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Classifying Invariant Structures of Step Traces

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Abstract

In the study of behaviours of concurrent systems, traces are sets of behaviourally equivalent action sequences. Traces can be represented by causal partial orders. Step traces, on the other hand, are sets of behaviourally equivalent step sequences, each step being a set of simultaneous actions. Step traces can be represented by relational structures comprising non-simultaneity and weak causality.

In this paper, we propose a classification of step alphabets as well as the corresponding step traces and relational structures representing them. We also explain how the original trace model fits into the overall framework.

Keywords: trace, independence, partial order, interleaving, trace of step sequences, simultaneity, sequentialisation, serialisability, invariant structure

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1. Introduction

Mazurkiewicz traces [1, 2] are a well-established, classical, and basic model for representing and structuring sequential observations of concurrent behaviour; see, e.g., [3]. The fundamental assumption underlying trace theory is that independent events (occurrences of actions) may be observed in any order. Sequences that differ only w.r.t. the ordering of independent events are identified as belonging to the same concurrent run of the system under consideration. Thus a trace is an equivalence class of sequences comprising all (sequential) observations of a single concurrent run. The dependencies between the events of a trace are invariant among (common to) all elements of the trace. They define an acyclic dependence graph which — through its transitive closure — determines the underlying causality structure of the trace as a (labelled) partial order [4]. In fact, this partial order can also be obtained as the intersection of the labelled total orders corresponding to the sequences forming the trace. Moreover, the sequences belonging to the trace correspond exactly to the linearisations (saturations) of this partial order. In [5], the necessary connection between causal structures (partial orders) and observations (total orders) is provided by showing that each partial order is the intersection of all its linearisations (Szpilrajn's property). Consequently, each trace can also be viewed as a labelled partial order which is unique up to isomorphism, i.e., up to the names of the underlying elements; see, e.g., [3, 6]. Thus, to capture the essence of equivalence between different observations of the same run of a concurrent system, Mazurkiewicz traces bring together two mathematical ideas, both based on a notion of independence between events expressed as a binary independence relation ind over actions. On the one hand, there are equations $ab = ba$ generating the equivalence by expressing the commutativity of occurrences of certain actions as determined by the independence relation. As a result, sequences $wabu$ and $wbau$ of action occurrences are considered equivalent whenever $\langle a, b\rangle \in \mathsf{ind}$, irrespective of what w and u are. On the other hand, there is a common acyclic dependence relation that underlies equivalent observations and is defined by the ordering of the occurrences of dependent actions, and its transitive closure interpreted as a causal partial order representing the trace to which $wabu$ and $wbau$ both belong. In a nutshell, the main concepts of trace theory are as follows:

- a *trace alphabet* comprising a finite set of actions Σ and an independence relation ind on Σ;
- a set of *equations* $ab = ba$, where $\langle a, b\rangle \in \mathsf{ind}$, defining a relation $\equiv$ of behavioural equivalence on action sequences, each equivalence class of $\equiv$ being a *trace*;
- an action-labelled *total order* representing in a unique way a finite action sequence;
- an action-labelled *dependence graph* (acyclic relation) derived from an action sequence which is common and unique to each trace;
- an action-labelled *causal partial order* derived from the dependence graph representing in a unique way a trace; and
- the operation of *transitive closure* which allows one to derive causal partial orders from dependence graphs.

Being based on equating independence and lack of ordering as well as assuming that no actions can be simultaneous, the model of Mazurkiewicz traces with the corresponding partial order interpretation of concurrency is not always sufficient. In [7], a generalisation of the theory of traces is presented for the case that actions could occur and may be observed as occurring simultaneously (a common assumption made, e.g., by concurrency models inspired by bio-chemical reactions as in [8, 9]; see also [10] for other examples). Thus observations consist of sequences of *steps*, i.e., sets of one or more actions that occur simultaneously. To retain the philosophy underlying Mazurkiewicz traces, the extended set-up is based on a few explicit and simple design choices.

Instead of the independence relation ind, *step alphabets* use two basic relations between pairs of actions: *simultaneity* sim indicating actions that may occur together in a step, and *sequentialisation* seq indicating equivalent orders of executing two different actions. The two relations are applied to identify step sequences as observations of the same concurrent run. The equations they determine are of the form $AB = BA$ and $AB = A \uplus B$, where A and B are steps, and the resulting equivalence classes of step sequences are called *step traces*.

Step sequences have been used to represent operational semantics of concurrent systems for long time [11, 12] and they are still popular [13]. The fundamental difference between models like [11, 12, 13] and the approach of this paper is that we group step sequences that are considered equivalent into step traces. Each step trace uniquely defines some relational structure, in the similar way as each trace uniquely defines a causal partial order.

The main aim of this paper is to investigate different classes of step traces obtained by restrictions on the simultaneity and sequentialisation relations, and to identify the corresponding relational structures. The proposed hierarchy of families of step traces includes new non-trivial classes of traces as well as the original Mazurkiewicz traces, comtraces [14, 15], and g-comtraces [16].

Modelling concurrency with relational structures stems from the results of [10, 17] and [18]. The basic idea is that general concurrent causal behaviour is represented by *a pair* of relations, instead of just one, as in the standard (causal partial order) approach (see, e.g., [4]). Depending on the assumptions for the chosen model of concurrency details vary, but basically there are two versions: one in which the two relations are interpreted as standard *causality* (dependence or precedence) and *weak causality* (not later than), respectively (see, e.g., [10, 14, 17]) and an extended, general, version (suggested in [10, 19] but eventually defined in [20]) with the two relations:[1] *mutual exclusion* and *weak causality*. The first version has a relatively well developed theory and substantial applications (see, e.g., [10, 14, 17, 21, 22, 23]). The second one, however, is relatively new and as such the starting point for this paper where we identify the invariant structures that characterise the subfamilies of step traces.

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[1] Causality being a derived notion.

The paper is organised as follows. In the next section, we present basic notions and definitions. In Sections 3 and 4, we recall the main definitions and results concerning step alphabets, step traces, and relational structures. In Sections 5—9, we present the main results of the paper, providing a characterisation of the relationships between the interesting subclasses of step traces and the corresponding relational structures. Section 10 concludes the paper.

This paper is an extended and refined version of a paper presented at the LATA'15 conference [24]. We have also streamlined some notions and notations used there as well as in previous papers, e.g. [7, 20]. Most of the proofs are included in the appendix.

2. Preliminaries

Throughout the paper, we assume that:

- Σ is an *alphabet of actions* taken to be a finite nonempty set; an *event* is a pair $\langle a, i\rangle$ such that $a \in \Sigma$ and $i \geq 1$; $\ell(\langle a, i\rangle) = a$ is the default labelling of an event $\langle a, i\rangle$; and an *event domain* is any set of events $\Delta = \{\langle a, i\rangle \mid a \in \Sigma \wedge 1 \leq i \leq k_a\}$, where, for every $a \in \Sigma$, $k_a \geq 0$.

- $\mathbb{S}$ is the set of *steps* over Σ comprising all the nonempty subsets of Σ; SSEQ is the set of all finite sequences of steps (*step sequences* Σ); and, if $u = A_1 \ldots A_k$ is a step sequence, then $occ(u)$ comprises all events $\langle a, i\rangle$ such that i does not exceed the number of occurrences of a within u, and $j = pos_u(\langle a, i\rangle)$ is such that the i-th occurrence of a is in A_j.

- The symmetric closure of a binary relation R is $R^{sym} = R \cup R^{-1}$; R is transitive if $R \circ R \subseteq R$; R is a preorder relation if it is irreflexive and $R \cup id_X$ is transitive, where $id_X = \{\langle x, x\rangle \mid x \in X\}$; R is an equivalence relation if it is symmetric, transitive and reflexive; R is a partial order relation if it is irreflexive and transitive; and R is a total order relation if it is a partial order relation such that we have $R^{sym} = (X \times X) \setminus id_X$.

- Given a binary relation $R \subseteq X \times X$, R^+ is the transitive closure of R; R^* is the reflexive transitive closure of R; $R^{\curlywedge} = R^* \setminus id_X$ is the irreflexive transitive closure of R; $R^{\circledast} = R^* \cap (R^*)^{-1}$ is the largest equivalence relation contained in R^*; and R is acyclic if R^+ is asymmetric.
- A labelled directed graph is triple $\langle X, R, \ell \rangle$ comprising a finite set of vertices X, an irreflexive binary relation R on X comprising arcs, and a labelling $X \xrightarrow{\ell} \Sigma$. It is a partial order / total order / preorder / acyclic graph if R is a partial order / total order / preorder / acyclic relation. The graph is complete if $R = (X \times X) \setminus id_X$, and a clique is any nonempty subset $Y \subseteq X$ such that $R|_{Y \times Y} = (Y \times Y) \setminus id_Y$. We say that $x, y \in X$ lie on a cycle if $\langle x, y \rangle, \langle y, x \rangle \in R^+$.

We often identify a singleton step $\{a\}$ with its only member, tacitly assuming that $\Sigma \subset \mathbb{S}$. Moreover, we denote non-singleton steps by listing their elements within parentheses.

3. Step traces

We start by recalling the basic definitions and results from [7]. A *step alphabet* is a triple $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where sim (*simultaneity*) and seq (*sequentialisation*) are irreflexive relations over Σ such that sim and $\mathsf{seq} \setminus \mathsf{sim}$ are symmetric. The family of all step alphabets will be denoted by Θ. Simultaneity defines legal *steps* over the alphabet θ, $\mathbb{S}_\theta = \{A \subseteq \Sigma \mid A \neq \varnothing \wedge (A \times A) \setminus id_\Sigma \subseteq \mathsf{sim}\}$, and the strings in $\mathsf{SSEQ}_\theta = \mathbb{S}_\theta^*$ are called *step sequences* over θ. Sequentialisation, on the other hand, defines ways in which steps can be sequentialised and identifies pairs of actions which can be interleaved, leading to the following *equations* over θ, where $A, B \in \mathbb{S}_\theta$:

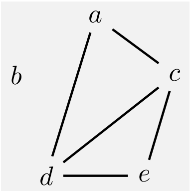
$$\begin{array}{lll l} AB = BA & \text{if} & A \times B \subseteq \mathsf{seq} \cap \mathsf{seq}^{-1} & (\textit{interleaving}) \\ AB = A \cup B & \text{if} & A \times B \subseteq \mathsf{sim} \cap \mathsf{seq} & (\textit{serialisability}) \end{array}$$

The above equations induce a relation $\approx$ on step sequences such that $u \approx v$ if there exist $w, t \in \mathsf{SSEQ}$ and $A, B \in \mathbb{S}$ satisfying: (i) $u = wABt$ and $u = wBAt$

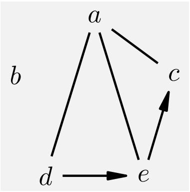
and $AB = BA$; or (ii) $u = wABt$ and $u = w(A \cup B)t$ and $AB = (A \cup B)$. We then define a relation $\equiv$ on step sequences as the reflexive, symmetric, and transitive closure of $\approx$. The equivalence classes of $\equiv$ containing step sequences in SSEQ_θ are *step traces* over θ, and their set is denoted by STR_θ. The trace containing $u \in \mathsf{SSEQ}_\theta$ will be denoted by $[\![u]\!]$. For a step trace $\tau = [\![u]\!] \in \mathsf{STR}_\theta$, for some step sequence u over θ, we use $occ(\tau) = occ(u)$ to denote the set of action occurrences in τ (note that this is well-defined, as all step sequences in τ have the same set of action occurrences). Step traces involve only legal steps, i.e., if $\tau \in \mathsf{STR}_\theta$ then $\tau \subseteq \mathsf{SSEQ}_\theta$. See [7] for more details and for an alternative, but equivalent, approach for defining step traces.

Example 3.1. *Consider* $\theta_0 = \langle\{a, b, c, d, e\}, \mathsf{sim}, \mathsf{seq}\rangle$, *a step alphabet with simultaneity and sequentialisation relations given below, where each undirected edge stands for two arrows in opposite directions:*

$\mathsf{sim} =$



$\mathsf{seq} =$



θ_0 *generates, e.g., the interleaving equations* $ae = ea$ *and* $a(ce) = (ce)a$, *and serialisability equations* $(ac) = ac$, $(ac) = ca$, *and* $(ce) = ec$. *However,* $(ce) = ce$ *is not an equation generated by* θ_0. *We also have:*

$$\begin{array}{lcl lcl}
[\![ace]\!] & = & \{ace, cae, cea, (ac)e\} & [\![abc]\!] & = & \{abc\} \\
[\![acd]\!] & = & \{acd, cad, cda, (ac)d, c(ad)\} & [\![aeb]\!] & = & \{aeb, eab\} \\
[\![(cde)]\!] & = & \{(cde)\} & [\![a(cd)]\!] & = & \{a(cd), (cd)a, (acd)\} \\
[\![dec]\!] & = & \{dec, (de)c, d(ce)\} & [\![a(cde)]\!] & = & \{a(cde), (cde)a\}\ .
\end{array}$$

◇

3.1. Classifying step alphabets

An immediate semantically meaningful classification of step alphabets is obtained by looking at the consequences of assuming that some of the three relations $\mathsf{sim} \setminus \mathsf{seq}$, $\mathsf{seq} \setminus \mathsf{sim}$, and $\mathsf{sim} \cap \mathsf{seq}$ are empty. This leads to eight classes of step alphabets, shown in Figure 1, where $\mathsf{sim} \triangle \mathsf{seq} = (\mathsf{sim} \setminus \mathsf{seq}) \cup (\mathsf{seq} \setminus \mathsf{sim})$ denotes the symmetric difference of sim and seq, and subscripts indicate the empty relationships. Thus, for example, $\Theta_{\mathsf{sim} \cap \mathsf{seq}}$ comprises all step alphabets with disjoint relations sim and seq. One can observe that:

- Θ is the family of all step alphabets.
- $\Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ comprises step alphabets such that the serialisability equations are rich enough to split any step in every possible way.
- $\Theta_{\mathsf{seq} \setminus \mathsf{sim}}$ comprises step alphabets without true interleaving (the interleaving equations can be realised through serialisation of steps). In the literature, alphabets in $\Theta_{\mathsf{seq} \setminus \mathsf{sim}}$ are called *comtrace alphabets* [10].
- $\Theta_{\mathsf{sim} \cap \mathsf{seq}}$ comprises step alphabets where the only manipulation of steps is through interleaving equations.
- Θ_{seq} comprises step alphabets generating step traces consisting of a single step sequence.
- Θ_{sim} comprises step alphabets which define only singleton steps. Alphabets in Θ_{sim} correspond to trace alphabets after dropping the empty relation sim and treating $\mathsf{seq} = \mathsf{seq}^{-1}$ as the independence relation.
- $\Theta_{\mathsf{sim} \triangle \mathsf{seq}}$ comprises step alphabets with serialisability equations that are rich enough to split and reorder steps in every possible way. Alphabets in $\Theta_{\mathsf{sim} \triangle \mathsf{seq}}$ can be seen as suitable trace alphabets for step sequence semantics of safe Petri nets (see [25]).
- $\Theta_{\mathsf{sim} \cup \mathsf{seq}}$ comprises step alphabets generating traces consisting of a single sequence.

So, the alphabets in $\Theta_{\mathsf{sim} \cup \mathsf{seq}}$ and Θ_{seq} are of little interest. The alphabets in Θ have been considered in [7]. Hence, we will focus on a closer investigation of Θ_{sim}, $\Theta_{\mathsf{sim} \triangle \mathsf{seq}}$, $\Theta_{\mathsf{sim} \setminus \mathsf{seq}}$, $\Theta_{\mathsf{seq} \setminus \mathsf{sim}}$, and $\Theta_{\mathsf{sim} \cap \mathsf{seq}}$. To the best of our knowledge, $\Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ and $\Theta_{\mathsf{sim} \cap \mathsf{seq}}$ lead to new subclasses of step traces, whereas the other

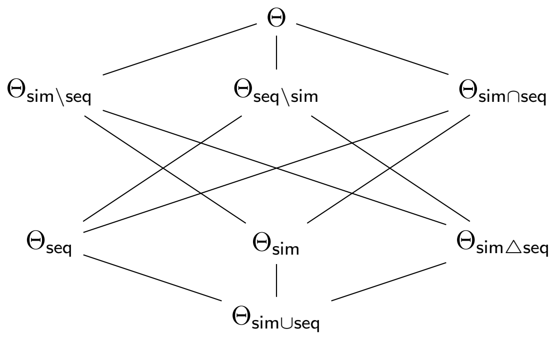


Figure 1: Inclusion diagram of the eight types of step alphabets.

three have to some extent already been identified in the literature (as recalled above).

4. Relational structures for step traces

The order theoretic treatment of step traces is based on *relational structures* $\langle\Delta, \rightleftharpoons, \sqsubset, \ell\rangle$ comprising a finite *domain* Δ, two binary relations $\rightleftharpoons$ and $\sqsubset$ on Δ, and a domain labelling $\Delta \xrightarrow{\ell} \Sigma$. Two domain elements, x and y, are *equilabelled* if $\ell(x) = \ell(y)$.

To represent observational and causal relationships in the behaviours of concurrent systems we use the *order structures* OR from [7, 20] which are an extension of ideas first proposed in [10, 17, 18]. Individual observations (step sequences) are represented by *saturated structures* SR, and causal relationships are represented by *invariant structures* IR.

4.1. Order structures

Referring to the set-up of Mazurkiewicz traces, order structures correspond to (labelled) acyclic relations.

An *order (relational) structure* is a relational structure $or = \langle\Delta, \rightleftharpoons, \sqsubset, \ell\rangle$ that is *separable*, meaning that the *mutex* relation $\rightleftharpoons$ is symmetric, the *weak* *causality* relation $\sqsubset$ is irreflexive[2], and $\rightleftharpoons \cap \sqsubset^{\circledast} = \varnothing$ (which implies that $\rightleftharpoons$ is also irreflexive); and that is *label-ordered*, meaning that any two distinct equilabelled events are related by both $\rightleftharpoons$ and $\sqsubset^{sym}$.

Intuitively, Δ is the set of events that have happened during some execution of a concurrent system with their labels giving the names of the corresponding actions; $x \rightleftharpoons y$ means that x occurred *not simultaneously* with y, and $x \sqsubset y$ that x occurred *not later* than y, i.e., *before or simultaneously* with y. Hence if $x \sqsubset y$ and $x \rightleftharpoons y$, then x must have occurred *before* y. We will therefore refer to the intersection $\sqsubset \cap \rightleftharpoons$ as *causality* (or *precedence*), denoting it by $\prec$. Note that $x \sqsubset y \sqsubset x$ intuitively means that x and y were observed as *simultaneous.*

Separability excludes situations where events forming a weak causality cycle in $\sqsubset^{\circledast}$, are also involved in the mutex relationship.

To improve clarity of explanations of definitions involving order structures, we will provide some of their properties referring explicitly to the following three derived labelled directed graphs: $\langle \Delta, \rightleftharpoons, \ell \rangle$, $\langle \Delta, \sqsubset, \ell \rangle$, and $\langle \Delta, \prec, \ell \rangle$.

In terms of graph representation of an order structure, any two equilabelled events are connected by an arc in both $\langle \Delta, \sqsubset, \ell \rangle$ and $\langle \Delta, \prec, \ell \rangle$ but they do not lie on a cycle, and in $\langle \Delta, \rightleftharpoons, \ell \rangle$ each set of equilabelled events is a clique. Moreover, no two $\rightleftharpoons$–connected events lay on a $\sqsubset$–cycle (see separability).

Label-orderedness in combination with separability implies *label-linearity*, i.e., for all actions, $\prec$ restricted to the elements labelled by this action, is a total order relation (see [7]). Label-linearity is the only condition involving event labels that we need on account of [7]. Although label-linearity is sufficient for the purposes of this paper, in general one can develop quite involved characterisation of all 'good' labellings for the order structures corresponding to general step

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[2] One could assume that $\sqsubset$ is reflexive obtaining an equivalent model (see [26]). In our view, assuming reflexivity or irreflexivity has its own advantages and disadvantages in the technical treatment.

traces (see [27]).

An *extension* of the order structure $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ is any order structure $\langle \Delta, \rightleftharpoons', \sqsubset', \ell \rangle$ such that $\rightleftharpoons \subseteq \rightleftharpoons'$ and $\sqsubset \subseteq \sqsubset'$.

4.2. Saturated structures

Referring to the set-up of Mazurkiewicz traces, saturated structures correspond to total orders, i.e., those acyclic relations which cannot be extended without violating their acyclicity.

A *saturated (relational) structure* is a relational structure $sr = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ satisfying, for all $x, y, z \in \Delta$:

$$\begin{array}{rcrcll}
x \neq y & \wedge & x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(S1)} \\
 & & x \rightleftharpoons y & \Longrightarrow & x \sqsubset^{sym} y & \textit{(S2)} \\
x \neq y & \wedge & x \not\rightleftharpoons y & \Longleftrightarrow & x \sqsubset y \sqsubset x & \textit{(S3)} \\
x \neq y & \wedge & \ell(x) = \ell(y) & \Longrightarrow & x \rightleftharpoons y & \textit{(S4)}
\end{array}$$

It follows that every saturated structure is separable and label-ordered and hence an order structure. In fact, the saturated structures are the only order structures which cannot be extended without violating separability. We denote by $\mathsf{or2SR}(or)$ the set of all saturated extensions of $or \in \mathsf{OR}$.

In terms of graph representation, any two events are either simultaneously connected in $\langle \Delta, \prec, \ell \rangle$ and in one direction in $\langle \Delta, \sqsubset, \ell \rangle$, or connected in both directions in $\langle \Delta, \sqsubset, \ell \rangle$.

4.3. Invariant structures

Referring to the set-up of Mazurkiewicz traces, invariant structures correspond to partial orders, i.e., those acyclic relations which cannot be extended without reducing their set of total order extensions.

An *invariant (relational) structure* is a relational structure $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ satisfying, for all $x, y, z \in \Delta$:

$$\begin{array}{rcrcll}
 & & & & x \not\sqsubset x & \textit{(I1)} \\
x \neq y & \wedge & x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(I2)} \\
 & & x \rightleftharpoons y & \Longrightarrow & y \rightleftharpoons x \neq y & \textit{(I3)} \\
x \prec z \sqsubset y & \vee & x \sqsubset z \prec y & \Longrightarrow & x \rightleftharpoons y & \textit{(I4)} \\
z \rightleftharpoons y & \wedge & z \sqsubset x \sqsubset z & \Longrightarrow & x \rightleftharpoons y & \textit{(I5)} \\
z \rightleftharpoons z' \wedge x \sqsubset z \sqsubset y & \wedge & x \sqsubset z' \sqsubset y & \Longrightarrow & x \rightleftharpoons y & \textit{(I6)} \\
x \neq y & \wedge & \ell(x) = \ell(y) & \Longrightarrow & x \prec^{sym} y & \textit{(I7)}
\end{array}$$

By *(I1)*, *(I3)*, and *(I5)*, every invariant structure is separable. Also, the labelling axiom *(I7)* guarantees that invariant structures are label-ordered. Hence invariant structures are order structures. Furthermore, invariant structures are the only order structures which cannot be extended without reducing their set of saturated extensions (see [7]).

Proposition 4.1. $\mathsf{SR} \subset \mathsf{IR} \subset \mathsf{OR}$.

Proof Follows from the general results proven in [7] together with

$$\begin{array}{lcll}
or & = & \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle\}, \\ \{\langle x,y\rangle, \langle y,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle & \in \ \mathsf{OR} \setminus \mathsf{IR} \\
ir & = & \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle y,z\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle & \in \ \mathsf{IR} \setminus \mathsf{SR}\,. \quad \square
\end{array}$$

Invariant structures are exactly those order structures or for which $or = \bigcap \mathsf{or2SR}(or)$ (since we always have $\mathsf{or2SR}(or) \neq \varnothing$, the intersection is well-defined), where the intersection of relational structures with the same domain and labelling is defined component-wise. In other words, invariant structures are exactly those order structures which can be represented by their saturated extensions. This fundamental property is a counterpart of Szpilrajn's Theorem [5] which implies that partial order relations are exactly those acyclic relations which can be represented by their total order extensions.

4.4. Order structure closure

> Referring to the set-up of Mazurkiewicz traces, order structure closure corresponds to transitive closure of an acyclic relation.

The *order structure closure* $\mathsf{OR} \xrightarrow{\mathsf{or2ir}} \mathsf{IR}$ is a mapping, for every structure $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}$, defined by:

$$\mathsf{or2ir}(or) = \langle \Delta, \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} \cup cross^{sym}, \sqsubset^{\curlywedge}, \ell \rangle$$

where $cross = \{\langle x, y \rangle \mid \exists z, w : z \rightleftharpoons w \wedge x \sqsubset^{*} z \sqsubset^{*} y \wedge x \sqsubset^{*} w \sqsubset^{*} y\}$. Order structure closure involves two components: the closure of mutex relation $\rightleftharpoons$ and the closure of the weak causality relation $\sqsubset$. The latter is simply the irreflexive transitive closure. The former is more involved and comprises two operations (see Figure 2). In order to calculate all new mutex pairs, one adds all the missing arcs between any two mutually exclusive equivalence classes of $\sqsubset^{\circledast}$, and connects any two events which are at the corners of a weak causality diamond with a mutex inside.

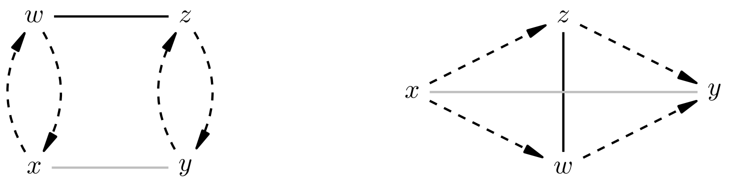


Figure 2: Closure rules for new mutex pairs $\langle x, y \rangle$ (denoted by light-gray edges) with $\langle x, y \rangle \in$ *cross* illustrated on the right. Solid edges denote the $\rightleftharpoons$ relation and dashed arcs the $\sqsubset^{*}$ relation.

Order structure closure is the unique mapping $\mathsf{OR} \xrightarrow{\mathsf{f}} \mathsf{IR}$ such that $\mathsf{f}(ir) = ir$, for every $ir \in \mathsf{IR}$, and $\mathsf{or2SR}(or) = \mathsf{or2SR} \circ \mathsf{f}(or)$, for every $or \in \mathsf{OR}$ (see [7]). This corresponds to the fact that transitive closure is the unique mapping from acyclic relations to partial orders which preserves the total order extensions.

In terms of graph representation of an invariant structure, $\langle \Delta, \sqsubset, \ell \rangle$ is a preorder, and $\langle \Delta, \prec, \ell \rangle$ is a partial order. Moreover, there are several mutex arcs in $\langle \Delta, \rightleftharpoons, \ell \rangle$ implied by the definition of the order structure closure illustrated in Figure 2.

4.5. Step sequences and saturated structures

Referring to the set-up of Mazurkiewicz traces, step sequences and saturated order structures are related in a similar way as action sequences and labelled total orders.

Let $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$ be a step alphabet. The set SR_θ of saturated order structures corresponding to the step sequences over θ comprises all saturated structures $sr = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ such that Δ is an event domain, ℓ is the default labelling of events, and, for all distinct $\langle a,i \rangle, \langle a,j \rangle, \langle b,k \rangle \in \Delta$:

$$\langle a,i \rangle \prec \langle a,j \rangle \iff i < j \quad \text{and} \quad \langle a,i \rangle \sqsubset^{\circledast} \langle b,k \rangle \implies \langle a,b \rangle \in \mathsf{sim}\ . \tag{1}$$

There are two mappings that allow switching between SR_θ and SSEQ_θ, the step sequences over θ. The first mapping, $\mathsf{SR}_\theta \xrightarrow{\mathsf{sr2sseq}} \mathsf{SSEQ}_\theta$, is defined, for every $sr = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{SR}_\theta$, by $\mathsf{sr2sseq}(sr) = \ell(\Delta_1) \ldots \ell(\Delta_k)$, where $\Delta_1 \ldots \Delta_k$ is the unique sequence such that $\Delta = \Delta_1 \uplus \cdots \uplus \Delta_k$, $\rightleftharpoons = \bigcup_{i \neq j} \Delta_i \times \Delta_j$, and $\sqsubset = \bigcup_{i \leq j} \Delta_i \times \Delta_j \setminus id_\Delta$. The second mapping, $\mathsf{SSEQ}_\theta \xrightarrow{\mathsf{sseq2sr}} \mathsf{SR}_\theta$, is defined, for every $u \in \mathsf{SSEQ}_\theta$, by $\mathsf{sseq2sr}(u) = \langle occ(u), \rightleftharpoons, \sqsubset, \ell \rangle$, where, for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$ we have:

$$k \neq m \implies \alpha \rightleftharpoons \beta \quad \text{and} \quad k \leq m \wedge \alpha \neq \beta \implies \alpha \sqsubset \beta\ .$$

As demonstrated in [7], $\mathsf{SR}_\theta \xrightarrow{\mathsf{sr2sseq}} \mathsf{SSEQ}_\theta \xrightarrow{\mathsf{sseq2sr}} \mathsf{SR}_\theta$ are inverse bijections.

4.6. Dependence structures

Referring to the set-up of Mazurkiewicz traces, dependence structures of step sequences correspond to dependence graphs of action sequences.

Given a step alphabet $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, the dependencies between the events underlying a step sequence $u \in \mathsf{SSEQ}_\theta$ are given by the mapping $\mathsf{SSEQ}_\theta \xrightarrow{\mathsf{sseq2or}_\theta} \mathsf{OR}$ defined, for every $u \in \mathsf{SSEQ}_\theta$, by $\mathsf{sseq2or}_\theta(u) = \langle occ(u), \rightleftharpoons, \sqsubset, \ell \rangle$, where for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$:

$$
\begin{array}{llll}
\alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \cap \mathsf{seq} & \wedge \;\; k < m \\
 & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \cap \mathsf{seq}^{-1} & \wedge \;\; k > m \\
\alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \cap \mathsf{seq}^{-1} & \wedge \;\; k < m \\
 & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sim} \setminus \mathsf{seq}^{-1} & \wedge \;\; k = m \; .
\end{array}
\tag{2}
$$

We refer to $\mathsf{sseq2or}_\theta(u)$ as the *dependence structure* of u (induced by θ). Crucially, if $u \equiv w$, then $\mathsf{sseq2or}_\theta(u) = \mathsf{sseq2or}_\theta(w)$, and so dependence structures can be lifted to the level of step traces through $\mathsf{sseq2or}_\theta([\![u]\!]) = \mathsf{sseq2or}_\theta(u)$ (see [7]). Hence there are two kinds of order structures capturing causal dependencies in the step sequences of SSEQ_θ and the traces in STR_θ, namely dependence structures and their closures, i.e., $\mathsf{OR}_\theta = \mathsf{sseq2or}_\theta(\mathsf{SSEQ}_\theta)$ and $\mathsf{IR}_\theta = \mathsf{or2ir}(\mathsf{OR}_\theta)$.

In what follows, for every set Θ' of step alphabets, $\mathsf{OR}_{\Theta'} = \bigcup_{\theta \in \Theta'} \mathsf{OR}_\theta$ and $\mathsf{IR}_{\Theta'} = \bigcup_{\theta \in \Theta'} \mathsf{IR}_\theta$.

4.7. Step traces and invariant structures

Referring to the set-up of Mazurkiewicz traces, step traces and invariant structures are related in a similar way as traces and causal partial orders.

Given a step alphabet θ, the step traces in STR_θ can be identified with the invariant structures in IR_θ, and a suitable correspondence is established by the pair of inverse bijections $\mathsf{STR}_\theta \xrightarrow{\mathsf{or2ir} \circ \mathsf{sseq2or}_\theta} \mathsf{IR}_\theta \xrightarrow{\mathsf{sr2sseq} \circ \mathsf{or2SR}} \mathsf{STR}_\theta$.

As shown in [7], one needs relational structures as complicated as the order structures in OR for the modelling of the dependencies underlying step sequences and step traces. More precisely, for any order structure *or* with an injective labelling, there is a step alphabet θ and a step sequence $u \in \mathsf{SSEQ}_\theta$ such that *or* is isomorphic to $\mathsf{sseq2or}_\theta(u)$. Thus step traces can generate all the causal *patterns* (i.e., an order structures without labels) of the dependence structures underpinning invariant structures.[3]

4.8. About the rest of this paper

Our main aim is to investigate different classes of step alphabets and the corresponding order structures. In the rest of this paper, we will discuss how the restriction to these subclasses of step alphabets leads to simplifications in the descriptions of their corresponding order structures, order structure closure operation, and invariant structures. Such simplifications can, in particular, lead to a more concise and efficient treatment of the algorithmic aspects involving step traces and their order structures.

For example, $\mathsf{sim} \subseteq \mathsf{seq}$ implies that each step can be split into sequences in every possible way, to be able to split a step into at least one sequence it is enough to require acyclicity of the relation $\mathsf{sim} \setminus \mathsf{seq}$ [25], and $\mathsf{sim} \cap \mathsf{seq} = \varnothing$ means that there are no serialisability equations at all.

In the subsequent sections, we will investigated five subclasses of step alphabets: Θ_{sim}, $\Theta_{\mathsf{sim}\setminus\mathsf{seq}}$, $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$, $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$, and $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$. For each subclass, we first describe the effect of the restriction on the equations defined and the resulting equivalence classes, i.e., step traces. Then we identify a distinguishing property of the order structures associated as dependence structures with these step traces and propose an axiomatisation for the corresponding invariant structures. We moreover simplify the order structure closure operation for each case. The main results in each section show that indeed the order structures and invariant structures associated with the subclass of step alphabets are included in the proposed classes of structures (e.g., Theorem 5.6 in Section 5), and that the

---

[3] Note that, for each order (or invariant) structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ and each injective labelling ℓ' of Δ, it is the case that $\langle \Delta, \rightleftharpoons, \sqsubset, \ell' \rangle$ is also an order (resp. invariant) structure.

proposed classes of structures cannot be smaller (e.g., Theorem 5.8 in Section 5).

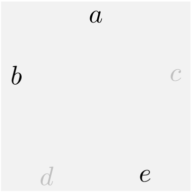
In order to streamline the presentation, we do not provide all the proofs in the paper proper. We do this only for two subclasses of step alphabets, viz. Θ_{sim} (as this class corresponds to the case of Mazurkiewicz trace alphabets), and $\Theta_{\mathsf{sim}\setminus\mathsf{seq}}$ (as this class has not yet been investigated in the literature). For the remaining three classes of step alphabets, the structure of the proofs is similar, and so they all have been moved to the appendix.

5. Relational structures for the alphabets in Θ_{sim}

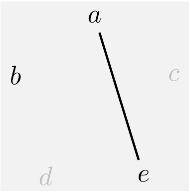
A step alphabet $\mu = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle \in \Theta_{\mathsf{sim}}$ has $\mathsf{sim} = \varnothing$ and $\mathsf{seq} = \mathsf{seq}^{-1}$, by the symmetry of $\mathsf{sim} \setminus \mathsf{seq}$. Hence the only legal steps according to μ are singletons and so the step sequences in SSEQ_μ correspond one-to-one to the sequences in Σ^*, and the saturated structures in SR_μ correspond one-to-one to the sequences in Σ^*. Indeed, since $\mathsf{sim} = \varnothing$, we have from (1) that for every $sr = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{SR}_\mu$, it is the case that $\sqsubset^{\circledast} = id_\Delta$, and so $\prec$ is a total order relation. Secondly, there are no serialisability equations. Thus, one may consider μ as a trace alphabet $\langle \Sigma, \mathsf{seq} \rangle$ with seq playing the role of the standard independence relation ind.

Example 5.1. *Recall the step alphabet θ_0 of Example 3.1. We restrict Σ to $\{a, b, e\}$. Then the resulting step alphabet $\mu_0 \in \Theta_{\mathsf{sim}}$ has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



$\mathsf{seq} =$



with

$$
\begin{array}{lclllcl}
[\![abe]\!] & = & \{abe\} & & [\![aeb]\!] & = & \{aeb, eab\} \\
[\![bae]\!] & = & \{bae, bea\} & & [\![aee]\!] & = & \{aee, eae, eea\}\ .
\end{array}
$$

◇

Recall that $\mathsf{OR}_{\Theta_{\mathsf{sim}}} = \bigcup_{\theta \in \Theta_{\mathsf{sim}}} \mathsf{OR}_\theta$ comprises the order structures that are as dependence structures associated with the step sequences and step traces over the alphabets of Θ_{sim} and reflect their causal dependencies. The corresponding family of invariant structures is $\mathsf{IR}_{\Theta_{\mathsf{sim}}} = \bigcup_{\theta \in \Theta_{\mathsf{sim}}} \mathsf{IR}_\theta$, where $\mathsf{IR}_\theta = \mathsf{or2ir}(\mathsf{OR}_\theta)$.

The definition of the dependence structure of a step sequence $u \in \mathsf{SSEQ}_\mu$ can be simplified by replacing (2), for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$, with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & k \neq m \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \ \wedge \ k < m \ . \end{array} \tag{3}$$

Hence these order structures have the property that $x \neq y \iff x \rightleftharpoons y$. Let now $\mathsf{OR}_{\mathsf{sim}}$ consist of all order structures $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}$ that satisfy this additional property; in other words $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$.

In terms of graph representation for $\mathsf{OR}_{\mathsf{sim}}$, $\langle \Delta, \sqsubset, \ell \rangle = \langle \Delta, \prec, \ell \rangle$ are acyclic graphs, and $\langle \Delta, \rightleftharpoons, \ell \rangle$ is complete.

Then we propose the following axiomatisation for their corresponding invariant structures.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR}_{\mathsf{sim}}$ if, for all $x, y, z \in \Delta$:

$$\begin{array}{rcrl} & & x \not\sqsubset x & \textit{(A1)} \\ x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(A2)} \\ x \neq y & \Longleftrightarrow & x \rightleftharpoons y & \textit{(A3)} \\ x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \sqsubset^{sym} y & \textit{(A4)} \end{array}$$

In terms of graph representation for $\mathsf{IR}_{\mathsf{sim}}$, $\langle \Delta, \sqsubset, \ell \rangle = \langle \Delta, \prec, \ell \rangle$ are also partial orders, and they capture all the relevant causal relationships.

We will now first establish that the relational structures defined by these axioms are indeed invariant structures. Moreover, all elements of $\mathsf{IR}_{\mathsf{sim}}$ are order structures belonging to $\mathsf{OR}_{\mathsf{sim}}$. Next we introduce a simplified order structure closure and, using this operation, we prove that $\mathsf{IR}_{\mathsf{sim}}$ consists exactly of the closures of the order structures in $\mathsf{OR}_{\mathsf{sim}}$.

Lemma 5.2. $\mathsf{IR}_{\mathsf{sim}} \subseteq \mathsf{IR}$.

Proof We first note that *(I1)* is simply *(A1)*. To show *(I2)* we observe that:

$$x \neq y \ \wedge \ x \sqsubset z \sqsubset y \ \Longrightarrow_{(A2)} \ x \sqsubset y \ .$$

To show *(I3)* we observe that:

$$x \rightleftharpoons y \ \Longrightarrow_{(A3)} \ x \neq y \ \Longrightarrow \ x \neq y \ \wedge \ y \neq x \ \Longrightarrow_{(A3)} \ x \neq y \ \wedge \ y \rightleftharpoons x \ .$$

To show *(I4)* we observe that:

$$\begin{aligned} x = y \ \wedge \ (x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y) \quad & \Longrightarrow \quad && x \prec z \sqsubset x \ \vee \ x \sqsubset z \prec x \\ & \Longrightarrow_{(A2)} && x \sqsubset x \\ & \Longrightarrow_{(A1)} && \mathit{false} \end{aligned}$$

and so we have:

$$x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y \ \Longrightarrow \ x \neq y \ \Longrightarrow_{(A3)} \ x \rightleftharpoons y \ .$$

To show *(I5)* we observe that:

$$z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \Longrightarrow_{(A2)} \ z \sqsubset z \ \Longrightarrow_{(A1)} \ \mathit{false} \ .$$

To show *(I6)* we observe that:

$$x = y \ \wedge \ x \sqsubset z \sqsubset y \ \Longrightarrow \ x \sqsubset z \sqsubset x \ \Longrightarrow_{(A2,A1)} \ \mathit{false}$$

and so we have:

$$z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \ \Longrightarrow \ x \neq y \ \Longrightarrow_{(A3)} \ x \rightleftharpoons y \ .$$

We finally note that *(I7)* follows from *(A3)* and *(A4)*. ☐

Lemma 5.3. $\mathsf{IR}_{\mathsf{sim}} \subseteq \mathsf{OR}_{\mathsf{sim}}$.

Proof Follows from Lemma 5.2, $\mathsf{IR} \subseteq \mathsf{OR}$, and *(A3)*. ☐

For closure we propose to consider a simplified order closure operation $\mathsf{or2ir}_{\mathsf{sim}}$ transforming order structures from $\mathsf{OR}_{\mathsf{sim}}$ into invariant structures in $\mathsf{IR}_{\mathsf{sim}}$ and corresponding to the transitive closure of an acyclic relation. This closure operation will then be shown to be the restriction of the standard closure operation for order structures. More precisely, $\mathsf{OR}_{\mathsf{sim}} \xrightarrow{\mathsf{or2ir}_{\mathsf{sim}}} \mathsf{IR}_{\mathsf{sim}}$ is such that, for every $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}}$, we have $\mathsf{or2ir}_{\mathsf{sim}}(or) = \langle \Delta, \rightleftharpoons, \sqsubset^+, \ell \rangle$.

Lemma 5.4. $\mathsf{or2ir}_{\mathsf{sim}}(\mathsf{OR}_{\mathsf{sim}}) \subseteq \mathsf{IR}_{\mathsf{sim}}$.

Proof Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}}$ and $ir = \mathsf{or2ir}_{\mathsf{sim}}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(A1)* suppose that $x \mathrel{\widehat{\sqsubset}} x$ which means $x \sqsubset^+ x$. Since $\sqsubset$ is irreflexive, there is $y \neq x$ satisfying $x \sqsubset^* y \sqsubset^* x$. Hence, by the separability of or, $x \not\rightleftharpoons y$, contradicting the definition of $\mathsf{OR}_{\mathsf{sim}}$.
To show *(A2)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \implies x \sqsubset^+ z \sqsubset^+ y \implies x \sqsubset^+ y \implies x \mathrel{\widehat{\sqsubset}} y \,.$$

We then observe that *(A3)* follows from $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$. Finally, *(A4)* follows from the label-linearity of or, as shown below:

$$x \neq y \ \wedge\ \ell(x) = \ell(y) \implies x \prec^{sym} y \implies x \mathrel{\widehat{\sqsubset}}^{sym} y \,.$$

Hence $ir \in \mathsf{IR}_{\mathsf{sim}}$. $\square$

Proposition 5.5. $\mathsf{or2ir}_{\mathsf{sim}}$ *is a surjection with* $\mathsf{or2ir}_{\mathsf{sim}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}}}$.

Proof We first show that $\mathsf{or2ir}_{\mathsf{sim}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}}}$. Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}}$ and $ir = \mathsf{or2ir}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. In this case $\sqsubset^{\circledast} = id_\Delta$ which follows directly from $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$ and the separability of or. As a result, we also have $\sqsubset^{\curlywedge} = \sqsubset^+$. Hence

$$\mathsf{or2ir}(or) = \langle \Delta, \rightleftharpoons \cup\, cross^{sym}, \sqsubset^+, \ell \rangle \,,$$

where $cross = \{\langle x, y \rangle \mid \exists z, w : z \rightleftharpoons w \wedge x \sqsubset^* z \sqsubset^* y \wedge x \sqsubset^* w \sqsubset^* y\}$. Moreover, $cross$ is irreflexive (as $\widehat{\rightleftharpoons}$ is irreflexive) and $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$. We therefore obtain $\mathsf{or2ir}(or) = \langle \Delta, \rightleftharpoons, \sqsubset^+, \ell \rangle$.

We then observe that $\mathsf{or2ir}_{\mathsf{sim}}(\mathsf{OR}_{\mathsf{sim}}) = \mathsf{IR}_{\mathsf{sim}}$ follows from Lemmas 5.2, 5.3, and 5.4, $\mathsf{or2ir}_{\mathsf{sim}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}}}$, and the fact that $\mathsf{or2ir}$ is the identity on IR, as then we obtain $\mathsf{or2ir}_{\mathsf{sim}}(\mathsf{OR}_{\mathsf{sim}}) \subseteq \mathsf{IR}_{\mathsf{sim}}$ and $\mathsf{or2ir}_{\mathsf{sim}}(\mathsf{OR}_{\mathsf{sim}}) \supseteq \mathsf{or2ir}_{\mathsf{sim}}(\mathsf{IR}_{\mathsf{sim}}) = \mathsf{or2ir}(\mathsf{IR}_{\mathsf{sim}}) = \mathsf{IR}_{\mathsf{sim}}$. □

Based on the above facts we can now present, as a main result, the full picture.

Theorem 5.6.

$$\begin{array}{ccccc}
\mathsf{OR}_{\Theta_{\mathsf{sim}}} & \subset & \mathsf{OR}_{\mathsf{sim}} & \subset & \mathsf{OR} \\
\cup & & \cup & & \cup \\
\mathsf{IR}_{\Theta_{\mathsf{sim}}} & \subset & \mathsf{IR}_{\mathsf{sim}} & \subset & \mathsf{IR}
\end{array}$$

Proof Let us consider one by one all the inclusions:

- $\mathsf{IR} \subset \mathsf{OR}$ follows from the general results proven in [7] and

$$or = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle, \langle z,y\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle y,z\rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OR} \setminus \mathsf{IR}\,.$$

- $\mathsf{IR}_{\mathsf{sim}} \subset \mathsf{OR}_{\mathsf{sim}}$ follows from $or \in \mathsf{OR}_{\mathsf{sim}} \setminus \mathsf{IR}_{\mathsf{sim}}$ and Lemma 5.3.
- $\mathsf{IR}_{\Theta_{\mathsf{sim}}} \subset \mathsf{OR}_{\Theta_{\mathsf{sim}}}$ follows from $or \in \mathsf{OR}_{\Theta_{\mathsf{sim}}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}}}$ and the general results proven in [7].
- $\mathsf{OR}_{\mathsf{sim}} \subset \mathsf{OR}$ follows from the definition of $\mathsf{OR}_{\mathsf{sim}}$ and

$$or' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OR} \setminus \mathsf{OR}_{\mathsf{sim}}\,.$$

- $\mathsf{IR}_{\mathsf{sim}} \subset \mathsf{IR}$ follows from $or' \in \mathsf{IR} \setminus \mathsf{IR}_{\mathsf{sim}}$ and Lemma 5.2.
- $\mathsf{OR}_{\Theta_{\mathsf{sim}}} \subset \mathsf{OR}_{\mathsf{sim}}$ can be proven by taking $\mu \in \Theta_{\mathsf{sim}}$, $u \in \mathsf{SSEQ}_{\mu}$, and $or = \mathsf{sseq2or}_{\mu}(u)$. We know that $or \in \mathsf{OR}$. Suppose that $\alpha, \beta \in occ(u)$ and $\alpha \neq \beta$. Then, by $\mathsf{sim} = \varnothing$, $pos_u(\alpha) \neq pos_u(\beta)$. Hence, by (3), we have $\alpha \rightleftharpoons_{or} \beta$, and so $or \in \mathsf{OR}_{\mathsf{sim}}$. Moreover, we note that

$$or'' = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OR}_{\mathsf{sim}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}}}.$$

- $\mathsf{IR}_{\Theta_{\mathsf{sim}}} \subset \mathsf{IR}_{\mathsf{sim}}$ follows from $or'' \in \mathsf{IR}_{\mathsf{sim}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}}}$, $\mathsf{OR}_{\Theta_{\mathsf{sim}}} \subseteq \mathsf{OR}_{\mathsf{sim}}$ and Lemma 5.4.

Moreover, note that $or \in \mathsf{OR}_{\mathsf{sim}} \setminus \mathsf{IR}$ and $or' \in \mathsf{IR} \setminus \mathsf{OR}_{\mathsf{sim}}$ which justifies that IR and $\mathsf{OR}_{\mathsf{sim}}$ are not related. Similarly, there is no inclusion between $\mathsf{IR}_{\mathsf{sim}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}}}$ since $or \in \mathsf{OR}_{\Theta_{\mathsf{sim}}} \setminus \mathsf{IR}_{\mathsf{sim}}$ and $or'' \in \mathsf{IR}_{\mathsf{sim}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}}}$. $\square$

As a consequence we prove our initial intuition correct by demonstrating that also the invariant structures in $\mathsf{IR}_{\mathsf{sim}}$ are characterised by the additional property that mutex coincides with non-equality.

Proposition 5.7. *For every relational structure* $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,

$$ir \in \mathsf{IR}_{\mathsf{sim}} \quad \Longleftrightarrow \quad (ir \in \mathsf{IR} \ \wedge \ \forall x, y \in \Delta : x \neq y \Longleftrightarrow x \rightleftharpoons y) \ .$$

Proof ($\Longrightarrow$) Follows from Theorem 5.6 and *(A3)*.

($\Longleftarrow$) Note that *(A3)* is the additional property; *(I1)* and *(A1)* are the same axioms; and *(A4)* follows from *(I7)*. To prove *(A2)*, assume that $x \sqsubset z \sqsubset y$. Then $x \neq z$ by *(I1)*, and so $x \rightleftharpoons z$. Hence $x \rightleftharpoons y$, by *(I4)*, and thus $x \neq y$. Consequently, $x \sqsubset y$ by *(I2)*, and *(A2)* follows. $\square$

Altogether we have identified $\mathsf{OR}_{\mathsf{sim}}$ and $\mathsf{IR}_{\mathsf{sim}}$ through a structural (not related to labels) property as the right classes of order structures and invariant structures for the step traces over step alphabets in Θ_{sim}. The next result shows that we cannot optimise this any further. When the labelling is ignored, for every relational structure $or \in \mathsf{OR}_{\mathsf{sim}}$ there is a step trace defined by a step alphabet in Θ_{sim} with the order structure underlying or as its causal pattern.

Theorem 5.8. *If $or \in \mathsf{OR}_{\mathsf{sim}}$ has an injective labelling, then there are $\mu \in \Theta_{\mathsf{sim}}$ and $u \in \mathsf{SSEQ}_{\mu}$ such that or is isomorphic to $\mathsf{sseq2or}_{\mu}(u)$.*

Proof Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [7] it follows that there exists $sr \in \mathsf{or2SR}(os)$ which, directly by the definition of $\mathsf{OR}_{\mathsf{sim}}$, satisfies

$\rightleftharpoons_{sr} = (\Delta \times \Delta) \setminus id_{\Delta}$. Hence $u = \mathsf{sr2sseq}(sr)$ is a sequence of singleton steps. Let $\mu = \langle \Sigma, \varnothing, \mathsf{seq} \rangle$, where:

$$\mathsf{seq} = \left\{ \langle a, b \rangle \in \Sigma \times \Sigma \;\middle|\; \begin{array}{lll} pos_u(\langle a,1 \rangle) < pos_u(\langle b,1 \rangle) & \wedge & \langle a,1 \rangle \not\sqsubset \langle b,1 \rangle \;\; \vee \\ pos_u(\langle b,1 \rangle) < pos_u(\langle a,1 \rangle) & \wedge & \langle b,1 \rangle \not\sqsubset \langle a,1 \rangle \end{array} \right\} .$$

Clearly, $\mu \in \Theta_{\mathsf{sim}}$ and $u \in \mathsf{SSEQ}_{\mu}$. It is easy to check that $or = \mathsf{sseq2or}_{\mu}(u)$. $\square$

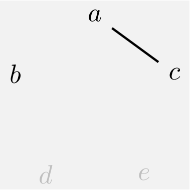
Corollary 5.9. *If* $ir \in \mathsf{IR}_{\mathsf{sim}}$ *has an injective labelling, then there are* $\mu \in \Theta_{\mathsf{sim}}$ *and* $u \in \mathsf{SSEQ}_{\mu}$ *such that* ir *is isomorphic to* $\mathsf{or2ir}_{\mathsf{sim}} \circ \mathsf{sseq2or}_{\mu}(u)$.

6. Relational structures for the alphabets in $\Theta_{\mathsf{sim} \setminus \mathsf{seq}}$

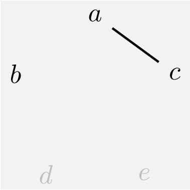
A step alphabet $\kappa = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle \in \Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ has $\mathsf{sim} \setminus \mathsf{seq} = \varnothing$ which is equivalent to $\mathsf{sim} \subseteq \mathsf{seq} \cap \mathsf{seq}^{-1}$, by the symmetry of sim. As a consequence, if $(a, b) \in \mathsf{seq} \setminus (\mathsf{seq}^{-1} \cap \mathsf{sim})$, then $(b, a) \in (\mathsf{seq} \setminus (\mathsf{seq}^{-1} \cap \mathsf{sim}))^{-1} \subseteq \mathsf{sim} \setminus \mathsf{seq} = \varnothing$. Hence $\mathsf{seq} \setminus (\mathsf{seq}^{-1} \cap \mathsf{sim}) = \varnothing$ and $\mathsf{seq} = \mathsf{seq}^{-1}$ is symmetric. And so, all steps over κ can be serialised in any order and combination of substeps.

Example 6.1. *Recall again the step alphabet* θ_0 *of Example 3.1. We restrict* Σ *to* $\{a, b, c\}$*. The resulting step alphabet* $\kappa_0 \in \Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ *has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



$\mathsf{seq} =$



with $[\![abc]\!] = \{abc\}$ *and* $[\![(ac)b]\!] = \{(ac)b, acb, cab\}$. $\diamond$

The definition of the dependence structure of a step sequence $u \in \mathsf{SSEQ}_{\kappa}$ can be simplified by replacing (2), for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$, with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \;\; \wedge \;\; k < m . \end{array} \tag{4}$$

Hence these order structures have the property that $x \sqsubset^{sym} y \Longrightarrow x \rightleftharpoons y$. Let $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ consist of all $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}$ that have this property.

In terms of graph representation for $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$, $\langle \Delta, \sqsubset, \ell \rangle = \langle \Delta, \prec, \ell \rangle$ are acyclic graphs, while the relationships captured by $\langle \Delta, \rightleftharpoons, \ell \rangle$ are more complicated than in the previous case.

For the corresponding invariant structures we thus propose the following axiomatisation.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ if, for all $x, y, z \in \Delta$:

$$\begin{array}{rclr}
x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(B1)} \\
x \sqsubset^{sym} y & \Longrightarrow & x \rightleftharpoons y & \textit{(B2)} \\
x \rightleftharpoons y & \Longrightarrow & y \rightleftharpoons x \neq y & \textit{(B3)} \\
x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \sqsubset^{sym} y & \textit{(B4)}
\end{array}$$

In terms of graph representation for $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$, $\langle \Delta, \sqsubset, \ell \rangle = \langle \Delta, \prec, \ell \rangle$ are partial orders, and this time they do not capture all the relevant causal relationships between events, while the implied mutex relationships captured by $\langle \Delta, \rightleftharpoons, \ell \rangle$ are less involved than in the general case (as the closure operation is much simpler).

In what follows, we first establish that these relational structures are invariant structures and moreover order structures belonging to $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$. Then, we introduce a simplified closure operation and prove, using this operation, that $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ consists exactly of the closures of the order structures in $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$.

Lemma 6.2. $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}} \subseteq \mathsf{IR}$.

Proof To show *(I1)* we observe that:

$$x \sqsubset x \ \Longrightarrow_{\textit{(B2)}} \ x \rightleftharpoons x \ \Longrightarrow_{\textit{(B3)}} \ x \neq x \ \Longrightarrow \ \textit{false}\,.$$

To show *(I2)* we observe that:

$$x \neq y \ \wedge \ x \sqsubset z \sqsubset y \ \Longrightarrow_{\mathit{(B1)}} \ x \sqsubset y \ .$$

We then note that *(I3)* is simply *(B3)*, and to show *(I4)* we observe that:

$$x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y \ \Longrightarrow_{\mathit{(B1)}} \ x \sqsubset y \ \Longrightarrow_{\mathit{(B2)}} \ x \rightleftharpoons y \ .$$

To show *(I5)* we observe that:

$$z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \Longrightarrow_{\mathit{(B1)}} \ z \sqsubset z \ \Longrightarrow_{\mathit{(B2,B3)}} \ \mathit{false} \ .$$

To show *(I6)* we observe that:

$$z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \ \Longrightarrow_{\mathit{(B1)}} \ x \sqsubset y \ \Longrightarrow_{\mathit{(B2)}} \ x \rightleftharpoons y \ .$$

We finally note that *(I7)* follows from *(B2)* and *(B4)*. ☐

Lemma 6.3. $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}} \subseteq \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$.

Proof Follows from Lemma 6.2, $\mathsf{IR} \subseteq \mathsf{OR}$, and *(B2)*. ☐

The simplified closure operation $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}} \xrightarrow{\mathsf{or2ir}_{\mathsf{sim}\backslash\mathsf{seq}}} \mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ is defined, for every $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$, by:

$$\mathsf{or2ir}_{\mathsf{sim}\backslash\mathsf{seq}}(or) = \langle \Delta, \rightleftharpoons \cup (\sqsubset^{+})^{sym}, \sqsubset^{+}, \ell \rangle \ .$$

Lemma 6.4. $\mathsf{or2ir}_{\mathsf{sim}\backslash\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}) \subseteq \mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$.

Proof Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $ir = \mathsf{or2ir}_{\mathsf{sim}\backslash\mathsf{seq}}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. $ir \in \mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$. To show *(B1)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \ \Longrightarrow \ x \sqsubset^{+} z \sqsubset^{+} y \ \Longrightarrow \ x \sqsubset^{+} y \ \Longrightarrow \ x \mathrel{\widehat{\sqsubset}} y \ .$$

To show *(B2)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} y \ \Longrightarrow \ x \sqsubset^{+} y \ \Longrightarrow \ x \mathrel{\widehat{\rightleftharpoons}} y \ .$$

To show *(B3)* we observe that:

$$x \mathrel{\widehat{\rightleftharpoons}} y \ \Longrightarrow \ x \rightleftharpoons y \ \vee \ x (\sqsubset^{+})^{sym} y \ \Longrightarrow \ y \rightleftharpoons x \ \vee \ y (\sqsubset^{+})^{sym} x \ \Longrightarrow \ y \mathrel{\widehat{\rightleftharpoons}} x \ .$$

Moreover, $x \widehat{\rightleftharpoons} y \implies x \neq y$ follows from the general results proved in [7]. Finally, *(B4)* follows from the label-linearity of or, as shown below:

$$x \neq y \;\wedge\; \ell(x) = \ell(y) \implies x \mathrel{\widehat{\prec}}^{sym} y \implies x \mathrel{\widehat{\sqsubset}}^{sym} y \ .$$

Hence $ir \in \mathsf{IR}_{\mathsf{sim} \backslash \mathsf{seq}}$. $\square$

Proposition 6.5. $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}}$ *is a surjection with* $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}}$.

Proof We first show that $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}}$. Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}$ and $ir = \mathsf{or2ir}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. We first observe that in such a case $\sqsubset^{\circledast} = id_\Delta$ which follows from $x \sqsubset^{sym} y \implies x \rightleftharpoons y$ and the separability of or. As a result, we also have $\sqsubset^{\curlywedge} = \sqsubset^{+}$. Hence

$$\mathsf{or2ir}(or) = \langle \Delta, \rightleftharpoons \;\cup\; cross^{sym}, \sqsubset^{+}, \ell \rangle \ ,$$

where $cross = \{\langle x, y \rangle \mid \exists z, w : z \rightleftharpoons w \wedge x \sqsubset^{*} z \sqsubset^{*} y \wedge x \sqsubset^{*} w \sqsubset^{*} y\}$. We will now show that $(\rightleftharpoons \;\cup\; cross^{sym}) = (\rightleftharpoons \;\cup\; (\sqsubset^{+})^{sym})$.

Suppose first that $\langle x, y \rangle \in cross$ which means that $x \neq y$ (which follows from the general theory), and there is z such that $x \sqsubset^{*} z \sqsubset^{*} y$. Hence $x \sqsubset^{+} y$ showing that the $(\subseteq)$ inclusion holds. To show the reverse inclusion, suppose that $x \sqsubset^{+} y$. If $x \sqsubset y$ then, by the definition of $\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}$, we have $x \rightleftharpoons y$. Otherwise, there is z such that $x \sqsubset z \sqsubset^{*} y$. Then, again by the definition of $\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}$, $z \rightleftharpoons x$. We therefore obtain that $\langle x, y \rangle \in cross$, after taking $w = x$. Hence $\mathsf{or2ir}(or) = \langle \Delta, \rightleftharpoons \;\cup\; (\sqsubset^{+})^{sym}, \sqsubset^{+}, \ell \rangle$. We then observe that $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}}(\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}) = \mathsf{IR}_{\mathsf{sim} \backslash \mathsf{seq}}$ follows from Lemmas 6.2, 6.3, and 6.4, $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}}$, and the fact that $\mathsf{or2ir}$ is the identity on IR, as then we obtain $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}}(\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}) \subseteq \mathsf{IR}_{\mathsf{sim} \backslash \mathsf{seq}}$ and $\mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}}(\mathsf{OR}_{\mathsf{sim} \backslash \mathsf{seq}}) \supseteq \mathsf{or2ir}_{\mathsf{sim} \backslash \mathsf{seq}}(\mathsf{IR}_{\mathsf{sim} \backslash \mathsf{seq}}) = \mathsf{or2ir}(\mathsf{IR}_{\mathsf{sim} \backslash \mathsf{seq}}) = \mathsf{IR}_{\mathsf{sim} \backslash \mathsf{seq}}$.. $\square$

Now, we can present as a main result the full picture relating $\mathsf{OR}_{\Theta_{\mathsf{sim} \backslash \mathsf{seq}}} = \bigcup_{\theta \in \Theta_{\mathsf{sim} \backslash \mathsf{seq}}} \mathsf{OR}_\theta$, the order structures that are as dependence structures associated with the step sequences and step traces over the alphabets of $\Theta_{\mathsf{sim} \backslash \mathsf{seq}}$, and the corresponding family of invariant structures $\mathsf{IR}_{\Theta_{\mathsf{sim} \backslash \mathsf{seq}}} = \bigcup_{\theta \in \Theta_{\mathsf{sim} \backslash \mathsf{seq}}} \mathsf{IR}_\theta$, where $\mathsf{IR}_\theta = \mathsf{or2ir}(\mathsf{OR}_\theta)$, with the newly introduced order structures and invariant structures.

Theorem 6.6.

$$\begin{array}{ccccc} \mathsf{OR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} & \subset & \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}} & \subset & \mathsf{OR} \\ \cup & & \cup & & \cup \\ \mathsf{IR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} & \subset & \mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}} & \subset & \mathsf{IR} \end{array}$$

Proof Let us consider one by one all the inclusions:

- $\mathsf{IR} \subset \mathsf{OR}$ was already justified in the proof of Theorem 5.6. Note, however, that we also have

$$or = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \{\langle x,y\rangle, \langle y,z\rangle\}, \\ \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OR} \setminus \mathsf{IR}\,.$$

- $\mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}} \subset \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}}$ follows from $or \in \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}} \setminus \mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}}$ and Lemma 6.3.

- $\mathsf{IR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} \subset \mathsf{OR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}}$ follows from $os \in \mathsf{OR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}}$ and the general results proved in [7].

- $\mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}} \subset \mathsf{OR}$ follows from the definition of $\mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}}$ and

$$or' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OR} \setminus \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}}\,.$$

- $\mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}} \subset \mathsf{IR}$ follows from $or' \in \mathsf{IR} \setminus \mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}}$ and Lemma 6.2.

- $\mathsf{OR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} \subset \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}}$ can be shown by taking $\kappa \in \Theta_{\mathsf{sim}\setminus\mathsf{seq}}$, $u \in \mathsf{SSEQ}_\kappa$, and $or = \mathsf{sseq2or}_\kappa(u)$. Since we know from the general theory that $or \in \mathsf{OR}$, we only need to show that $\sqsubset_{or}^{sym} \subseteq \rightleftharpoons_{or}$. This, however, follows from (4). Hence $or \in \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}}$. Moreover, we note that

$$or'' = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}}\,.$$

- $\mathsf{IR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} \subseteq \mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}}$ follows from Lemma 6.4 $or'' \in \mathsf{IR}_{\mathsf{sim}\setminus\mathsf{seq}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} \subseteq \mathsf{OR}_{\mathsf{sim}\setminus\mathsf{seq}}$.

Moreover, note that $or \in \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{IR}$ and $or' \in \mathsf{IR} \setminus \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ which justifies that IR and $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ are not related. Similarly, $or \in \mathsf{OR}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \setminus \mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $or'' \in \mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$, hence there is no inclusion between $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$. ☐

As a consequence of the last result, we can now prove our intuition that led to the definition of $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ correct, by demonstrating that also the invariant structures in $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ are characterised by the additional property that weak ordering implies mutual exclusion.

Proposition 6.7. *For every relational structure* $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,

$$ir \in \mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}} \quad \Longleftrightarrow \quad (ir \in \mathsf{IR} \ \wedge \ \forall x, y \in \Delta : x \sqsubset^{sym} y \implies x \rightleftharpoons y) \, .$$

Proof ($\Longrightarrow$) Follows from Theorem 6.6 and *(B2)*.

($\Longleftarrow$) Note that *(B2)* is the additional property; *(I3)* and *(B3)* are the same axioms; and *(B4)* follows from *(I7)*. To prove *(B1)*, assume that $x \sqsubset z \sqsubset y$. Then $x \rightleftharpoons z$ by the additional property. Hence $x \rightleftharpoons y$ by *(I4)*. Thus $x \neq y$ by *(I3)*, and *(B2)* follows. ☐

Summarising, we have identified $\mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $\mathsf{IR}_{\mathsf{sim}\backslash\mathsf{seq}}$ through a structural property as suitable subclasses of OR and IR for the relational structures associated with the step traces over step alphabets in $\Theta_{\mathsf{sim}\backslash\mathsf{seq}}$. As the next theorem shows, this result is optimal in the sense that for every relational structure in $or \in \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$, there is a step trace defined by a step alphabet in $\Theta_{\mathsf{sim}\backslash\mathsf{seq}}$ with the unlabelled order structure underlying or as its causal pattern.

Theorem 6.8. *If a structure* $or \in \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}}$ *has an injective labelling, then there are* $\kappa \in \Theta_{\mathsf{sim}\backslash\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\kappa$ *such that* or *is isomorphic to* $\mathsf{sseq2or}_\kappa(u)$.

Proof Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [7] it follows that there exists $sr \in \mathsf{or2SR}(os)$. Let $u = \mathsf{sseq2sr}^{-1}(sr)$, and $\kappa = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where:

$$\begin{array}{rcll}
\mathsf{sim} & = & \{\langle a,b\rangle \in \Sigma \times \Sigma & \mid \ (pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle) \ \wedge \ a \neq b) \vee \\
 & & & \ (pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle) \ \wedge \ \langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle)\} \\
\mathsf{seq} & = & \{\langle a,b\rangle \in \Sigma \times \Sigma & \mid \ (pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle) \ \wedge \ a \neq b) \\
 & & & \ \vee (pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle) \ \wedge \ \langle a,1\rangle \not\sqsubset \langle b,1\rangle) \\
 & & & \ \vee (pos_u(\langle b,1\rangle) < pos_u(\langle a,1\rangle) \ \wedge \ \langle b,1\rangle \not\sqsubset \langle a,1\rangle)\} \ .
\end{array}$$

We then observe that sim is symmetric since $\rightleftharpoons$ is symmetric, and $\mathsf{seq} \setminus \mathsf{sim}$ is symmetric because sim and seq are symmetric. Hence κ is a step alphabet. To show $\kappa \in \Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ we need to show that $\mathsf{sim} \subseteq \mathsf{seq}$.

Let $\langle a,b\rangle \in \mathsf{sim}$. If $pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)$ and $a \neq b$ then clearly we have $\langle a,b\rangle \in \mathsf{seq}$. Moreover, if $pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle)$ and $\langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle$ then, by $or \in \mathsf{OR}_{\mathsf{sim} \setminus \mathsf{seq}}$, $pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle)$ and $\langle a,1\rangle \not\sqsubset^{sym} \langle b,1\rangle$. Hence $\langle a,b\rangle \in \mathsf{seq}$, and so $\kappa \in \Theta_{\mathsf{sim} \setminus \mathsf{seq}}$.

We then observe that $u \in \mathsf{SSEQ}_\kappa$ as $pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)$ and $a \neq b$ together imply $\langle a,b\rangle \in \mathsf{sim}$, and it is easy to check that $or = \mathsf{sseq2or}_\kappa(u)$. $\square$

Corollary 6.9. *If $ir \in \mathsf{IR}_{\mathsf{sim} \setminus \mathsf{seq}}$ has an injective labelling, then there are $\mu \in \Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ and $u \in \mathsf{SSEQ}_\mu$ such that ir is isomorphic to $\mathsf{or2ir}_{\mathsf{sim} \setminus \mathsf{seq}} \circ \mathsf{sseq2or}_\mu(u)$.*

We conclude this section showing that the step traces defined by step alphabets in $\Theta_{\mathsf{sim} \setminus \mathsf{seq}}$ are histories satisfying the concurrency paradigm π_2 of [10].

Proposition 6.10. *Let τ be a step trace over a step alphabet $\kappa \in \Theta_{\mathsf{sim} \setminus \mathsf{seq}}$. Let $\alpha, \beta \in occ(\tau)$ be distinct action occurrences of τ. Then*

$$\begin{gathered}
(\exists v \in \tau : \ pos_v(\alpha) = pos_v(\beta)) \\
\Longrightarrow \\
(\exists u \in \tau : \ pos_u(\alpha) < pos_u(\beta)) \ \wedge \ (\exists w \in \tau : \ pos_w(\alpha) > pos_w(\beta)).
\end{gathered}$$

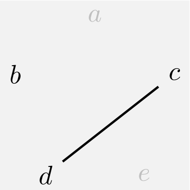
Proof Let $ir = \mathsf{or2ir} \circ \mathsf{sseq2or}_\kappa(v)$. From $pos_v(\alpha) = pos_v(\beta)$ it follows directly that $\langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{sim}$ and there is $sr \in \mathsf{or2SR}(ir)$ such that $\alpha \sqsubset_{sr} \beta \sqsubset_{sr} \alpha$. Hence, $\alpha \not\rightleftharpoons_{ir} \beta$. Moreover, by the simplified form of the $\mathsf{sseq2or}_\kappa$ mapping and the order structure closure, $\alpha \not\sqsubset_{ir} \beta$ and $\beta \not\sqsubset_{ir} \alpha$. This, by the general results proved in [7], means that there are $sr', sr'' \in \mathsf{or2SR}(ir)$ such that $\alpha \prec_{sr'} \beta$ and $\beta \prec_{sr''} \alpha$. Then the conclusion holds by taking $u = \mathsf{sseq2or}_{\kappa}^{-1}(sr')$ and $w = \mathsf{sseq2or}_{\kappa}^{-1}(sr'')$. $\square$

7. Relational structures for the alphabets in $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$

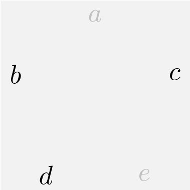
A step alphabet $\nu \in \Theta_{\mathsf{sim}\cap\mathsf{seq}}$ is the one satisfying $\mathsf{sim} \cap \mathsf{seq} = \varnothing$, and so we have $\mathsf{seq} = \mathsf{seq}^{-1}$. For the alphabets in $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$ steps can be only manipulated through the interleaving equations.

Example 7.1. *Let us recall the step alphabet* θ_0 *of Example 3.1 and restrict* Σ *to* $\{b, c, d\}$*. The resulting step alphabet* $\nu_0 \in \Theta_{\mathsf{sim}\cap\mathsf{seq}}$ *has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



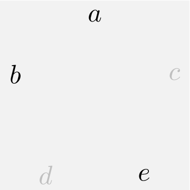
$\mathsf{seq} =$



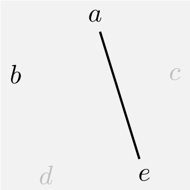
with $[\![b(cd)]\!] = \{b(cd)\}$ *and* $[\![bcd]\!] = \{bcd\}$.

One can also obtain another example of an alphabet from $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$ *by taking* θ_0 *and restricting* Σ *to* $\{a, b, e\}$*. The resulting step alphabet* ν_1 *has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



$\mathsf{seq} =$



with $[\![aeb]\!] = \{aeb, eab\}$ *and* $[\![abe]\!] = \{abe\}$. $\diamond$

The definition of the dependence structure of a step sequence $u \in \mathsf{SSEQ}_{\nu}$ can be simplified by replacing (2), for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and

$pos_u(\beta) = m$, with:

$$\begin{aligned} &\alpha \rightleftharpoons \beta \;\;\textit{if}\;\; k \neq m \\ &\alpha \sqsubset \beta \;\;\textit{if}\;\; \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \;\wedge\; k \leq m \;\wedge\; \alpha \neq \beta\,. \end{aligned} \tag{5}$$

The order structures $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ are all those $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}$ for which $x \neq y \Longrightarrow x \rightleftharpoons y \;\vee\; x \sqsubset y \sqsubset x$, and the axiomatisation of the corresponding invariant structures becomes simpler.

In terms of graph representation for $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$, any two events are either connected in $\langle \Delta, \rightleftharpoons, \ell \rangle$, or connected in both directions in $\langle \Delta, \sqsubset, \ell \rangle$.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ if, for all $x, y, z \in \Delta$:

$$\begin{array}{rcll} & & x \not\rightleftharpoons x & \textit{(C1)} \\ x \neq y \wedge x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(C2)} \\ x \not\rightleftharpoons y \wedge x \neq y & \Longleftrightarrow & x \sqsubset y \sqsubset x & \textit{(C3)} \\ x \neq y \;\wedge\; \ell(x) = \ell(y) & \Longrightarrow & x \prec^{sym} y & \textit{(C4)} \end{array}$$

In terms of graph representation for $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$, the part of the order structure closure responsible for mutex relation is trivial.

The definitions of $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ and $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ are sound.

The simplified order structure closure $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} \xrightarrow{\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}} \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ is such that $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(or) = \langle \Delta, \rightleftharpoons, \sqsubset^{\curlywedge}, \ell \rangle$, for every $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$.

Proposition 7.2. $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}$ *is a surjection with* $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}}$.

Theorem 7.3.

$$\begin{array}{ccccc} \mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} & \subset & \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} & \subset & \mathsf{OR} \\ \cup & & \cup & & \cup \\ \mathsf{IR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} & \subset & \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} & \subset & \mathsf{IR} \end{array}$$

The next result demonstrates the correctness of the reduction from the axioms *(I1)–(I7)* to *(C1)–(C4)* when an additional, equivalent to $\mathsf{sim} \cap \mathsf{seq} = \varnothing$ in the case of invariant structures over a given step alphabet, property is assumed.

Proposition 7.4. *For every relational structure* $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$*,*

$$ir \in \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \quad \iff \quad (ir \in \mathsf{IR} \ \wedge\ \forall x, y \in \Delta : x \neq y \implies x \rightleftharpoons y \vee x \sqsubset y \sqsubset x)\ .$$

The step alphabets in $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$ can generate all the causal patterns involving causal relationships captured by the structures in $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$.

Theorem 7.5. *If a structure* $or \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ *has an injective labelling, then there are* $\nu \in \Theta_{\mathsf{sim}\cap\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\nu$ *such that or is isomorphic to* $\mathsf{sseq2or}_\nu(u)$*.*

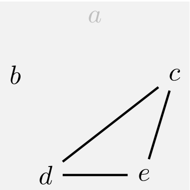
Corollary 7.6. *If* $ir \in \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ *has an injective labelling, then there are* $\mu \in \Theta_{\mathsf{sim}\cap\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\mu$ *such that ir is isomorphic to* $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}} \circ \mathsf{sseq2or}_\mu(u)$*.*

8. Relational structures for the alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$

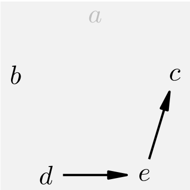
A step alphabet $\sigma = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ is the one satisfying $\mathsf{seq} \setminus \mathsf{sim} = \varnothing$ and therefore we have $\mathsf{seq} \cup \mathsf{seq}^{-1} \subseteq \mathsf{sim}$. Alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ do not allow true interleaving, and swapping of steps can be achieved by splitting and joining steps. In [10], such alphabets are referred to as *comtrace alphabets.*

Example 8.1. *Let us recall the step alphabet* θ_0 *of Example 3.1 and restrict* Σ *to* $\{b, c, d, e\}$*. The resulting step alphabet* $\sigma_0 \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ *has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



$\mathsf{seq} =$

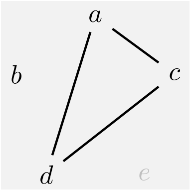


with

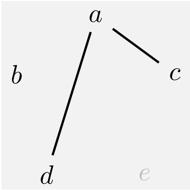
$$\begin{aligned} [\![(cde)]\!] &= \{(cde)\} & [\![(ce)]\!] &= \{(ce), ec\} \\ [\![(de)]\!] &= \{(de), de\} & [\![dec]\!] &= \{dec, (de)c, d(ce)\}\ . \end{aligned}$$

One can also obtain another example of an alphabet from $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ *by taking* θ_0 *and restricting* Σ *to* $\{a, b, c, d\}$*. The resulting step alphabet* σ_1 *has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



$\mathsf{seq} =$



with $[\![acd]\!] = \{acd, cad, cda, (ac)d, c(ad)\}$, $[\![a(cd)]\!] = \{a(cd), (cd)a, (acd)\}$, *and* $[\![abc]\!] = \{abc\}$. ◇

The definition of the dependence structure of a step sequence $u \in \mathsf{SSEQ}_\sigma$ can be simplified by replacing (2), for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$, with:

$$\begin{array}{rlll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} & \wedge\ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq}^{-1} & \wedge\ k > m \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \cap \mathsf{seq}^{-1} & \wedge\ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sim} \setminus \mathsf{seq}^{-1} & \wedge\ k = m\ .] \end{array} \tag{6}$$

The order structures $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ needed to reflect causal dependencies in the step traces over the concurrent alphabets of $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ are all those order structures $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}$ for which $x \rightleftharpoons y \Longrightarrow x \sqsubset^{sym} y$. The corresponding invariant structures can then be provided with a simpler definition.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ if

$$\begin{array}{rcrcll} & & & & x \not\sqsubset x & \textit{(D1)} \\ x \neq y & \wedge & x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(D2)} \\ & & x \rightleftharpoons y & \Longrightarrow & x \sqsubset^{sym} y \wedge y \rightleftharpoons x & \textit{(D3)} \\ x \prec z \sqsubset y & \vee & x \sqsubset z \prec y & \Longrightarrow & x \rightleftharpoons y & \textit{(D4)} \\ x \neq y & \wedge & \ell(x) = \ell(y) & \Longrightarrow & x \rightleftharpoons y & \textit{(D5)} \end{array}$$

In terms of graph representation for both $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$, any two events are connected in $\langle\Delta,\rightleftharpoons,\ell\rangle$ iff they are connected in $\langle\Delta,\prec,\ell\rangle$.

The definitions of $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ are sound.

The simplified order structure closure $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}} \xrightarrow{\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}} \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ is such that, for every $or = \langle\Delta,\rightleftharpoons,\sqsubset,\ell\rangle \in \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$:

$$\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}(or) = \langle\Delta,(\sqsubset^{*}\circ\prec\circ\sqsubset^{*})^{sym},\sqsubset^{\curlywedge},\ell\rangle\ .$$

Proposition 8.2. $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}$ *is a surjection with* $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}}$.

Theorem 8.3.

$$\begin{array}{ccccc}
\mathsf{OR}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} & \subset & \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}} & \subset & \mathsf{OR} \\
\cup & & \cup & & \cup \\
\mathsf{IR}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} & \subset & \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}} & \subset & \mathsf{IR}
\end{array}$$

The next result demonstrates the correctness of the reduction from the axioms *(I1)–(I7)* to *(D1)–(D5)* when an additional property, equivalent to $\mathsf{seq}\setminus\mathsf{sim} = \varnothing$ in the case of invariant structures over a given step alphabet, is assumed.

Proposition 8.4. *For every relational structure* $ir = \langle\Delta,\rightleftharpoons,\sqsubset,\ell\rangle$,

$$ir \in \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}} \quad\Longleftrightarrow\quad (ir \in \mathsf{IR}\ \wedge\ \forall x,y\in\Delta : x \rightleftharpoons y \implies x \sqsubset^{sym} y)\ .$$

Step traces over the step alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ can generate all the causal patterns involving causal relationships captured by the structures in $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$.

Theorem 8.5. *If a structure or* $\in \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ *has an injective labelling, then there are* $\sigma \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ *and* $u \in \mathsf{SSEQ}_{\sigma}$ *such that or is isomorphic to* $\mathsf{sseq2or}_{\sigma}(u)$.

Corollary 8.6. *If* $ir \in \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ *has an injective labelling, then there are* $\mu \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ *and* $u \in \mathsf{SSEQ}_{\mu}$ *such that ir is isomorphic to* $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}} \circ \mathsf{sseq2or}_{\mu}(u)$.

An example of a system model for which the step alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ and invariant structures $\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ provide a suitable semantical treatment are the elementary net systems with inhibitor arcs [14]. Note that every causal pattern can be obtained as a closure of dependence structure for a computation in an elementary net system with inhibitor arcs.

Finally, as shown below, traces generated by the alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ are histories satisfying the concurrency paradigm π_3 of [10] by which actions that can be executed in any order can also be executed simultaneously (but not necessarily vice versa).

Proposition 8.7. *Let α and β be two action occurrences of a step trace τ generated by $\sigma \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$. Then*

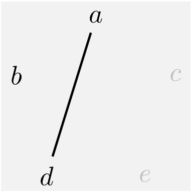
$$\begin{gathered}(\exists u \in \tau : \ pos_u(\alpha) < pos_u(\beta)) \ \wedge \ (\exists w \in \tau : \ pos_w(\alpha) > pos_w(\beta)) \\ \Longrightarrow \\ (\exists v \in \tau : \ pos_v(\alpha) = pos_v(\beta))\end{gathered}$$

9. Relational structures for the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$

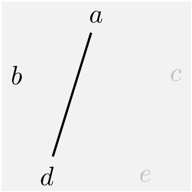
A step alphabet $\omega = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ satisfies $\mathsf{sim}\triangle\mathsf{seq} = \varnothing$, and therefore we have $\mathsf{sim} = \mathsf{seq} = \mathsf{seq}^{-1}$. For the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ the interleaving equations are not really needed, and the serialisability equations are rich enough to split and reorder steps in every possible way. As a result, all steps can be completely sequentialised.

Example 9.1. *Let us recall the step alphabet θ_0 of Example 3.1 and restrict Σ to $\{a, b, d\}$. The resulting step alphabet $\omega_0 \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ has the following simultaneity and sequentialising relations:*

$\mathsf{sim} =$



$\mathsf{seq} =$



with $[\![abd]\!] = \{abd\}$ and $[\![adb]\!] = \{adb, dab, (ad)b\}$. ◇

The definition of the dependence structure of a step sequence $u \in \mathsf{SSEQ}_\omega$ can be simplified by replacing (2), for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$, with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \ \wedge \ k < m \ . \end{array} \tag{7}$$

The order structures $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ are all those $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}$ for which $x \rightleftharpoons y \iff x \sqsubset^{sym} y$.

In terms of graph representation for $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$, any two events are connected in $\langle \Delta, \rightleftharpoons, \ell \rangle$ iff they are connected in the acyclic graphs $\langle \Delta, \sqsubset, \ell \rangle = \langle \Delta, \prec, \ell \rangle$.

The corresponding invariant structures can also be provided with a simpler definition. A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ if, for all $x, y, z \in \Delta$:

$$\begin{array}{rcll} & & x \not\sqsubset x & \textit{(E1)} \\ x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(E2)} \\ x \rightleftharpoons y & \Longleftrightarrow & x \sqsubset^{sym} y & \textit{(E3)} \\ x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \sqsubset^{sym} y & \textit{(E4)} \end{array}$$

In terms of graph representation for $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$, any two events are connected in $\langle \Delta, \rightleftharpoons, \ell \rangle$ iff they are connected in the partial orders $\langle \Delta, \sqsubset, \ell \rangle = \langle \Delta, \prec, \ell \rangle$ and, similarly as in $\mathsf{IR}_{\mathsf{sim}}$, they fully capture all the relevant causal relationships between events.

The definitions of $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ are sound.

The simplified order structure closure $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}} \xrightarrow{\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}} \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ is such that, for every $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$:

$$\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(or) = \langle \Delta, (\sqsubset^+)^{sym}, \sqsubset^+, \ell \rangle \ .$$

Proposition 9.2. $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}$ *is a surjection with* $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}}$.

Theorem 9.3.

$$\begin{array}{ccccc}
\mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} & \subset & \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}} & \subset & \mathsf{OR} \\
\cup & & \cup & & \cup \\
\mathsf{IR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} & \subset & \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} & \subset & \mathsf{IR}
\end{array}$$

The next result demonstrates the correctness of the reduction from the axioms *(I1)–(I7)* to *(E1)–(E4)* when an additional, equivalent to $\mathsf{sim}\triangle\mathsf{seq} = \varnothing$ in the case of invariant structures over a given step alphabet, property is assumed.

Proposition 9.4. *For every relational structure* $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,

$$ir \in \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} \iff (ir \in \mathsf{IR} \wedge \forall x, y \in \Delta : x \rightleftharpoons y \iff x \sqsubset^{sym} y) \,.$$

The step alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ can generate all the causal patterns involving causal relationships captured by the structures in $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Theorem 9.5. *If a structure or* $\in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ *has an injective labelling, then there are* $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\omega$ *such that or is isomorphic to* $\mathsf{sseq2or}_\omega(u)$.

Corollary 9.6. *If* $ir \in \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ *has an injective labelling, then there are* $\mu \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\mu$ *such that ir is isomorphic to* $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}} \circ \mathsf{sseq2or}_\mu(u)$.

Finally, as shown below, the step traces generated by the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ are histories satisfying the true concurrency paradigm π_8 of [10] and a system model for which this subclass provides a suitable semantical treatment are the elementary net systems with step sequence semantics. Note that every causal pattern (without labels) can be obtained as the closure of a dependence structure for a computation in an elementary net system with step sequence semantics.

Proposition 9.7. *Let* α *and* β *be distinct action occurrences* α *and* β *of a step trace* τ *generated by* $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$. *Then*

$$(\exists v \in \tau : \; pos_v(\alpha) = pos_v(\beta))$$

$$\iff$$

$$(\exists u \in \tau : \; pos_u(\alpha) < pos_u(\beta)) \;\wedge\; (\exists w \in \tau : \; pos_w(\alpha) > pos_w(\beta))$$

10. Concluding remarks

It may come as a surprise that invariant structures $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ are in a one-to-one correspondence with partial orders, similarly as for $\mathsf{IR}_{\mathsf{sim}}$, even though the actual definition of the two classes of order structures is different. The reason why these two structures differ is that the defining subclasses of alphabets, Θ_{sim} and $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$, are based on different models of observations. The former only admits sequential observations whereas the latter admits true step sequences. That the underlying causal structures are partial orders comes from the fact that in the case of $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ simultaneity always implies the possibility of sequentialisation.

In [7] we introduced and investigated how to extend the trace theory to the case of step sequences, and we established that the general traces defined through step alphabets are indeed the most general in terms of their underlying order structures. In this paper, we have continued our investigations and identified for the five natural subclasses of step traces their corresponding – simplified – invariant order structures.

As observed in [7], there are invariant structures that cannot be generated by any step alphabet. One reason is that the latter can only capture *static* dependencies between actions, whereas in the former different occurrences of the same pair of actions may exhibit different causality dependencies. Another reason is that the order-theoretic properties of invariant structure are orthogonal to the properties of their labellings. A characterisation of 'good' labellings for the order structures corresponding to general step traces has been addressed in [27]. In our ongoing work we aim at similar characterisations for each subclass of invariant structures considered in this paper.

We have considered an extension of Mazurkiewicz traces taking steps as the smallest units of observation, and to represent observational and causal relationships in the behaviours of concurrent systems we used the *order structures* from [28] which are an extension of an idea first proposed in [10, 17, 18]. A direct predecessor of order structures were the *stratified order structures* (i.e., those generated by $\Theta_{\mathsf{seq\setminus sim}}$), introduced independently in [17] and [29], and then applied, e.g., in [30, 31]. The approach presented here allows classifications fitting both established (e.g., comtraces [14] and ST-traces [32, 33]), and as yet uninvestigated trace models.

There are differences with other concurrency models that at first sight might seem related to step traces. First of all, there exist other generalisations of traces. Semi-traces originally introduced as rewriting systems by [34] and later investigated in, e.g., [35, 36] are generated by semi-commutations. The rewriting rules that change the order of two adjacent action occurrences can be one-directional, $ab \to ba$, rather than bi-directional. This cannot be done in the model discussed in this paper. Conversely, there are no partial order models which can deal with weak causality [10, 14]. Approaches other than steps, either do not support weak causality [13, 32, 37], or, as [21, 33, 38], can equivalently be modelled with the comtraces of [14] (i.e., the model of $\Theta_{\mathsf{seq\setminus sim}}$). We are also not aware of a model that can express a mutex situation represented here by the interleaving equation ($AB = BA$ and $A \cap B = \varnothing$) other than those following [16]. Other extensions of Mazurkiewicz traces consider infinite sequences, leading to complex traces or infinite traces as in, e.g., [39, 40]. Finally, it should be noted that the extension of Mazurkiewicz traces discussed in this paper is a *static* one, in contrast to the context or history dependent traces from, e.g., [41, 42, 43].

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Appendix I. Proofs for the alphabets in $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$

Lemma Appendix I.1. $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \subseteq \mathsf{IR}$.

Proof We first note that:

$$x \sqsubset y \sqsubset x \wedge x \rightleftharpoons y \implies_{(C3)} x \neq y \wedge x \neq y \wedge x \rightleftharpoons y \implies \mathit{false} \quad (*)$$

Hence, by *(C1)*,

$$x \rightleftharpoons y \quad \Longleftrightarrow \quad x \neq y \ \wedge \ \neg(x \sqsubset y \sqsubset x) \quad (**)$$

To show *(I1)* we observe that:

$$x \sqsubset x \implies x \sqsubset x \sqsubset x \implies_{(C3)} x \neq x \wedge x \neq x \implies \mathit{false}.$$

Then we note that *(I2)* is simply *(C2)*. To show *(I3)* we observe that:

$$\begin{aligned} x \rightleftharpoons y \quad & \Longrightarrow_{(**)} \quad x \neq y \ \wedge \ \neg(x \sqsubset y \sqsubset x) \\ & \Longrightarrow \quad x \neq y \ \wedge \ (y \neq x \ \wedge \ \neg(y \sqsubset x \sqsubset y)) \\ & \Longrightarrow_{(**)} \quad x \neq y \ \wedge \ y \rightleftharpoons x \ . \end{aligned}$$

To show *(I4)* we observe that:

$$\begin{aligned} x \not\rightleftharpoons y \ \wedge \ x \prec z \sqsubset y \quad & \Longrightarrow_{(**)} \quad (x = y \ \vee \ x \sqsubset y \sqsubset x) \ \wedge \ x \prec z \sqsubset x \\ & \Longrightarrow_{(C1)} \quad (x = y \ \vee \ x \sqsubset y \sqsubset x) \wedge \\ & \qquad\qquad x \sqsubset z \sqsubset y \ \wedge \ x \rightleftharpoons z \ \wedge \ z \neq x \\ & \Longrightarrow \quad x \sqsubset z \sqsubset x \ \wedge \ x \rightleftharpoons z \ \vee \\ & \qquad\qquad x \sqsubset z \sqsubset y \sqsubset x \ \wedge \ x \rightleftharpoons z \ \wedge \ z \neq x \\ & \Longrightarrow_{(C2)} \quad x \sqsubset z \sqsubset x \ \wedge \ x \rightleftharpoons z \ \vee \ x \sqsubset z \sqsubset x \ \wedge \ x \rightleftharpoons z \\ & \Longrightarrow \quad x \sqsubset z \sqsubset x \ \wedge \ x \rightleftharpoons z \\ & \Longrightarrow_{(C3)} \quad \textit{false} \ . \end{aligned}$$

Similarly, $x \not\rightleftharpoons y \ \wedge \ x \sqsubset z \prec y \ \Longrightarrow \ \textit{false}$. Hence we have:

$$x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y \ \Longrightarrow \ x \rightleftharpoons y \ .$$

To show *(I5)* we first observe that:

$$\begin{aligned} & z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \wedge \ x \sqsubset y \sqsubset x \\ & \qquad \Longrightarrow_{(C1)} \quad z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset y \sqsubset x \sqsubset z \ \wedge \ z \neq y \\ & \qquad \Longrightarrow_{(C2)} \quad z \rightleftharpoons y \ \wedge \ z \sqsubset y \sqsubset z \\ & \qquad \Longrightarrow_{(*)} \quad \textit{false} \ , \end{aligned}$$

$$\begin{aligned} & z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \wedge \ x = y \\ & \qquad \Longrightarrow \quad z \rightleftharpoons y \ \wedge \ z \sqsubset y \sqsubset z \\ & \qquad \Longrightarrow_{(*)} \quad \textit{false} \ . \end{aligned}$$

Hence we have:

$$z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \Longrightarrow \ \neg(y \sqsubset x \sqsubset y) \ \wedge \ x \neq y \ \Longrightarrow_{(**)} \ x \rightleftharpoons y \ .$$

To show *(I6)* we observe that:

$$\begin{aligned}
& z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \;\wedge\; x \sqsubset y \sqsubset x \\
& \qquad \Longrightarrow_{\textit{(C1)}} \quad z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \;\wedge\; x \sqsubset y \sqsubset x \wedge \\
& \qquad \phantom{\Longrightarrow_{\textit{(C1)}}} \quad z \neq z' \;\wedge\; z \neq x \;\wedge\; y \neq z \\
& \qquad \Longrightarrow \quad z \rightleftharpoons z' \;\wedge\; z \sqsubset y \sqsubset x \sqsubset z' \sqsubset y \sqsubset x \sqsubset z \wedge \\
& \qquad \quad z \neq z' \;\wedge\; z \neq x \;\wedge\; y \neq z \\
& \qquad \Longrightarrow_{\textit{(C2)}} \quad z \rightleftharpoons z' \;\wedge\; z \sqsubset x \sqsubset z' \sqsubset y \sqsubset z \;\wedge\; z \neq z' \\
& \qquad \Longrightarrow_{\textit{(C2)}} \quad z \rightleftharpoons z' \;\wedge\; z \sqsubset z' \sqsubset z \\
& \qquad \Longrightarrow_{\textit{(*)}} \quad \textit{false}
\end{aligned}$$

$$\begin{aligned}
& z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \;\wedge\; x = y \\
& \qquad \Longrightarrow_{\textit{(C1)}} \quad z \rightleftharpoons z' \;\wedge\; z \sqsubset x \sqsubset z' \sqsubset x \sqsubset z \;\wedge\; z \neq z' \\
& \qquad \Longrightarrow_{\textit{(C2)}} \quad z \rightleftharpoons z' \;\wedge\; z \sqsubset z' \sqsubset z \\
& \qquad \Longrightarrow_{\textit{(*)}} \quad \textit{false}\,.
\end{aligned}$$

Hence we have:

$$z \rightleftharpoons z' \wedge x \sqsubset z \sqsubset y \wedge x \sqsubset z' \sqsubset y \;\Longrightarrow\; \neg(y \sqsubset x \sqsubset y) \wedge x \neq y \;\Longrightarrow_{\textit{(**)}}\; x \rightleftharpoons y\,.$$

We finally note that *(I7)* is simply *(C4)*. □

Lemma Appendix I.2. $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \subseteq \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$.

Proof Follows from Lemma Appendix I.1, $\mathsf{IR} \subseteq \mathsf{OR}$, and *(C3)*. □

Lemma Appendix I.3. $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}) \subseteq \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$.

Proof

Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ and $ir = \mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(C1)* we observe that $\widehat{\rightleftharpoons} \;= \rightleftharpoons$, and to show *(C2)*, we observe that:

$$x \neq y \;\wedge\; x \,\widehat{\sqsubset}\, z \,\widehat{\sqsubset}\, y \;\Longrightarrow\; x \neq y \;\wedge\; x \sqsubset^{\Yup} z \sqsubset^{\Yup} y \;\Longrightarrow\; x \sqsubset^{\Yup} y \;\Longrightarrow\; x \,\widehat{\sqsubset}\, y\,.$$

To show *(C3)* we observe that:

$$\sqsubset^{\circledast} = \sqsubset^{*} \;\cap (\sqsubset^{*})^{-1} \;=\; (\sqsubset^{\Yup} \uplus id_{\Delta}) \cap (\sqsubset^{\Yup} \uplus id_{\Delta})^{-1} \;=\; (\sqsubset^{\Yup} \cap (\sqsubset^{\Yup})^{-1}) \uplus id_{\Delta},$$

hence

$$\widehat{\rightleftharpoons} \; = \rightleftharpoons = \; (\Delta \times \Delta) \setminus \sqsubset^{\circledast} = (\Delta \times \Delta) \setminus (\sqsubset^{\Yup} \cap (\sqsubset^{\Yup})^{-1} \uplus id_{\Delta}),$$

and so

$$x \; \widehat{\not\rightleftharpoons} \; y \; \wedge \; x \neq y \iff x \; \widehat{\sqsubset} \; y \; \widehat{\sqsubset} \; x.$$

Finally, *(C4)* follows from the label-linearity of or, as shown below:

$$x \neq y \; \wedge \; \ell(x) = \ell(y) \implies x \prec^{sym} y \implies x \; \widehat{\precsim}^{sym} y \; .$$

Hence $ir \in \mathsf{IR}_{\mathsf{sim} \cap \mathsf{seq}}$ $\square$

Proof of Proposition 7.2

We show that $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}}$. Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ and $ir = \mathsf{or2ir}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. We first observe that in such a case we have $\rightleftharpoons = (\Delta \times \Delta) \setminus \sqsubset^{\circledast}$, which follows from $x \neq y \Rightarrow x \rightleftharpoons y \vee x \sqsubset y \sqsubset z$ and the separability of or. By the general theory we know that

$$(\sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} \cup \sqsubset^{\circledast} \circ cross^{sym} \circ \sqsubset^{\circledast}) \cap \sqsubset^{\circledast} = \; \varnothing.$$

and since $\rightleftharpoons \subseteq \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast}$ we obtain $\mathsf{or2ir}(or) = \langle \Delta, \rightleftharpoons, \sqsubset^{\Yup}, \ell \rangle$.

We observe that $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}) = \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ follows from Lemmas Appendix I.1, Appendix I.2, and Appendix I.3, $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}}$, and the fact that $\mathsf{or2ir}$ is the identity on IR, as then we obtain $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}) \subseteq \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ and $\mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}) \supseteq \mathsf{or2ir}_{\mathsf{sim}\cap\mathsf{seq}}(\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}) = \mathsf{or2ir}(\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}) = \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$. $\square$

Proof of Theorem 7.3

Let us consider one by one all the inclusions:

- $\mathsf{IR} \subset \mathsf{OR}$ was already justified in the proof of Theorem 5.6. Note, however, that we also have

$$or = \left\langle \begin{array}{c} \{x, y, z\}, \{\langle y, z \rangle, \langle z, y \rangle, \langle x, z \rangle, \langle z, x \rangle\}, \\ \{\langle x, y \rangle, \langle y, x \rangle, \langle y, z \rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OR} \setminus \mathsf{IR} \; .$$

- $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \subset \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ follows from $or \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} \setminus \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ and Lemma Appendix I.2.

- $\mathsf{IR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} \subset \mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}}$ follows from $os \in \mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}}$ and the general results proven in [7].

- $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} \subset \mathsf{OR}$ follows from the definition of $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ and

$$or' = \langle \{x, y\}, \varnothing, \{\langle x, y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OR} \setminus \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} \ .$$

- $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \subset \mathsf{IR}$ follows from $or' \in \mathsf{IR} \setminus \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ and Lemma Appendix I.1.

- $\mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} \subset \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ can be shown by taking $\nu \in \Theta_{\mathsf{sim}\cap\mathsf{seq}}$, $u \in \mathsf{SSEQ}_\nu$, and $or = \mathsf{sseq2or}_\nu(u) = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$. Since we know that $or \in \mathsf{OR}$, we only need to demonstrate that:

$$(\Delta \times \Delta) \setminus id_\Delta \ \subseteq \rightleftharpoons \cup \, (\sqsubset \cap \sqsubset^{-1}) \ .$$

  The above holds since, by (5), $pos_u(\alpha) = pos_u(\beta) \wedge \alpha \neq \beta$ implies $\alpha \sqsubset \beta \sqsubset \alpha$, and $pos_u(\alpha) \neq pos_u(\beta)$ implies $\alpha \rightleftharpoons \beta$. Hence $or \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$. Moreover, we note that

$$or'' = \left\langle \begin{array}{c} \{x, y, z\}, \\ \{\langle x, y\rangle, \langle y, x\rangle, \langle x, z\rangle, \langle z, x\rangle, \langle y, z\rangle, \langle z, y\rangle\}, \\ \{\langle x, y\rangle, \langle x, z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} .$$

- $\mathsf{IR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} \subset \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ follows from Lemma Appendix I.3, $or'' \in \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} \subseteq \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$.

Moreover, note that $or \in \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}} \setminus \mathsf{IR}$ and $or' \in \mathsf{IR} \setminus \mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ which justifies that IR and $\mathsf{OR}_{\mathsf{sim}\cap\mathsf{seq}}$ are not related. Similarly, $or \in \mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}} \setminus \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ and $or'' \in \mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}}$, hence there is no inclusion between $\mathsf{IR}_{\mathsf{sim}\cap\mathsf{seq}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}\cap\mathsf{seq}}}$. $\square$

Proof of Proposition 7.4

($\Longrightarrow$) Follows from Theorem 7.3 and *(C3)*.

($\Longleftarrow$) Note that *(I2)* and *(C2)* as well as *(I7)* and *(C4)* are the same axioms; and *(C1)* follows from *(I3)*. To prove *(C3)*, assume that $x \sqsubset y \sqsubset x$. Then $x \neq y$ by *(I1)* and $x \not\rightleftharpoons y$ by separability (or directly by *(I5)* and *(C1)*). Conversely, assume that $x \not\rightleftharpoons y$ and $x \neq y$. Then by additional property $x \sqsubset y \sqsubset x$, which concludes the proof. $\square$

Proof of Theorem 7.5

Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [7] it follows that there exists $sr \in \mathsf{or2SR}(os)$ which, by the definition of $\mathsf{OR}_{\mathsf{sim} \cap \mathsf{seq}}$ and separability of OR satisfies $(\Delta \times \Delta) = id_\Delta \uplus \rightleftharpoons_{sr} \uplus (\sqsubset_{sr} \cap \sqsubset_{sr}^{-1})$. Let $\nu = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where:

$$
\begin{array}{rcl}
\mathsf{sim} & = & \{\langle a, b\rangle \in \Sigma \times \Sigma \quad | \quad pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)\} \\
\mathsf{seq} & = & \{\langle a, b\rangle \in \Sigma \times \Sigma \quad | \quad (pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle) \;\wedge\; \langle a,1\rangle \not\sqsubset \langle b,1\rangle) \\
& & \qquad\qquad\qquad\quad \vee (pos_u(\langle b,1\rangle) < pos_u(\langle a,1\rangle) \;\wedge\; \langle b,1\rangle \not\sqsubset \langle a,1\rangle)\} \;.
\end{array}
$$

Clearly, $\nu \in \Theta_{\mathsf{sim} \cap \mathsf{seq}}$ and $u \in \mathsf{SSEQ}_\nu$. It is easy to check that $or = \mathsf{sseq2or}_\nu(u)$. $\square$

Appendix II. Proofs for the alphabets in $\Theta_{\mathsf{seq} \setminus \mathsf{sim}}$

Lemma Appendix II.1. $\mathsf{IR}_{\mathsf{seq} \setminus \mathsf{sim}} \subseteq \mathsf{IR}$.

Proof We first note that *(I1)*, *(I2)* and *(I4)* are respectively *(D1)*, *(D2)* and *(D4)*. To show *(I3)* we observe that:

$$x \rightleftharpoons y \;\Longrightarrow_{(D3)}\; x \sqsubset^{sym} y \;\wedge\; y \rightleftharpoons x \;\Longrightarrow_{(D1)}\; x \neq y \;\wedge\; y \rightleftharpoons x \;.$$

To show *(I5)* we observe that:

$$
\begin{array}{rll}
z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z & \Longrightarrow_{(D3)} & z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z \;\wedge\; z \sqsubset^{sym} y \;\wedge\; y \rightleftharpoons z \\
& \Longrightarrow & x \sqsubset z \prec y \;\vee\; y \prec z \sqsubset x \\
& \Longrightarrow_{(D4)} & x \rightleftharpoons y \;\vee\; y \rightleftharpoons x \\
& \Longrightarrow_{(D3)} & x \rightleftharpoons y \;.
\end{array}
$$

To show *(I6)* we observe that:

$$\begin{aligned}
& z \rightleftharpoons z' \ \wedge\ x \sqsubset z \sqsubset y \ \wedge\ x \sqsubset z' \sqsubset y \\
& \qquad \Longrightarrow_{\mathit{(D3)}} \quad z \rightleftharpoons z' \ \wedge\ x \sqsubset z \sqsubset y \ \wedge\ x \sqsubset z' \sqsubset y \ \wedge\ z' \sqsubset^{sym} z \ \wedge\ z' \rightleftharpoons z \\
& \qquad \Longrightarrow_{\mathit{(D1)}} \quad (x \sqsubset z \prec z' \sqsubset y \ \vee\ x \sqsubset z' \prec z \sqsubset y) \ \wedge\ x \neq z \ \wedge\ y \neq z \\
& \qquad \Longrightarrow_{\mathit{(D2,D4)}} \quad x \sqsubset z \prec y \ \vee\ x \prec z \sqsubset y \\
& \qquad \Longrightarrow_{\mathit{(D4)}} \quad x \rightleftharpoons y \ .
\end{aligned}$$

We finally note that *(I7)* follows from *(D3)* and *(D5)*. □

Lemma Appendix II.2. $\mathsf{IR}_{\mathsf{seq\backslash sim}} \subseteq \mathsf{OR}_{\mathsf{seq\backslash sim}}$.

Proof Follows from Lemma Appendix II.1, $\mathsf{IR} \subseteq \mathsf{OR}$, and *(D3)*. □

Lemma Appendix II.3. $\mathsf{or2ir}_{\mathsf{seq\backslash sim}}(\mathsf{OR}_{\mathsf{seq\backslash sim}}) \subseteq \mathsf{IR}_{\mathsf{seq\backslash sim}}$.

Proof

Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{seq\backslash sim}}$ and $ir = \mathsf{or2ir}_{\mathsf{seq\backslash sim}}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(D1)*, we observe that:

$$x \mathrel{\widehat{\sqsubset}} x \implies x \sqsubset^{\curlywedge} x \implies \mathit{false} \ .$$

To show *(D2)*, we observe that:

$$x \neq y \ \wedge\ x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \implies x \neq y \ \wedge\ x \sqsubset^{\curlywedge} z \sqsubset^{\curlywedge} y \implies x \sqsubset^{\curlywedge} y \implies x \mathrel{\widehat{\sqsubset}} y \ .$$

To show *(D3)* we observe that all we need is to prove that $x \widehat{\rightleftharpoons} y \implies x \widehat{\sqsubset}^{sym} y$, in the following way:

$$\begin{aligned}
x \mathrel{\widehat{\rightleftharpoons}} y \quad & \Longrightarrow \quad x(\sqsubset^{*} \circ \prec \circ \sqsubset^{*})^{sym} y \implies x \neq y \ \wedge\ x(\sqsubset^{+})^{sym} y \\
& \Longrightarrow \quad x(\sqsubset^{\curlywedge})^{sym} y \implies x \mathrel{\widehat{\sqsubset}}^{sym} y \ ,
\end{aligned}$$

where $x \mathrel{\widehat{\rightleftharpoons}} y \implies x \neq y$ follows from Lemma Appendix II.1 and *(I3)*. Finally, *(D5)* follows from the label-linearity of *or*, as shown below:

$$x \neq y \ \wedge\ \ell(x) = \ell(y) \implies x \mathrel{\widehat{\prec}}^{sym} y \implies x \mathrel{\widehat{\rightleftharpoons}} y \ .$$

Hence $ir \in \mathsf{IR}_{\mathsf{seq\backslash sim}}$. □

Proof of Proposition 8.2

We first show that $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}}$. Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $ir = \mathsf{or2ir}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. We first observe that

$$\sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} = \sqsubset^{\circledast} \circ \prec^{sym} \circ \sqsubset^{\circledast} \quad \textit{and} \quad \textit{cross} = \sqsubset^{*} \circ \prec \circ \sqsubset^{*}$$

which follows from $x \rightleftharpoons y \Longrightarrow x \sqsubset^{sym} y$. Hence

$$\widehat{\rightleftharpoons} \ = \sqsubset^{\circledast} \circ (\sqsubset^{*} \circ \prec \circ \sqsubset^{*})^{sym} \circ \sqsubset^{\circledast} = \ (\sqsubset^{*} \circ \prec \circ \sqsubset^{*})^{sym} \ .$$

We then observe that $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}) = \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ follows directly from Lemmas Appendix II.1, Appendix II.2, and Appendix II.3, $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}}$, and the fact that $\mathsf{or2ir}$ is the identity on IR, as then we obtain $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}) \subseteq \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $\mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}) \supseteq \mathsf{or2ir}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}) = \mathsf{or2ir}(\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}) = \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$. ☐

Proof of Theorem 8.3

Let us consider one by one all the inclusions:

- $\mathsf{IR} \subset \mathsf{OR}$ was already justified in the proof of Theorem 5.6. Note, however, that we also have

$$or = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle\}, \{\langle x,y\rangle, \langle y,z\rangle\}, \\ \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OR} \setminus \mathsf{IR} \ .$$

- $\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}} \subset \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ follows from $or \in \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}} \setminus \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and Lemma Appendix II.2.

- $\mathsf{IR}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \subset \mathsf{OR}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$ follows from $os \in \mathsf{OR}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \setminus \mathsf{IR}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$ and the general results proven in [7].

- $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}} \subset \mathsf{OR}$ follows from the definition of $\mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and

$$or' = \langle \{x,y\}, \{\langle x,y\rangle\}, \varnothing, \{x \mapsto a, y \mapsto b\} \rangle \in \mathsf{OR} \setminus \mathsf{OR}_{\mathsf{seq}\setminus\mathsf{sim}} \ .$$

- $\mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}} \subset \mathsf{IR}$ follows from $or' \in \mathsf{IR} \setminus \mathsf{IR}_{\mathsf{seq}\setminus\mathsf{sim}}$ and Lemma Appendix II.1.

- $\mathsf{OR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} \subset \mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}}$ can be proven by taking $\sigma \in \Theta_{\mathsf{seq}\backslash\mathsf{sim}}$, $u \in \mathsf{SSEQ}_\sigma$ and $or = \mathsf{sseq2or}_\sigma(u) = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since we know that $or \in \mathsf{OR}$, we only need to show that $\rightleftharpoons \subseteq \sqsubset^{sym}$. This, however, follows from (6). Hence $or \in \mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}}$. Moreover, we note that
$$or'' = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}} \setminus \mathsf{OR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} .$$
- $\mathsf{IR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} \subseteq \mathsf{IR}_{\mathsf{seq}\backslash\mathsf{sim}}$ follows from Lemma Appendix II.3, $or'' \in \mathsf{IR}_{\mathsf{seq}\backslash\mathsf{sim}} \setminus \mathsf{IR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}}$ and $\mathsf{OR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} \subseteq \mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}}$.

Moreover, note that $or \in \mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}} \setminus \mathsf{IR}$ and $or' \in \mathsf{IR} \setminus \mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}}$ which justifies that IR and $\mathsf{OR}_{\mathsf{seq}\backslash\mathsf{sim}}$ are not related. Similarly, $or \in \mathsf{OR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} \setminus \mathsf{IR}_{\mathsf{seq}\backslash\mathsf{sim}}$ and $or'' \in \mathsf{IR}_{\mathsf{seq}\backslash\mathsf{sim}} \setminus \mathsf{OR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}}$, hence there is no inclusion between $\mathsf{IR}_{\mathsf{seq}\backslash\mathsf{sim}}$ and $\mathsf{OR}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}}$. ☐

Proof of Proposition 8.4

($\Longrightarrow$) Follows from Theorem 8.3 and *(D3)*.

($\Longleftarrow$) Note that *(I1)* and *(D1)* as well as *(I2)* and *(D2)*, and *(I4)* and *(D4)* are the same axioms; and *(D5)* follows from *(I7)*. To prove *(D3)*, assume that $x \rightleftharpoons y$. Then $x \sqsubset^{sym} y$ by additional property, while $y \rightleftharpoons x$ by *(I3)*. ☐

Proof of Theorem 8.5

Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [7] it follows that there exists $sr \in \mathsf{or2SR}(os)$. Let $u = \mathsf{sseq2sr}^{-1}(sr)$, and $\sigma = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where:

$$\begin{array}{rcll} \mathsf{sim} & = & \{\langle a,b\rangle \in \Sigma \times \Sigma & \mid \; (pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle) \;\wedge\; a \neq b) \vee \\ & & & \quad\; (pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle) \;\wedge\; \langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle)\} \\ \mathsf{seq} & = & \{\langle a,b\rangle \in \Sigma \times \Sigma & \mid \; (pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle) \;\wedge\; a \neq b \;\wedge\; \langle b,1\rangle \not\sqsubset \langle a,1\rangle) \\ & & & \quad\; \vee (pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle) \;\wedge\; \langle a,1\rangle \not\sqsubset \langle b,1\rangle) \\ & & & \quad\; \vee (pos_u(\langle b,1\rangle) < pos_u(\langle a,1\rangle) \;\wedge\; \langle b,1\rangle \not\sqsubset \langle a,1\rangle)\} . \end{array}$$

We then observe that sim is symmetric since $\rightleftharpoons$ is symmetric, and $\mathsf{seq} \setminus \mathsf{sim}$ is symmetric because it is empty (it follows from $\mathsf{seq} \subseteq \mathsf{sim}$, as we show below). Hence σ is a step alphabet. To show $\sigma \in \Theta_{\mathsf{seq} \setminus \mathsf{sim}}$ we need to show that $\mathsf{seq} \subseteq \mathsf{sim}$.

Let $\langle a, b\rangle \in \mathsf{seq}$. If $pos_u(\langle a, 1\rangle) = pos_u(\langle b, 1\rangle)$ then, clearly, $\langle a, b\rangle \in \mathsf{sim}$. If $pos_u(\langle a, 1\rangle) < pos_u(\langle b, 1\rangle)$ and $\langle a, 1\rangle \not\sqsubset \langle b, 1\rangle$ then, by $or \in \mathsf{OR}_{\mathsf{seq} \setminus \mathsf{sim}}$, we obtain $\langle a, 1\rangle \not\rightleftharpoons \langle b, 1\rangle$ or $\langle a, 1\rangle \rightleftharpoons \langle b, 1\rangle \ \wedge \ \langle b, 1\rangle \sqsubset \langle a, 1\rangle$.
Moreover, by $pos_u(\langle a, 1\rangle) < pos_u(\langle b, 1\rangle)$, we obtain $\langle b, 1\rangle \not\sqsubset \langle a, 1\rangle$ and so we have $\langle a, 1\rangle \not\rightleftharpoons \langle b, 1\rangle$. Hence $\langle a, b\rangle \in \mathsf{sim}$, and so $\sigma \in \Theta_{\mathsf{seq} \setminus \mathsf{sim}}$.

We then observe that $u \in \mathsf{SSEQ}_\sigma$ as $pos_u(\langle a, 1\rangle) = pos_u(\langle b, 1\rangle)$ and $a \neq b$ together imply $\langle a, b\rangle \in \mathsf{sim}$, and it is easy to check that $or = \mathsf{sseq2or}_\sigma(u)$. ☐

Proof of Proposition 8.7

Let $ir = \mathsf{or2ir} \circ \mathsf{sseq2or}_\kappa(u) = \mathsf{or2ir} \circ \mathsf{sseq2or}_\kappa(w)$. From $pos_u(\alpha) < pos_u(\beta)$ it follows that there is $sr_u \in \mathsf{or2SR}(ir)$ such that $\alpha \prec_{sr_u} \beta$. Similarly, from $pos_w(\alpha) > pos_w(\beta)$ it follows that there is $sr_w \in \mathsf{or2SR}(ir)$ such that $\beta \prec_{sr_w} \alpha$. Hence, $\alpha \not\sqsubset_{ir} \beta \not\sqsubset_{ir} \alpha$. Moreover, by $ir \in \mathsf{OR}_{\mathsf{seq} \setminus \mathsf{sim}}$, $\alpha \not\rightleftharpoons_{ir} \beta$. This, by the general results proved in [7], there is $sr_v \in \mathsf{or2SR}(ir)$ such that $\alpha \sqsubset_{sr_v} \beta \sqsubset_{sr_v} \alpha$. Then the conclusion holds by taking $v = \mathsf{sseq2or}_\sigma^{-1}(sr_v)$. ☐

Appendix III. Proofs for the alphabets in $\Theta_{\mathsf{sim} \triangle \mathsf{seq}}$

Lemma Appendix III.1. $\mathsf{IR}_{\mathsf{sim} \triangle \mathsf{seq}} \subseteq \mathsf{IR}$.

Proof We first note that *(I1)* is simply *(E1)*. To show *(I2)* we observe that

$$x \neq y \ \wedge \ x \sqsubset z \sqsubset y \implies_{\mathit{(E2)}} x \sqsubset y \ .$$

To show *(I3)* we observe that

$$x \rightleftharpoons y \implies_{\mathit{(E3)}} x \sqsubset^{sym} y \implies_{\mathit{(E3)}} y \rightleftharpoons x \ .$$

and we observe that if $x \rightleftharpoons x$ then we obtain a contradiction as follows:

$$x \rightleftharpoons x \implies_{\mathit{(E3)}} x \sqsubset^{sym} x \implies x \sqsubset x \implies_{\mathit{(E1)}} x \neq x \ .$$

To show *(I4)* we observe that:

$$x \prec z \sqsubset y \;\vee\; x \sqsubset z \prec y \;\Longrightarrow_{(E2)}\; x \sqsubset y \;\Longrightarrow_{(E3)}\; x \rightleftharpoons y\ .$$

To show *(I5)* we observe that:

$$z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z \;\Longrightarrow_{(E2)}\; z \sqsubset z \;\Longrightarrow_{(E1)}\; \textit{false}\ .$$

To show *(I6)* we observe that:

$$z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \;\Longrightarrow_{(E2)}\; x \sqsubset y \;\Longrightarrow_{(E3)}\; x \rightleftharpoons y\ .$$

We finally note that *(I7)* follows from *(E3)* and *(E4)*. □

Lemma Appendix III.2. $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} \subseteq \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Proof Follows from Lemma Appendix III.1, $\mathsf{IR} \subseteq \mathsf{OR}$, and *(E3)*. □

Lemma Appendix III.3. $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}) \subseteq \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Proof
Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $ir = \mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(E1)* we observe that $x \mathrel{\widehat{\sqsubset}} x$ together with $x \not\sqsubset x$ imply that there are y, z such that $x \sqsubset^{*} y \sqsubset z \sqsubset^{*} x$. Hence, by the definition of $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$, $y \rightleftharpoons z$, contradicting the separability of or.
To show *(E2)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \;\Longrightarrow\; x \sqsubset^{+} z \sqsubset^{+} y \;\Longrightarrow\; x \sqsubset^{+} y \;\Longrightarrow\; x \mathrel{\widehat{\sqsubset}} y\ .$$

To show *(E3)* we observe that:

$$x \mathrel{\widehat{\sqsubset}^{sym}} y \;\Longleftrightarrow\; x (\sqsubset^{+})^{sym} y \;\Longleftrightarrow\; x \mathrel{\widehat{\rightleftharpoons}} y\ .$$

Finally, *(E4)* follows from the label-linearity of or, as shown below:

$$x \neq y \;\wedge\; \ell(x) = \ell(y) \;\Longrightarrow\; x \mathrel{\widehat{\precsim}^{sym}} y \;\Longrightarrow\; x \mathrel{\widehat{\sqsubset}^{sym}} y\ .$$

Hence $ir \in \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$. □

Proof of Proposition 9.2

We show that $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}}$. Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $ir = \mathsf{or2ir}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. We first observe that in such a case we have $\sqsubset^{\circledast} = id_\Delta$ which follows from $x \sqsubset^{sym} y \iff x \rightleftharpoons y$ and the separability of or. As a result, we also have $\sqsubset^{\curlywedge} = \sqsubset^{+}$. Hence

$$\mathsf{or2ir}(or) = \langle \Delta, \rightleftharpoons \cup\, cross^{sym}, \sqsubset^{+}, \ell \rangle \ ,$$

where $cross = \{\langle x, y\rangle \mid \exists z, w : z \rightleftharpoons w \land x \sqsubset^{*} z \sqsubset^{*} y \land x \sqsubset^{*} w \sqsubset^{*} y\}$. We will now show that $(\rightleftharpoons \cup\, cross^{sym}) = (\sqsubset^{+})^{sym}$.
Suppose first that $\langle x, y\rangle \in cross$ which means that $x \neq y$ (which follows from the general theory), and there is z such that $x \sqsubset^{*} z \sqsubset^{*} y$. Hence $x \sqsubset^{+} y$ showing that the $(\subseteq)$ inclusion holds. To show the reverse inclusion, suppose that $x \sqsubset^{+} y$. If $x \sqsubset y$ then, by the definition of $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$, we have $x \rightleftharpoons y$. Otherwise, there is z such that $x \sqsubset z \sqsubset^{*} y$. Then, again by the definition of $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$, $z \rightleftharpoons x$. We therefore obtain that $\langle x, y\rangle \in cross$, after taking $w = x$. Hence

$$\mathsf{or2ir}(or) = \langle \Delta, (\sqsubset^{+})^{sym}, \sqsubset^{+}, \ell \rangle \ .$$

We observe that $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}) = \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ follows from Lemmas Appendix III.1, Appendix III.2, and Appendix III.3, $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}} = \mathsf{or2ir}|_{\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}}$, and the fact that $\mathsf{or2ir}$ is the identity on IR, as then we obtain $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}) \subseteq \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $\mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}) \supseteq \mathsf{or2ir}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}) = \mathsf{or2ir}(\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}) = \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$. ☐

Proof of Theorem 9.3

Let us consider one by one all the inclusions:

- $\mathsf{IR} \subset \mathsf{OR}$ was already justified in the proof of Theorem 5.6. Note, however, that we also have

$$or = \left\langle \begin{array}{c} \{x, y, z\}, \{\langle x, y\rangle, \langle y, x\rangle, \langle y, z\rangle, \langle z, y\rangle\}, \\ \{\langle x, y\rangle, \langle y, z\rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OR} \setminus \mathsf{IR} \ .$$

- $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} \subset \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ follows from $or \in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}} \backslash \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and Lemma Appendix III.2.
- $\mathsf{IR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subset \mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$ follows from $os \in \mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \setminus \mathsf{IR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$ and the general results proved in [7].
- $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}} \subset \mathsf{OR}$ follows from the definition of $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and

$$or' = \langle \{x, y\}, \varnothing, \{\langle x, y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OR} \setminus \mathsf{OR}_{\mathsf{sim}\backslash\mathsf{seq}} .$$

- $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} \subset \mathsf{IR}$ follows from $or' \in \mathsf{IR} \backslash \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and Lemma Appendix III.1.
- $\mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subset \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ can be proven by taking $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$, $u \in \mathsf{SSEQ}_\omega$, and $or = \mathsf{sseq2or}_\omega(u)$. Since $or \in \mathsf{OR}$, we only need to show that $\sqsubset^{sym}_{or} = \rightleftharpoons_{or}$. This, however, follows from (7). Moreover, we note that

$$or'' = \left\langle \begin{array}{c} \{x, y, z\}, \{\langle x, y\rangle, \langle y, x\rangle, \langle x, z\rangle, \langle z, x\rangle\}, \\ \{\langle x, y\rangle, \langle x, z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}} \backslash \mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} .$$

- $\mathsf{IR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subseteq \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ follows from Lemma Appendix III.3, $or'' \in \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} \backslash \mathsf{IR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subseteq \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Moreover, note that $or \in \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{IR}$ and $or' \in \mathsf{IR} \setminus \mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ which justifies that IR and $\mathsf{OR}_{\mathsf{sim}\triangle\mathsf{seq}}$ are not related. Similarly, $or \in \mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \setminus \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $or'' \in \mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$, hence there is no inclusion between $\mathsf{IR}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $\mathsf{OR}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$. $\square$

Proof of Proposition 9.4

($\Longrightarrow$) Follows from Theorem 9.3 and *(E3)*.

($\Longleftarrow$) Note that *(E3)* is the additional property; *(I1)* and *(E1)* are the same axioms; and *(E4)* follows from *(I7)*. To prove *(E2)* assume $x \sqsubset z \sqsubset y$. Then, by additional property $x \rightleftharpoons z$. Then $x \rightleftharpoons y$ by *(I5)* and thus, $x \neq y$ by *(I3)*. Hence $x \sqsubset y$ by *(I2)*, and *(E2)* follows. $\square$

Proof of Theorem 9.5

Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [7] it follows that there exists $sr \in \mathsf{or2SR}(os)$. Let $u = \mathsf{sseq2sr}^{-1}(sr)$, and $\omega = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where:

$$\mathsf{seq} = \mathsf{sim} \quad = \quad \{\langle a, b \rangle \in \Sigma \times \Sigma \quad | \quad (pos_u(\langle a, 1 \rangle) \neq pos_u(\langle b, 1 \rangle) \;\wedge\; \langle a, 1 \rangle \not\rightleftharpoons \langle b, 1 \rangle)\} \; .$$

We then observe that sim is symmetric since $\rightleftharpoons$ is symmetric. Hence ω is a step alphabet. Clearly, $\omega \in \Theta_{\mathsf{sim} \triangle \mathsf{seq}}$ and $u \in \mathsf{SSEQ}_\omega$. It is easy to check that $or = \mathsf{sseq2or}_\kappa(u)$. □

Proof of Proposition 9.7

Let $ir = \mathsf{or2ir} \circ \mathsf{sseq2or}_\omega(v)$. By $pos_v(\alpha) = pos_v(\beta)$, we obtain $\langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sim}$ and there is $sr \in \mathsf{or2SR}(ir)$ such that $\alpha \sqsubset_{sr} \beta \sqsubset_{sr} \alpha$. Hence, $\alpha \not\rightleftharpoons_{ir} \beta$. Moreover, by the order structure closure, $\alpha \not\sqsubset_{ir} \beta$ and $\beta \not\sqsubset_{ir} \alpha$. This, by the general results proved in [7], means that there are $sr', sr'' \in \mathsf{or2SR}(ir)$ such that $\alpha \prec_{sr'} \beta$ and $\beta \prec_{sr''} \alpha$. Then the first implication holds by taking $u = \mathsf{sseq2or}_\omega^{-1}(sr')$ and $w = \mathsf{sseq2or}_\omega^{-1}(sr'')$.

On the other hand, let $ir = \mathsf{or2ir} \circ \mathsf{sseq2or}_\omega(u) = \mathsf{or2ir} \circ \mathsf{sseq2or}_\omega(w)$. Then there exist $sr_u, sr_w \in \mathsf{or2SR}(ir)$ such that $\alpha \prec_{sr_u} \beta$ and $\beta \prec_{sr_w} \alpha$, and so, by the order structure closure, $\alpha \not\rightleftharpoons_{ir} \beta$. This, by the general results proved in [7], means that there exists $sr \in \mathsf{or2SR}(ir)$ such that $\alpha \sqsubset_{sr} \beta \sqsubset_{sr} \alpha$. Hence the second implication holds by taking $v = \mathsf{sseq2or}_\omega^{-1}(sr)$, which ends the proof. □