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LEVI DECOMPOSITION OF NILPOTENT CENTRALISERS IN CLASSICAL GROUPS

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ABSTRACT. We check that the connected centralisers of nilpotent elements in the orthogonal and symplectic groups have Levi decompositions in even characteristic. This provides a justification for the identification of the isomorphism classes of the reductive quotients as stated in [Liebeck, Seitz; Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras].

1. INTRODUCTION

Let G be a linear algebraic group over an arbitrary field k with unipotent radical $U := \mathscr{R}_u(G)$. Then U is by definition a subgroup of $G_{\bar{k}}$, where $G_{\bar{k}}$ is the base change of G to the algebraic closure $\bar{k}$ of k. In fact, the subgroup U is defined to be the largest smooth, connected, unipotent normal subgroup of $G_{\bar{k}}$. If G is smooth, we say G has a *Levi subgroup* L if $G_{\bar{k}} = L_{\bar{k}}U$ and $L_{\bar{k}} \cap U = \{1\}$, scheme-theoretically; that is to say, that the following conditions hold:

$$L_{\bar{k}}(\bar{k}) \cap U(\bar{k}) = \{1\}; \tag{1}$$

$$\mathrm{Lie}(L_{\bar{k}}) \cap \mathrm{Lie}(U) = 0. \tag{2}$$

The existence (or otherwise) of Levi subgroups is a central issue to address in understanding the subgroup structure of linear algebraic groups. When k is a field of characteristic 0, it is an old theorem of G. D. Mostow [Mos56] that all linear algebraic groups have Levi subgroups. Essentially, the proof relies on Lie's theorem and exponentiation, both of which fail over fields of characteristic $p > 0$. Indeed, algebraic groups need not have Levi subgroups over such fields. The points $G(W_2(k))$ of a reductive k-group G over the length 2 Witt vectors $W_2(k)$ furnish an example of such an algebraic group; see [CGP10, §A.6] for a full account. (Also note that a minimal dimensional faithful representation for $G = \mathrm{SL}_2(W_2(k))$ is constructed in [McN03].) In this case one has a short exact sequence $1 \to \mathfrak{g}^{[1]} \to G(W_2(k)) \to G \to 1$, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}^{[1]}$ is its first Frobenius twist as a G-module. Then the (unipotent) vector subgroup $\mathfrak{g}^{[1]} \subseteq G(W_2(k))$ coincides with the unipotent radical of the latter. One can see that this sequence corresponds to an element of the rational (Hochschild) cohomology group $\mathrm{H}^2(G, \mathfrak{g}^{[1]})$ and indeed one has a suite of examples of G-modules V where $\mathrm{H}^2(G, V) \neq 0$ each giving rise to a non-split extension of V by G such that V is the unipotent radical of the extension E with no Levi factor. By contrast, if G is a *connected* linear algebraic group over k with unipotent radical U which is defined over k then [McN14, Thm. B] (see also [Ste13, Thm. 3.3.5]) shows that one can find a filtration of U such that the sections have the structure of modules for G/U, and [McN10] points out that the vanishing of the second Hochschild cohomology of these modules is enough to guarantee a Levi subgroup.

Certain interesting situations arise over an imperfect field k since it is possible that the unipotent radical U may fail to be defined over k. This can happen in particular when one considers the

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case that G is a *pseudo-reductive* group. The main result of the monograph [CGP10] asserts that most pseudo-reductive groups arise from Weil restriction of a reductive group across an inseparable extension of k. Moreover, if G' is a reductive group that happens to be defined over k and k'/k is an inseparable extension, then the Weil restriction $\mathrm{R}_{k'/k}(G'_{k'})$ is a non-reductive linear algebraic group G whose unipotent radical U is not defined over k but which contains a canonical copy of G' as a Levi subgroup. For a general result on the existence of Levi subgroups in pseudo-reductive groups, see [CGP10, Thm. 3.4.6].

In [Jan04, Prop. 5.10], Jantzen shows, using arguments from [Ric67] that when the characteristic p of k is good for G the (smooth) centraliser $C_G(e)$ of a nilpotent elements $e \in \mathrm{Lie}(G)$ for G, a reductive group always has a Levi subgroup. In bad characteristic, this can apparently fail in the exceptional groups (see [LS12, p283])—though we confess we have not been able to elicit any explicit examples from the community.

Let $C_G(e)^\circ_{\mathrm{red}}$ be the unique smooth group whose k-points are the same as that of $C_G(e)^\circ$. This is then *the* centraliser in the sense of [Spr98]. In this short note we wish to make the observation:

Theorem. *Let G be a simple algebraic group of classical type over $k = \bar{k}$ of characteristic 2 and $e \in \mathrm{Lie}(G)$ a nilpotent element. Then $C_G(e)^\circ_{\mathrm{red}}$ has a Levi decomposition.*

Remarks 1.1. (i) Note that the centralisers of nilpotent elements in bad characteristic need not be smooth, so that $C_G(e)^\circ_{\mathrm{red}} \subsetneqq C_G(e)^\circ$. For example, if $G = \mathrm{SL}_p$ then the connected centraliser of a regular nilpotent element will contain the non-smooth centre Z of G while the group of k-points will consist only of unipotent elements. By contrast, if $G = \mathrm{PGL}_2$ over a field of characteristic 2 then the centraliser of a regular nilpotent element $e \in \mathrm{Lie}(U)$ for $U = \mathscr{R}_u(B)$ and B a Frobenius stable Borel subgroup will be the non-smooth group U_1U where U_1 is the non-smooth Frobenius kernel of the unipotent radical of the opposite Borel.

Though it seems one could define a Levi subgroup of a non-smooth group H as a subgroup L such that $L\mathscr{R}_u(H) = H$ and $L \cap \mathscr{R}_u(H) = 1$ as schemes, this appears to be avoided in the literature, so that Levi subgroups are only defined for smooth groups; see [CGP10, §3.4]. It remains to be seen how much value there is in the extension of the definition to non-smooth groups, and so we have not considered that question here—this is why the theorem is stated in terms of $C_G(e)^\circ_{\mathrm{red}}$.

(ii) Even when an algebraic group H admits a Levi subgroup for H°, there may fail to be one for H: it is easy to find a finite group F and a kF-module V such that $\mathrm{H}^2(F, V) \neq 0$. Then one can build an algebraic group H, a non-split extension of V by F such that $H^\circ = V$—so that H° has a Levi subgroup (the identity subgroup)—but such that H itself does not.

Nevertheless, the stated component groups in [LS12, Thm. 4.1] are all elementary abelian 2-groups and are found to be generated by involutions in $C_G(e)_{\mathrm{red}}$ in a series of arguments [LS12, Lem. 5.13–5.22,§5.5.1]. Therefore we in fact deduce the following corollary.

Corollary 1.2. *Let G be a simple algebraic group of classical type over $k = \bar{k}$ of characteristic 2 and $e \in \mathrm{Lie}(G)$ a nilpotent element. Then $C_G(e)_{\mathrm{red}}$ has a Levi decomposition, $C_G(e)_{\mathrm{red}} = L\mathscr{R}_u(C_G(e)_{\mathrm{red}})$. Here L itself a semidirect product $F \ltimes L^\circ$, with $F \subseteq L$ a finite subgroup.*

Most of the work in proving the theorem is done by [LS12], which finds a subgroup L of $C_G(e)^\circ$ satisfying (1) above. It remains to show that (2) holds. Chasing through the proof of [LS12, Prop. 5.11] and applying a result of Vasiu we show this is the case.

Having established the existence of a subgroup L satisfying (1), the authors of [LS12] do not appear to have made an attempt to justify their statement in [LS12, Thm. 5.6] that there is an isomorphism

$C_G(e)^\circ_{\mathrm{red}}/\mathscr{R}_u(C_G(e)^\circ_{\mathrm{red}}) \cong L$ as algebraic groups and indeed this map can fail to be an isomorphism of algebraic groups, precisely when (2) does not hold. Hence our theorem provides the missing justification.

2. Proof of the theorem

In this section k will denote an algebraically closed field of characteristic 2.

The following is a brief version of [LS12, Thm. 5.6]. As explained in the introduction, the proof in *op. cit.* only establishes the isomorphisms at the level of the abstract groups of points.

Theorem 2.1. *Let e be a nilpotent element of $\mathrm{Lie}(G)$ where $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$ and V is a vector space over k. Then there are integers m_i and a_i such that:*

- (i) *If $G = \mathrm{Sp}(V)$, then $C_G(e)^\circ_{\mathrm{red}}/\mathscr{R}_u(C_G(e)) \cong \prod_i \mathrm{Sp}_{2a_i}$.*
- (ii) *If $G = \mathrm{O}(V)$ then $C_G(e)^\circ_{\mathrm{red}}/\mathscr{R}_u(C_G(e)) \cong \prod_{m_i} \mathrm{Sp}_{2a_i} \times \prod_{m_i} I_{a_i}$, where I_{a_i} is either SO_{2a_i} or SO_{2a_i+1}.*

A technical condition related to the action of e on V determines the integers a_i and m_i and the condition by which one decides the isomorphism class of I_{a_i}. Then [LS12, Prop. 5.11] finds subgroups C such that $C_G(e)_{\mathrm{red}} = C\mathscr{R}_u(C_G(e))$.

To prove our theorem, we use [Vas05, Thm. 1.2]. Recall that for a field k of characteristic p, α_p denotes the height 1 group scheme whose representing Hopf algebra is $k[X]/(X^p)$, the comultiplication being determined by $\Delta(X) = 1 \otimes X + X \otimes 1$. (It is also the first Frobenius kernel of the smooth additive group $\mathbb{G}_a$.) For us, *loc. cit.* takes the form:

Theorem 2.2 (Vasiu)**.** *Let G be a reductive group over k. If G has a non-trivial normal unipotent subgroup scheme then $\mathrm{char}\, k = 2$ and G has a direct factor isomorphic to SO_{2n+1}. Furthermore, if $G = \mathrm{SO}_{2n+1}$ then $U \cong \alpha_2^{2n}$ is the unique such; and $\mathrm{Lie}(U)$ is a $2n$-dimensional module for SO_{2n+1} of high weight ϖ_1.*

Remark 2.3. In the theorem above, the $2n$-dimensional module $L(\varpi_1)$ is obtained as a quotient of the 'natural' Weyl module $V(\varpi_1)$ by the radical of its form; see [Jan03, II.8.21] for a brief discussion.

As is rather well-known (see [Vas05, 2.1]) we have that $\mathrm{SO}_{2n+1}/U \cong \mathrm{Sp}_{2n}$, where $U \cong \alpha_2^{2n}$ is its infinitesimal unipotent normal subgroup. The following is now immediate from the theorem and the fact that $L(\varpi_1)$ is irreducible.

Corollary 2.4. *Let G be a linear algebraic group over k admitting a reductive subgroup C such that $G = C\mathscr{R}_u(G)$. Then either the quotient map $q : G \to G/\mathscr{R}_u(G)$ restricts to an isomorphism on C or C contains a direct factor H isomorphic to SO_{2n+1} and the image of H under q is isomorphic to Sp_{2n}.*

Proof of Theorem. In [LS12, Prop. 5.11] a subgroup $C \subseteq C_G(e)$ is constructed such that $C_G(e)^\circ_{\mathrm{red}} = C\mathscr{R}_u(C_G(e))$. One finds that C contains direct factors of type SO_{2n+1} only if G is $\mathrm{O}(V)$ for some V, hence Corollary 2.4 implies $\mathrm{Lie}(C) \cap \mathrm{Lie}(\mathscr{R}_u(C_G(e)))$ is trivial when $G = \mathrm{Sp}_{2n}$.

Hence we assume G is $O(V)$ and C contains a direct factor isomorphic to SO_{2r+1}. The proof of [LS12, Prop. 5.11] proceeds by finding an orthonormal basis for V and describing explicitly the action of e on V. One finds that the action of e on V is constructed as a direct sum of non-isomorphic indecomposable ke-modules which are labelled $W(m_i)$ and $W_l(n)$; a basis of these modules and explicit action of e is given in [LS12, §5.1]. The multiplicity of the module $W(m_i)$ is labelled a_i, thus $W(m_i)^{a_i}$ appears as a direct ke-summand of V. Furthermore, a certain 1-dimensional torus $T \subset G$ associated to e is constructed which stabilises each of the indecomposable ke-modules above. Then C is constructed as a subgroup of $C_G(T, e) = C_G(T) \cap C_G(e)$. It turns out that the non-zero weight spaces of T on $C_G(e)$ are all of positive weight; denoting the corresponding subgroup by $C_G(e)_{>0}$ we have $C_G(e)_{>0} \subseteq \mathscr{R}_u(C_G(e))$. Thus it suffices to show that $\mathscr{R}_u(C_G(T, e)) \cap C = \{1\}$, scheme-theoretically.

We proceed by identifying, for each direct factor H of type SO_{2r+1} in $C_G(e)$, a $C_G(T, e)$-submodule of V on which H acts faithfully and on which $\mathscr{R}_u(C_G(T, e))$ acts trivially. This is enough to prove the theorem.

Since C contains a direct factor isomorphic to SO_{2r+1} we have from [LS12, Lem. 5.10] that V contains a summand of the form $W_l(n)$ with $2(n-l) \leq m_i \leq 2l-1$. Then following the proof of *loc. cit.* we obtain an action of SO_{2a_i+1} on the zero weight space Z_0 of the module $Z := W(m_i)^{(a_i)} \perp W_l(n)$. Given the explicit description of the modules $W(m_i)$ and $W_l(n)$ from [LS12, §5.1], we have that Z_0 is non-degenerate of dimension $2a_i + 2$. Then the proof of [LS12, Lem. 5.10] describes SO_{2a_i+1} as acting on Z_0 as the indecomposable module with successive factors being the trivial module k, $L(\varpi_1)$ and k again (or k, $L(2\varpi_1)$, k if $Y \cong \mathrm{SO}_3 = \mathrm{PGL}_2$). Since the natural module for SO_{2a_i+1} is isomorphic to the unique codimension 1-submodule of Z_0, we have that SO_{2a+1} acts faithfully on this module. As is well-known, SO_{2a_i+1} is contained in no parabolic subgroup of O_{2a_i+2}. Hence by the Borel–Tits theorem, the image of $C_G(T, e)$ in O_{2a_i+2} must be reductive. Thus its unipotent radical $\mathscr{R}_u(C_G(T, e))$ acts trivially on the faithful SO_{2a_i+1}-module Z_0 as required. □

3. A question

It is possible for a reductive subgroup L of an algebraic group $G = LU$ to satisfy (1) but not (2). This occurs specifically when $L = \mathrm{SO}_{2n+1} \subset G := \mathrm{Sp}_{2n} \ltimes V$ where V is the natural module for Sp_{2n}. Nevertheless, G evidently does have a Levi subgroup. In light of this, we raise the following question.

Question 3.1. *Suppose G is an algebraic group over $k = \bar{k}$ with unipotent radical U, and L is a subgroup which satisfies $G(k) = L(k)U(k)$. Must G have a Levi factor L' such that $G = L' \ltimes U$?*

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