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Cosmic Rays in Intermittent Magnetic Fields

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Abstract

The propagation of cosmic rays in turbulent magnetic fields is a diffusive process driven by the scattering of the charged particles by random magnetic fluctuations. Such fields are usually highly intermittent, consisting of intense magnetic filaments and ribbons surrounded by weaker, unstructured fluctuations. Studies of cosmic-ray propagation have largely overlooked intermittency, instead adopting Gaussian random magnetic fields. Using test particle simulations, we calculate cosmic-ray diffusivity in intermittent, dynamo-generated magnetic fields. The results are compared with those obtained from non-intermittent magnetic fields having identical power spectra. The presence of magnetic intermittency significantly enhances cosmic-ray diffusion over a wide range of particle energies. We demonstrate that the results can be interpreted in terms of a correlated random walk.

Key words: cosmic rays – diffusion – dynamo – magnetic fields

1. Introduction

Cosmic rays are charged relativistic particles (mostly protons) scattered, as they propagate, by random magnetic fields (Berezinskii et al. 1990). Over sufficiently long time and length scales, their propagation is diffusive (Cesarsky 1980). Assuming an interstellar magnetic field of strength 5 μG, the Larmor radius r_L of a cosmic-ray proton of energy 5 GeV is of the order of 10^{12} cm, much smaller than the correlation length of interstellar MHD turbulence ($\sim 10^{20}$ cm). Thus, cosmic rays closely follow field lines (for a significant time), and so the geometry and statistical properties of magnetic fields control their propagation. The dominant contribution to particle scattering is from magnetic irregularities at a scale comparable to r_L. In this Letter, we mostly discuss cosmic rays that propagate diffusively.

With some exceptions that are discussed below (see also Alouani-Bibi & le Roux 2014; Pucci et al. 2016), studies of cosmic-ray propagation employ random magnetic fields with Gaussian statistics that are completely described by the two-point correlation function or the power spectrum (e.g., Michalek & Ostrowski 1997; Giacalone & Jokipii 1999; Casse et al. 2002; Candia & Roulet 2004; Parizot 2004; DeMarco et al. 2007; Globus et al. 2008; Plotnikov et al. 2011; Harari et al. 2014; Snodin et al. 2016; Subedi et al. 2017). However, the interstellar and intergalactic magnetic fields have a more complicated structure. The fluctuation (small-scale) dynamo (Zeldovich et al. 1990; Wilkin et al. 2007) and random shock waves (Bykov & Toptygin 1987) produce highly intermittent, strongly non-Gaussian, essentially three-dimensional magnetic fields with random magnetic filaments and ribbons surrounded by weaker fluctuations. Filamentary and planar structures in the interstellar medium, consistent with the notion of spatial intermittency, have been detected in the radio (Section 5.2 in Haverkorn & Spangler 2013) and submillimeter (Zaroubi et al. 2015) ranges as well as in the neutral hydrogen distribution (Heiles & Troland 2005). In such a magnetic field, the propagation of charged particles is controlled not only by its power spectrum, but also by the size and separation of the magnetic structures. The influence of such a complex magnetic field upon cosmic-ray propagation is poorly understood. Existing theories, using the

quasilinear approach (Jokipii 1966; Berezinskii et al. 1990; Schlickeiser 2002), or its nonlinear extensions and alternative ideas (e.g., Vlad et al. 1998; Yan & Lazarian 2002; Matthaeus et al. 2003; Shalchi 2009), do not consider intermittency, or use the Corrsin hypothesis (Corrsin 1959), which assumes Gaussian statistics for the magnetic field. Recent test particle simulations used magnetic fields obtained from simulations of MHD turbulence (e.g., Dmitruk et al. 2004; Reville et al. 2008; Beresnyak et al. 2011; Lynn et al. 2012; Weidl et al. 2015; Cohet & Marcowith 2016) (see also Roh et al. 2016). These models are free from the assumption of Gaussian statistics, but they do not consider any effects of magnetic structures even if those were present. There have been no systematic attempts to examine the significance of realistic, physically realizable magnetic intermittency in 3D; this is our goal here. In intermittent magnetic fields, particle trapping can be important even in 3D. We note that the Kubo number, often used to delineate different transport regimes, depends only on second-order correlations and is therefore insensitive to intermittency.

We use test particle simulations (Giacalone & Jokipii 1999; Casse et al. 2002; Desiati & Zweibel 2014; Snodin et al. 2016), integrating the equation of motion for a large number of particles in a statistically isotropic, prescribed magnetic field, in the regime where cosmic-ray pressure is too low to excite significant MHD waves. The magnetic field is obtained as a solution of the induction equation with a prescribed velocity field that drives the fluctuation dynamo. This produces a realistic, intermittent magnetic field. The degree of intermittency depends on the magnetic Reynolds number R_m. As R_m increases, the magnetic structures occupy a smaller proportion of the volume. The intermittency introduces two distinct particle propagation regimes, one within a magnetic structure and another between them. Cosmic-ray particles are strongly scattered by the magnetic structures and move relatively freely between them. By comparing particle diffusion in an intermittent field with that in a magnetic field lacking structure, but with an identical power spectrum, we demonstrate that intermittency can significantly enhance diffusion, and so diffusion cannot be described in terms of the power spectrum alone.

2. Magnetic Field Produced by Dynamo Action

We generate intermittent, statistically isotropic, fully three-dimensional random magnetic fields b by solving the induction equation with a prescribed velocity field u:

$$\frac{\partial b}{\partial t} = \nabla \times (u \times b) + R_{m}^{-1} \nabla^2 b, \qquad \nabla \cdot b = 0, \qquad (1)$$

with periodic boundary conditions in a cubic domain of width $L = 2\pi$ and 256^3 or 512^3 mesh points. Equation (1) is written in a dimensionless form, expressing length in the units of the flow scale l_0 and time in the units of l_0/u_0, where u_0 is the rms flow speed. Here, $R_m = l_0 u_0/\eta$ is the magnetic Reynolds number[3] and η is the magnetic diffusivity, assumed to be constant. In a generic, three-dimensional, random flow, dynamo action occurs (i.e., the mean magnetic energy density grows exponentially with t) provided $R_m > R_{m,c}$, where $R_{m,c}$ is the critical magnetic Reynolds number (Zeldovich et al. 1990). Depending on the nature of the velocity field, typically $R_{m,c} \simeq 10$–100, and the magnetic field decays for $R_m < R_{m,c}$ (Brandenburg & Subramanian 2005). As $R_m \to \infty$, the magnetic structures produced by the dynamo become progressively more filamentary in nature, with the thickness of each filament of the order of $d = l_0 R_m^{-1/2}$ and a characteristic filament length (radius of curvature) of the order of l_0 (Zeldovich et al. 1990; Wilkin et al. 2007). The magnetic field used in our simulations is an eigenfunction obtained by renormalizing the exponentially growing solution of Equation (1) to have a constant rms field strength b_0. We expect the magnetic structure of the corresponding nonlinear dynamo to be similar to that of the marginal eigenfunction obtained at $R_m \approx R_{m,c}$ (Subramanian 1999). However, we consider a wider range of R_m to explore the effects of a variable degree of intermittency: it increases with R_m. To isolate robust features of cosmic-ray propagation independent of the particular form of intermittent magnetic field, we use two types of incompressible flow to drive the dynamo, both chaotic, but one of a single scale and the other multi-scale with a controlled power spectrum. The first flow (Willis 2012), henceforth referred to as flow W, is stationary:

$$u(x) = (2/\sqrt{3})(\sin y \cos z, \sin z \cos x, \sin x \cos y). \qquad (2)$$

It is a very efficient dynamo with $R_{m,c} \approx 11$, producing regularly spaced magnetic structures in the form of ellipsoids of identical size that become thinner as R_m increases and whose positions are determined solely by the flow geometry (so are independent of R_m). The second flow (KS) is time-dependent and multi-scale; it was employed for dynamo simulations (Wilkin et al. 2007) and as a Lagrangian model of turbulence (Fung et al. 1992):

$$u(x, t) = \sum_{n=0}^{N-1} (C_n \cos \phi_n + D_n \sin \phi_n), \qquad (3)$$

where $\phi_n = k_n \cdot x + \omega_n t$, with k_n a randomly oriented wave vector (of magnitude k_n) and ω_n a frequency specified below. The random vectors, C_n and D_n, are chosen to be orthogonal to k_n to ensure $\nabla \cdot u = 0$. We select $N = 40$ distinct wave

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[3] Some authors define R_m in terms of the wavenumber k_0, resulting in R_m values a factor of 2π smaller.

vectors, with magnitudes between $k_0 = 2\pi/L$ and $k_{N-1} \approx 8k_0$, so that the flow is periodic with the outer scale $l_0 = L$. The amplitudes of C_n and D_n are selected to produce an energy spectrum $E(k) \propto k^{-5/3}$ with $\int_{k_0}^{k_{N-1}} E(k)dk = u_0^2/2$. We take $\omega_n = [k_n^3 E(k_n)]^{1/2}$, which introduces a scale-dependent time variation. The dynamo in this flow has $R_{m,c} \approx 1000$ (Wilkin et al. 2007). The flow produces transient magnetic structures, consisting of filaments of various sizes, as illustrated in the leftmost panel of Figure 1.

To identify the effects of magnetic intermittency on cosmic-ray diffusion, we also consider random magnetic fields where the structures have been destroyed but the magnetic energy spectrum remains unchanged (Snodin et al. 2013). This is achieved by taking the spatial Fourier transform of $b(x)$ from Equation (1) and then multiplying each complex Fourier mode by $\exp[i\psi(k)]$, with $\psi(k)$ a random phase selected independently for each k. The inverse Fourier transform of the result produces a magnetic field with an unchanged spectrum but with little remaining structure, as demonstrated in the second from left panel of Figure 1. As shown in the second from right panel of Figure 1, the probability density functions (PDFs) of the field components for the intermittent fields produced by each flow (W and KS) have long, heavy tails, while the phase randomization produces nearly Gaussian random fields. Another aspect of this difference is also illustrated in the rightmost panel of Figure 1 where the fractional volume occupied by magnetic structures with $b/b_0 > \nu$ is shown as a function of ν: an intermittent magnetic field has more strong, localized structures with $\nu \gtrsim 1.4$ than a Gaussian field with an identical power spectrum.

To explore the effects of a mean magnetic field, we also consider particle propagation in a magnetic field given by $B = b + B_0$, where B_0 is an imposed uniform magnetic field. In such cases, the rms magnetic field $\tilde{b}_0$ quoted below includes the mean part, $\tilde{b}_0^2 = B_0^2 + b_0^2$.

3. Cosmic-ray Propagation

Using magnetic field realizations generated from Equation (1), or the corresponding randomized magnetic fields, we obtain an ensemble of cosmic-ray trajectories ($\geqslant 1000$ in number) by solving numerically the dimensionless equation of motion for the particle trajectories $x(t)$,

$$\ddot{x} = \alpha \dot{x} \times B(x), \qquad (4)$$

with $\alpha = ql_0 \tilde{b}_0/(\gamma mc v_0)$, q the particle charge, m its rest mass, $\tilde{b}_0$ the total rms field strength, γ the Lorentz factor, v_0 the particle speed, and c the speed of light. As in most cosmic-ray propagation models (Berezinskii et al. 1990; Schlickeiser 2002; Shalchi 2009), we neglect electric fields in Equation (4): they are negligible at the scales of interest ($\simeq 1$ kpc in galaxies and $\simeq 10$ kpc in galaxy clusters). Hence, the particle speed v_0 remains constant. Each particle is given a random initial position and propagation direction, but the same initial speed. The *characteristic* dimensionless Larmor radius, based on the rms magnetic field strength, is $r_L/l_0 = \alpha^{-1}$; we use this ratio to characterize the particle properties. When $B_0 = 0$, we calculate the isotropic diffusion coefficient $\kappa = \lim_{t\to\infty} \langle |\Delta x(t)|^2 \rangle/(6t)$, where $\Delta x(t)$ is the particle displacement, and the angular brackets denote averaging over particle displacements. In the
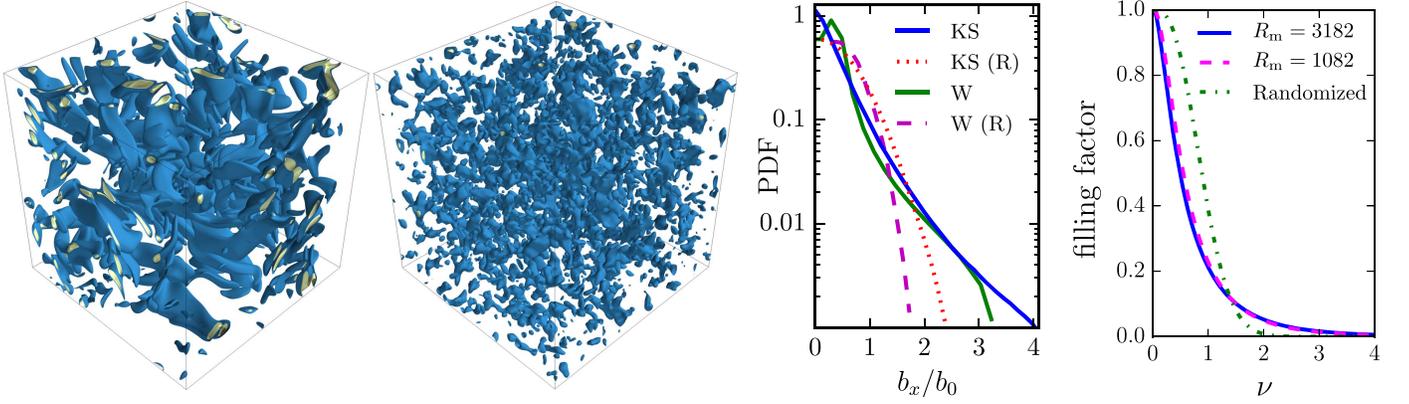
Figure 1. Isosurfaces of magnetic field strength $b^2/b_0^2 = 2.5$ (blue) and $b^2/b_0^2 = 5$ (yellow) with b_0 the rms magnetic field, for magnetic field generated by the KS flow (3) at $R_m = 1082$ (left) and for the same magnetic field after Fourier phase randomization as described in the text (second from left). Magnetic field generated by the W flow (2) is similarly affected (not shown). The second from right panel shows the PDFs of a magnetic field component b_x for the original (KS, W: solid) and randomized (KS (R), W (R): dashed) magnetic fields obtained with both velocity fields (only $b_x > 0$ is shown as the PDFs are essentially symmetric about $b_x = 0$). The randomized fields have almost perfectly Gaussian statistics, whereas magnetic intermittency leads to heavy tails. The panel on the right shows the fractional volume within magnetic structures where $b \geqslant \nu b_0$, with b_0 the rms field strength, as a function of ν for the intermittent magnetic field produced by the flow (3) (solid for $R_m = 3182$ and dashed for $R_m = 1082$) and its Gaussian counterpart (dashed–dotted for $R_m = 3142$ and 1082) obtained by Fourier phase randomization; the filling factor of the randomized fields is independent of R_m.

presence of a mean magnetic field directed along the z-axis, we introduce similarly defined parallel and perpendicular diffusion coefficients, which are given by $\kappa_\parallel = \lim_{t \to \infty} \langle \Delta z(t)^2 \rangle / (2t)$ and $\kappa_\perp = \lim_{t \to \infty} \langle [\Delta x(t)^2 + \Delta y(t)^2] \rangle / (4t)$.

4. Cosmic-ray Diffusivity

Figure 2(a) shows the dependence of the cosmic-ray diffusion coefficient on r_L/l_0 (proportional to the particle energy) for $B_0 = 0$. For $r_L/l_0 \gg 1$, we recover the asymptotic scaling $\kappa \propto r_L^2$ (high energy limit) in agreement with earlier results (Parker 1965; Aloisio & Berezinsky 2004; Parizot 2004; DeMarco et al. 2007; Globus et al. 2008; Beresnyak et al. 2011; Plotnikov et al. 2011; Harari et al. 2014; Snodin et al. 2016; Subedi et al. 2017). At lower energies, the dependence of κ on particle energy is weaker and is sensitive to magnetic structure. Magnetic intermittency is expected to be important at those energies where

$$r_L/l_0 \lesssim 1, \tag{5}$$

and the dependence $\kappa(r_L/l_0)$ in Figure 2(a) indeed deviates from the asymptotic form in this range. The role of magnetic intermittency is demonstrated in Figure 2(b), showing the ratio of the diffusivity κ calculated with a dynamo-generated magnetic field to that in the corresponding randomized field, κ_R ($B_0 = 0$ in panels (a) and (b)). At high energies (large r_L/l_0), $\kappa/\kappa_R \simeq 1$, suggesting that the magnetic structures play an insignificant role. However, κ/κ_R increases rapidly up to more than 2.5 at lower energies: magnetic structures enhance diffusion when inequality (5) is satisfied. We find that the ratio κ/κ_R at fixed r_L/l_0 increases with R_m for a given flow. At high values of r_L/l_0, the diffusivity still depends on R_m via changes in the magnetic correlation length (Figure 2(a)), but not via the R_m-dependent intermittency, as suggested by Figure 2(b), where κ/κ_R tends to unity as r_L/l_0 increases. One might expect a change in the diffusivity behavior at $r_L/l_0 \approx R_m^{-1/2}$, associated with the thickness of magnetic filaments, and this may explain the variation in slope of κ at low r_L/l_0 in
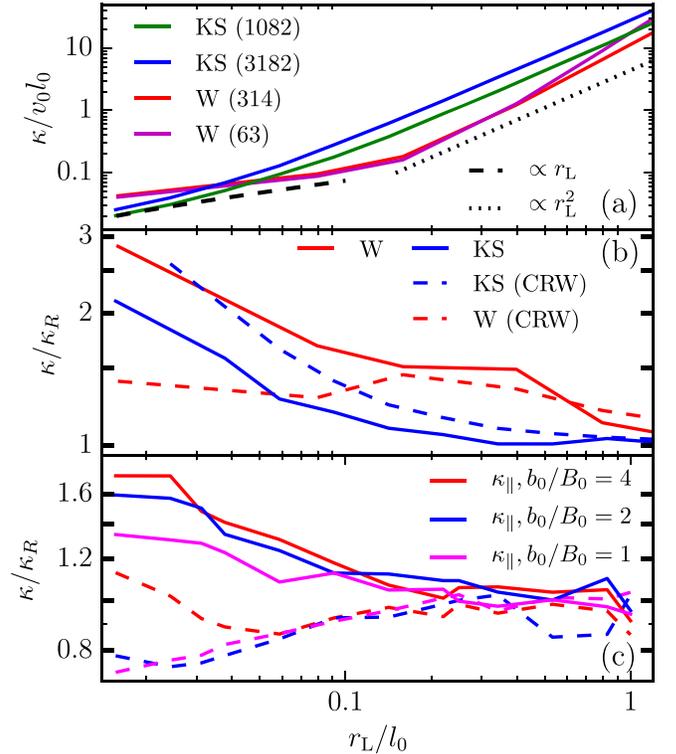

Figure 2. (a) Cosmic-ray diffusion coefficient for the W flow (2) (red, magenta) and the KS flow (3) (blue, green) as a function of r_L/l_0 for the values of R_m given in parenthesis in the legend. The dotted–dashed line shows the scaling $\kappa \propto r_L^2$ and $\kappa \propto r_L$, respectively. (b) The ratio of diffusion coefficients from intermittent, κ, and randomized, κ_R, magnetic fields for the two flows (solid lines, KS with $R_m = 3182$, W with $R_m = 314$). The dashed lines of the same color show the corresponding CRW model, Equation (7). (c) As in (b), but in the presence of a mean magnetic field B_0, of the relative strength specified in the legend, for the KS flow with $R_m = 3182$; $\kappa_\parallel$ and $\kappa_\perp$ are shown as solid and dashed lines, respectively.

Figure 2(a) (or the ratios in Figure 2(b)). However, at present, the role of this scale is unclear.

Figure 2(c) illustrates the effects of the mean magnetic field, presenting the ratio of the parallel and perpendicular
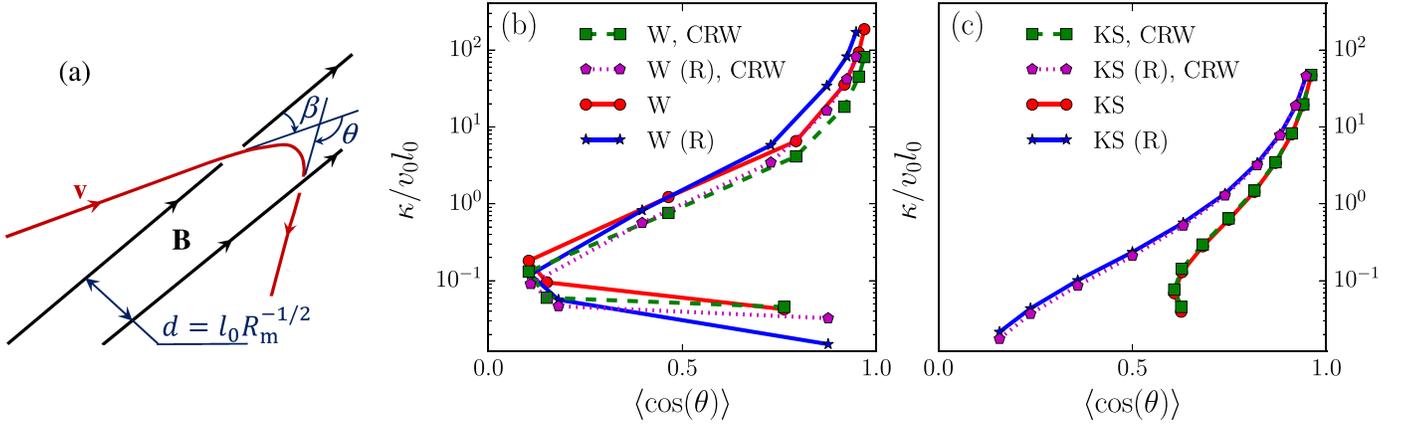
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Figure 3. (a) A charged particle with the pitch angle β is deflected by an angle θ in a magnetic structure of a thickness d, thus introducing correlation into the random particle trajectory. The dependence of the diffusivity in simulations on $\langle \cos \theta \rangle$ in the W flow and the randomized (R) magnetic field is shown in (b) (solid lines). The corresponding CRW approximations (Equation (7)) are shown with broken lines. (c) As in (b), but for the KS flow. These results are obtained for $B_0 = 0$.

diffusivities in the intermittent and Gaussian magnetic fields. A mean magnetic field somewhat reduces the effect of intermittency, but does not eliminate it even for $b_0/B_0 = 1$. Magnetic intermittency enhances $\kappa_\parallel$ (i.e., $\kappa_\parallel > (\kappa_\parallel)_R$) at all but the highest energies, but $\kappa_\perp < (\kappa_\perp)_R$ at lower energies for $b_0/B_0 = 2$ and $b_0/B_0 = 1$. The effects of the mean field will be discussed in detail elsewhere.

5. Cosmic-ray Propagation as a Correlated Random Walk (CRW)

The Brownian motion is a widely used model for diffusive processes. This is the simplest type of random walk where each step is made in a direction independent of the previous direction. In a continuum limit, it leads to the diffusion equation. However, a charged particle moves differently. As illustrated in Figure 3(a), the direction of its motion after deflection by a magnetic structure is correlated with the previous direction. The deflection angle θ is related to r_L, the angle between the velocity and magnetic field, β, and the magnetic structure width d:

$$\theta \simeq d/(r_L \sin \beta). \tag{6}$$

This is a correlated random walk (CRW) (Gillis 1955), a first-order Markov chain (since the correlation does not extend beyond two consecutive steps). For a symmetric probability distribution of θ, the CRW diffusivity depends on $\langle \cos \theta \rangle$, where angular brackets denote the ensemble average. The mean-square displacement in the CRW was obtained in 2D (Kareiva & Shigesada 1983) and implies the following 3D diffusivity (Equation (3.3.7) in Chen & Renshaw 1992):

$$\kappa = \frac{\langle l^2 \rangle}{6\tau} + \frac{\langle l \rangle^2}{3\tau} \frac{\langle \cos \theta \rangle}{1 - \langle \cos \theta \rangle}, \tag{7}$$

with $\tau = \langle l \rangle / v$, v the particle speed, and l the step length. To calculate $\langle \cos \theta \rangle$, we assume that the pitch angle β is uniformly distributed between 0 and π. Defining $a = d/r_L \simeq l_0 R_m^{-1/2}/r_L$, it can be shown that

$$\langle \cos \theta \rangle = \pi^{-1} \int_0^\pi \cos(a/\sin \beta) \mathrm{d}\beta$$
$$= 1 - \tfrac{1}{2}\pi a [J_0(a)\mathcal{H}_{-1}(a) - J_{-1}(a)\mathcal{H}_0(a)], \tag{8}$$

where $J_n(x)$ and $\mathcal{H}_n(x)$ are the Bessel and Struve functions (2.5.8.6 in Prudnikov et al. 1991). Finite length of the magnetic structures can be accounted for, but this represents a small correction and the integral cannot be taken analytically.

To derive $\langle \cos \theta \rangle$ in the simulations, the particle trajectories were sampled at a rate that is comparable to the local Larmor time; the sampling frequency does not affect the results much (cf. Codling & Hill 2005; Rosser et al. 2013). $\langle \cos \theta \rangle$ computed using Equation (8) and the same obtained from the simulations show reasonable qualitative agreement if we adopt $d = l_0 R_m^{-1/2}$ for the flow (2) and d as the thickness of the magnetic structures calculated using the Minkowski functionals (Wilkin et al. 2007) for the flow (3).

Figures 3(b) and (c) show the variation of κ with $\langle \cos \theta \rangle$, where κ is obtained numerically for both the intermittent and randomized magnetic fields, and in each case, the corresponding κ predicted from Equation (7) is also shown. For τ in Equation (7), we have used r_L/v_0, where r_L is the local Larmor radius. The agreement is remarkably good for the flow (2) and excellent for the less regular magnetic field resulting from the flow (3). This confirms directly that the cosmic-ray propagation is a CRW with the diffusivity given by Equation (7). This applies to *both* intermittent and Gaussian random magnetic fields (see also Figure 2(b)). We note that the first term in Equation (7) dominates at large r_L.

6. Conclusions

We have demonstrated that cosmic-ray propagation in random magnetic fields is affected by magnetic intermittency in the range of energies (5), or

$$\frac{E}{1 \text{ GeV}} \lesssim 10^9 \frac{l_0}{1 \text{ kpc}} \frac{B}{1 \mu\text{G}}.$$

In the interstellar medium, $l_0 \simeq 100$ pc and $B \simeq 10 \mu$G, and for ultra-relativistic protons, this energy range is $E \lesssim 10^9$ GeV. In galaxy clusters, $l_0 \simeq 10$ kpc, $B \simeq 2 \mu$G, and $E \lesssim 10^{10}$ GeV.

Assuming $R_{m,eff} = R_{m,c} = 100$ in the interstellar medium, we might expect some effect at $r_L/l_0 = 0.1$, which would correspond to 10^8 GeV protons using the above values. Such an effect might produce a knee or spectral break in the cosmic-ray energy spectrum near this energy. The influence of magnetic intermittency extends to below this energy (the effect of

intermittency on cosmic-ray diffusivity increases as energy decreases), but further investigation is needed to quantify this. Finally, we note that magnetic intermittency may also affect ultrahigh-energy cosmic rays that propagate non-diffusively and that their propagation can also be interpreted as a CRW.

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