrm{m},k} - \bar{\Phi}_{\mathrm{m,ss}}\right)\boldsymbol{\eta}_k + \left[\tilde{\mathbf{x}}_k^{\top} \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^{\top} \tag{49}$$

and since all the terms on the right-hand-side of (49) are bounded then $\lim_{k\to\infty} \|\boldsymbol{\gamma}_k\| \leq \nu_{\max}$. The solution of (48) is given as

$$\boldsymbol{\eta}_k = \hat{\Phi}_{\mathrm{m,ss}}^{k-k_0}\boldsymbol{\eta}_{k_0} + \sum_{i=k_0}^{k-1} \bar{\Phi}_{\mathrm{m,ss}}^{i}\boldsymbol{\gamma}_{k-i} \tag{50}$$

where $\boldsymbol{\eta}_{k_0}$ is the initial value of the vector $\boldsymbol{\eta}_k$. At steady state $\lim_{k\to\infty} \boldsymbol{\eta}_k$ is given as

$$\lim_{k\to\infty} \boldsymbol{\eta}_k = \bar{\gamma}_k \tag{51}$$

where $\bar{\gamma}_k = \lim_{k\to\infty} \sum_{i=0}^{k-1} \bar{\Phi}_{\mathrm{m,ss}}^{i}\boldsymbol{\gamma}_{k-i} \in O\left(T^2\right)$, [10]. Now, consider the definition of $\boldsymbol{\eta}_k$ given as

$$\boldsymbol{\eta}_k = \left[\,\mathbf{x}_k^{\top} \,\vdots\, \cdots \,\vdots\, \mathbf{x}_{k-2}^{\top}\,\right]^{\top} + \sum_{i=0}^{p_{\mathrm{m}}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i} \tag{52}$$

premultiplying (52) with $C^{\top} = \left[\,I \,\vdots\, [0] \,\vdots\, \cdots \,\vdots\, [0]\,\right] \in \mathbb{R}^{n\times 3n}$ gives

$$\mathbf{x}_k = C^{\top}\left(\boldsymbol{\eta}_k - \sum_{i=1}^{p_{\mathrm{m}}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i}\right). \tag{53}$$

The steady state value $\lim_{k\to\infty} \mathbf{x}_k$ is given as

$$\begin{aligned} \lim_{k\to\infty} \mathbf{x}_k &= C^{\top} \lim_{k\to\infty}\left(\boldsymbol{\eta}_k - \sum_{i=1}^{p_{\mathrm{m}}} \hat{\Omega}_{i,k}\mathbf{u}_{k-i}\right) \\ &= C^{\top}\left(\lim_{k\to\infty}\boldsymbol{\eta}_k - \sum_{i=1}^{p_{\mathrm{m}}} \hat{\Omega}_{i,\mathrm{ss}} \lim_{k\to\infty}\mathbf{u}_{k-i}\right). \end{aligned} \tag{54}$$

From (31) it is obtained that

$$\lim_{k\to\infty} \mathbf{u}_k = -L_{x,ss}^{\top} \lim_{k\to\infty} \boldsymbol{\eta}_k. \tag{55}$$

If $\|L_{x,ss}\| \in O(1)$ and, since, $\lim_{k\to\infty} \|\boldsymbol{\eta}_k\| \in O\left(T^2\right)$ then $\lim_{k\to\infty} \|\mathbf{u}_k\| \in O\left(T^2\right)$. From (54) and using the bounds on $\boldsymbol{\eta}_k$ and $\mathbf{u}_k$ while assumung that $\|\hat{\Omega}_{i,\mathrm{ss}}\| \in O(1)$ it is obtained that

$$\lim_{k\to\infty} \|\mathbf{x}_k\| \leq \epsilon \tag{56}$$

where $\epsilon \in O\left(T^2\right) + O(p_{\mathrm{m}}) \cdot O\left(T^2\right)$ and if the sampling interval T is selected in such a way that $p_{\mathrm{m}} \in O(1)$ then $\epsilon \in O\left(T^2\right)$.

■

IV. SIMULATION EXAMPLE

Consider an unstable 2^{nd}-order continuous-time plant with time-delay given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & -2.5 \\ -2.5 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t-\tau_1) \\ u_2(t-\tau_2) \end{bmatrix} \\ &\quad + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \sin\left(\frac{7}{10}\pi t\right) \end{aligned} \tag{57}$$

where two cases are considered: $\tau_1 = \tau_2 = 0$s (i.e. no delay) and $\tau_1 = 0.1, \tau_2 = 0.3$s with $\tau_p = 0.4$s. The sampling interval is selected as $T = 0.1$s which, for each of the two different time-delays, results in the discrete-time plants respectively given by

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1.14 & -0.28 \\ -0.28 & 1.14 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0.105 & -0.013 \\ -0.013 & 0.105 \end{bmatrix} \mathbf{u}_k + \boldsymbol{\omega}_k \quad (58)$$

and

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1.14 & -0.28 \\ -0.28 & 1.14 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0.105 & 0 \\ -0.013 & 0 \end{bmatrix} \mathbf{u}_{k-d_1} + \begin{bmatrix} 0 & -0.013 \\ 0 & 0.105 \end{bmatrix} \mathbf{u}_{k-d_2} + \boldsymbol{\omega}_k \quad (59)$$

where $d_1 = 1, d_2 = 3$ and $p = 4$, respectively. The initial condition of the plant is set at $\mathbf{x}(0) = [1\ 1]^\top$.

To investigate the adaptive performance of the regulator under no delay conditions, the closed-loop system is simulated using an uncertainty of 25% on the paramters of (58). The adaptive regulator is initialised with $P_0 = 7 \times 10^2 \times I_{p+3 \times p+3}$. In Fig. 1-2 the results are shown for the state regulation of x_1 and the control input profile of the closed-loop plant under a delay upper-bound $\tau_{\rm p} = 0$s, for $\tau_1 = \tau_2 = 0$s respectively. As expected, $x_1(t)$ is regulated to a bound of $O\left(0.1^2\right)$ asymptotically. Finally, the system is simulated for a delay of $\tau_1 = 0.1$s and $\tau_2 = 0.3$s with an upper-bound of $\tau_{\rm p} = 0.4$s. In Fig. 3-5 the results are shown for the state regulation of x_1, the control input profile and the elements of the matrix $\hat{\Phi}_3$ of the closed-loop plant respectively. As expected, the adaptive parameters converge to constant values at steady state while the state x_1 is regulated to a bound of $O\left(0.1^2\right)$.

V. CONCLUSIONS

In this paper, a discrete-time adaptive regulator is proposed for a MIMO linear plant which is subject to an unmeasurable disturbance and has uncertain input delays that may differ across input channels. An adaptive plant model incorporating a second-order disturbance observer is formulated which does not depend on knowledge of the time delays, only their upper bound. The adaptation laws based on recursive least squares incorporate a dead zone that ensures stability in the presence of disturbance. To facilitate control law design, state substitutes are used to transform the plant into a delay-free dynamics, which was shown to be controllable. It was further shown that the proposed adaptive regulator drives the plant state to zero asymptotically, within an $O(T^2)$ bound. Simulation results demonstrate the ability of the adaptive regulator to handle a delay-free plant, as well as mismatches between the delay upper-bound and the true time delay.

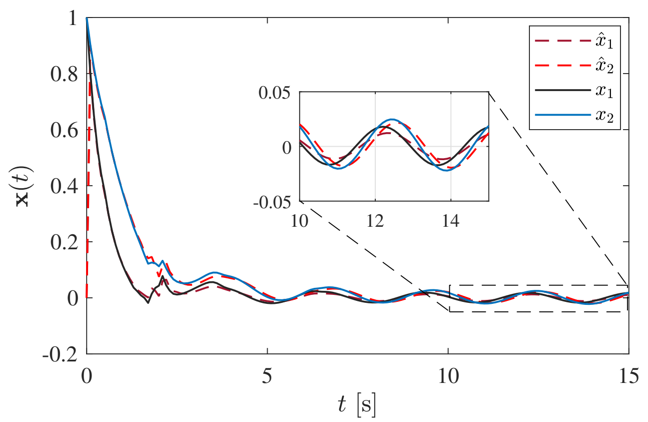

Fig. 1. State regulation of x_1 on the delay-free plant ($\tau_1 = \tau_2 = 0$s), using a delay upper-bound $\tau_{\rm p} = 0$s.

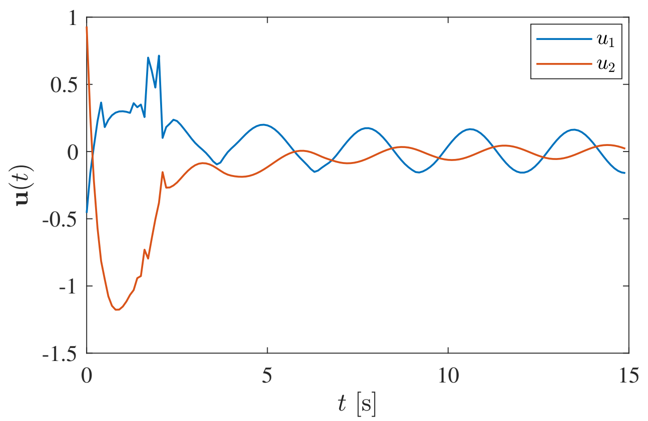

Fig. 2. Control input profile on the delay-free plant ($\tau_1 = \tau_2 = 0$s), using a delay upper-bound $\tau_{\rm p} = 0$s.

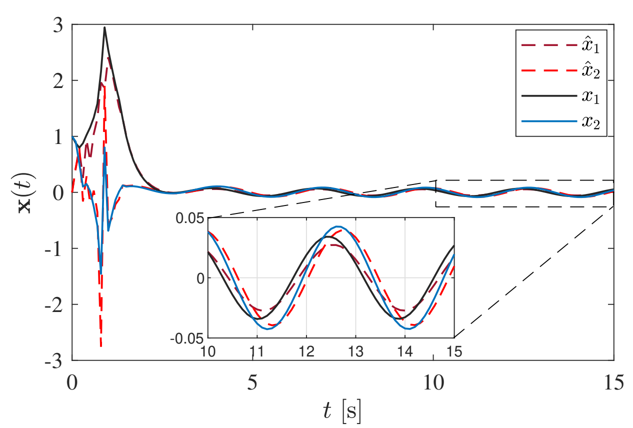

Fig. 3. State regulation x_1 on the plant with delay ($\tau_1 = 0.1$s and $\tau_2 = 0.3$s), using a delay upper-bound $\tau_{\rm p} = 0.4$s.

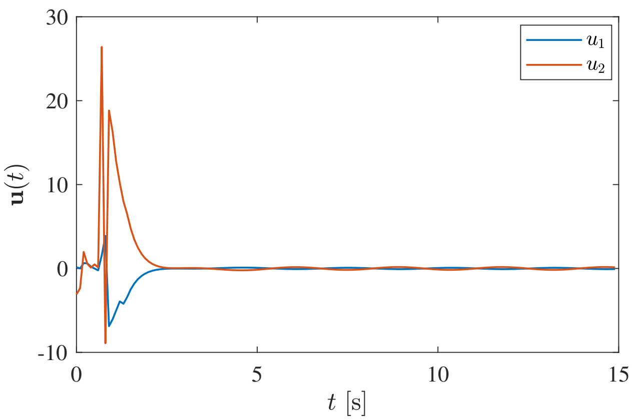

Fig. 4. Control input profile on the plant with delay ($\tau_1 = 0.1$s and $\tau_2 = 0.3$s), using a delay upper-bound $\tau_\mathrm{p} = 0.4$s.

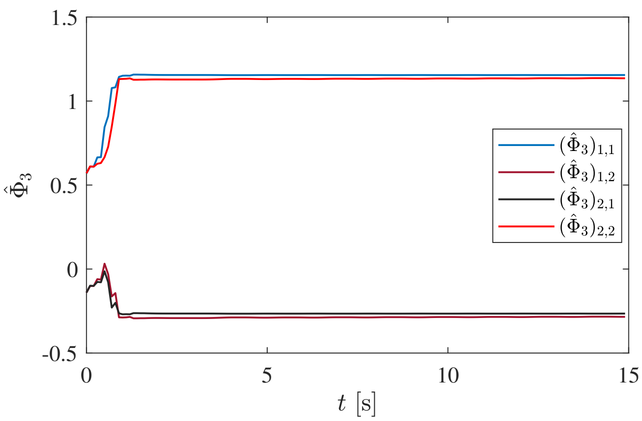

Fig. 5. Adaptive parameters $\hat{\Phi}_3$ of the plant with delay ($\tau_1 = 0.1$s and $\tau_2 = 0.3$s), using a delay upper-bound $\tau_\mathrm{p} = 0.4$s.

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