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Finite Section Method in a Space with Two Norms.

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FINITE SECTION METHOD IN A SPACE WITH TWO NORMS

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Don Sarason, in memoriam

Abstract. We compare the finite central truncations of a given matrix with respect to two non-equivalent Hilbert space norms. While the limit sets of the finite sections spectra are merely located via numerical range bounds, the weak $*$-limits of the counting measures of these spectra are proven in general to be gravi-equivalent with respect to the logarithmic potential in the complex plane. Classical methods of factorization of Volterra type or Wiener-Hopf type operators lead to a series of effective criteria of asymptotic equivalence, or uniform boundedness of the two sequences of truncations. Examples from function theory, integral equations and potential theory complement the theoretical results.

1. Introduction

Weaker norm estimates often simplify the analysis of concrete operators, helping to locate spectra, obtain resolvent estimates, prove stability and so on. It is known for instance that the double layer potential, also known as the Poincaré-Neumann operator, has real spectrum on Lebesgue L^2-space of the respective boundary, but the explanation is far from obvious: this linear transform is self-adjoint only with respect to a weaker topology Hilbert space (a Sobolev space of negative fractional order). Moreover, the spectrum of the Neumann-Poincaré operator is highly sensitive on the choice of the underlying normed space, see for instance [1, 13]. It is therefore natural to ask how the finite rank truncations of a given operator behave when modifying the inner space structure. Such an inquiry was started in the note [13], but soon it become clear that a general framework is lurking in the background.

The aim of the present article is to collect a series of general observations related to the comparison of finite central truncations of a prescribed infinite matrix,

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with respect to two non-equivalent Hilbert space norms. Norm relaxations in the spectral analysis on the Sobolev scale, L^p scale or on Gelfand pairs are ubiquitous in applied mathematics, see for instance [11]. Added to these is the high degree of sophistication reached by the finite central truncation technique as part of Galerkin approximation or Krylov subspace method, see for instance [19, 5, 15, 17]. And even more, operator algebra experts have recognized in the finite central truncation scheme some familiar approximation schemes [2, 4, 6]. By contrast, our approach is rather elementary, exploiting solely classical works of the Ukrainian school of functional analysis, which in its turn has roots in the theory of Volterra or Wiener-Hopf type operators [14, 3, 8, 9].

Quite specifically, we focus on the two sequences of compressions $T_n = P_nTP_n$ and $\tilde{T}_n = Q_nTQ_n$ of a given linear transformation T, where P_n and Q_n are projections onto the same finite dimensional subspace $H_n = P_nH$, but they are orthogonal with respect to two different norms. The projections P_n are orthogonal and converge monotonically to the identity on a Hilbert space H, while Q_n are orthogonal with respect to the inner product $\langle A\cdot, \cdot\rangle$ induced by a positive, in general non-invertible, linear operator on H. We compare the spectra of T_n and T_n regarded as linear transforms of H_n and try to link the weak-* limits of the counting measures of these spectra, when n tends to infinity. The last section incorporates some examples supporting this quest. We do not always expect simple answers, as the approximation theory of Toeplitz matrices amply testifies, even under a single inner product [5].

It turns out that bounds of the operator norm gap $\|T_n - \tilde{T}_n\|$ or the trace-class gap $|T_n - \tilde{T}_n|_1$ are at the key quantitative indicators to look for. Several competing factors contribute to the effective evaluation of these bounds: the adaptation of the chain of finite dimensional subspaces to the two norms, the matrix structure of the operator T on the given chain of subspaces, or the intrinsic properties of T such as compactness or quasi-diagonality. In complete analogy to the Cholesky factorization of Volterra type operators [7, 9], a weaker norm induced by a positive operator of the form

$$A = (I + L)D(I + L^*)$$

is "universally good" for all matrices T. Above D is block-diagonal with respect to the chain of subspaces (H_n) while L is strictly lower-triangular and compact. The departure from the standard theory is the non-invertibility of D. Several results of this note concur to the this picture. By "universally good" we mean the existence of a uniform bound for the operator norm gap $\|T_n - \tilde{T}_n\|$. If moreover,

$$\lim_n \frac{\dim H_n - \dim H_{n-1}}{\dim H_n} = 0,$$

and the matrix attached to T has a Hessenberg structure, that is at most the first block sub-diagonal is non-zero, then the trace-class norm gap is bounded, and in this case any two weak-* limits of the counting measures of the spectra of T_n

and $\tilde{T}_n$ have the same logarithmic potential at infinity (sometimes called gravi-equivalent measures). This scenario applies in particular to complex orthogonal polynomials. A special role is played by a compact operator A and the chain of subspaces spanned by its eigenvectors. These form a system of doubly orthogonal vectors with respect to the two norms, a concept well isolated and exploited in function theory and potential theory [10]. We analyze what kind of perturbations of such a doubly orthogonal system of vectors still produces a "universally good" chain of subspaces.

The general framework proposed in the subsequent sections provides only a basis for analyzing several specific situations. While our setting was motivated by the latter, in the present article we remain at the abstract level, leaving for future works the return to, and enhancement of, the original sources.

2. Preliminaries

Let H be a complex separable Hilbert space and $T \in \mathcal{L}(H)$ a linear bounded operator acting on H. The spectrum of T is denoted $\sigma(T)$ and the numerical range of T is $W(T) = \{\langle Tx, x\rangle, \quad \|x\| = 1\}$. By a Theorem of Hausdorff and Toeplitz we know that the closure of $W(T)$ is a compact set, containing $\sigma(T)$.

We endow H with a weaker pre-hilbertian space norm:

$$(x, y) = \langle Ax, y\rangle, \qquad x, y \in H,$$

where $A > 0$ is a positive, non-invertible bounded linear operator acting on H. Let K denote the Hilbert spec completion of H with respect to the new norm. We have $H \subset K$, with dense range inclusion. Sometimes we will call A-norm, or A-convergence the respective entities induced by the norm of the Hilbert space K denoting

$$\|x\|_A^2 = \langle Ax, x\rangle, \quad x \in H.$$

Let $H_n \subset H$ be an increasing sequence of finite-dimensional subspaces, whose union is dense in H. Unless otherwise stated we link the operator T to the chain of subspaces (H_n) by the assumption

$$T(H_n) \subset H_{n+1}, \quad n \geq 0.$$

This means that the block matrix decomposition of T with respect to the orthogonal direct sum $H = H_0 \oplus (H_1 \ominus H_0) \oplus (H_2 \ominus H_1) \oplus \ldots$ has only the first sub-diagonal non-zero. This structure is known in numerical analysis as a block Hessenberg matrix. Remark that we do not assume T to be bounded as a linear transformation from K to K. However, T can be regarded as a densely defined operator on K and the graph of T turns out to be closed in the A-norm.

We denote by P_n the orthogonal projection of H onto H_n, and by Q_n the orthogonal projection of K onto H_n. Note that $P_n \to I$ in the strong operator topology of $\mathcal{L}(H)$.

The two sets of projections satisfy:

$$P_nQ_n = Q_n, \quad Q_nP_n = P_n$$

when regarded as linear endomorphisms of H. The compression $A_n = P_nAP_n$ of the operator A to the subspace H_n is positive, hence invertible.

Lemma 2.1. *For every vector $x \in H$ one has*

$$Q_nx = A_n^{-1}P_nAx. \tag{2.1}$$

Proof. Indeed, for every vector $y \in H_n$ we obtain

$$(x - A_n^{-1}P_nAx, y) = \langle A(x - A_n^{-1}P_nAx), y\rangle =$$
$$\langle P_nAx - P_nAP_nA_n^{-1}P_nAx, y\rangle = \langle P_nAx - P_nAx, y\rangle = 0.$$

Moreover, for every $y \in H_n$:

$$A_n^{-1}P_nAy = A_n^{-1}A_ny = y.$$

□

We are concerned with the asymptotic behavior of the spectra of the finite central truncations $T_n = P_nTP_n$, versus $\tilde{T}_n = Q_nTQ_n$, regarded as endomorphisms of the finite dimensional space H_n. Besides the set theoretic distance between spectra, it is customary to look at the finite atomic counting measures μ_n, respectively $\tilde{\mu}_n$ defined on complex polynomials as:

$$\int d\mu_n = \frac{1}{\dim H_n}\text{trace } p(T_n), \quad p \in \mathbb{C}[z],$$

and similarly

$$\int d\tilde{\mu}_n = \frac{1}{\dim H_n}\text{trace } p(\tilde{T}_n), \quad p \in \mathbb{C}[z].$$

While normalized trace norm convergence

$$\lim_{n\to\infty} \frac{1}{\dim H_n}\text{trace}|T_n - \tilde{T}_n| = 0,$$

and the uniform bound

$$\sup_n \|\tilde{T}_n\| < \infty,$$

imply

$$lim_{n\to\infty}(\int pd\mu_n - \int pd\tilde{\mu}_n) = 0,$$

for every complex polynomial p, and hence the weak-$*$ limit points of the measures μ_n, respectively $\tilde{\mu_n}$, have the same complex moments, we will impose weaker convergence conditions, with more modest consequences.

Remark that the support of the counting measure μ_n coincides with the spectrum of T_n and in particular is contained in the numerical range $W(T)$, simply because by definition $W(T_n) \subset W(T)$.

Lemma 2.2. *Assume that* $\limsup \|T_n - \tilde{T}_n\| = r$*. Then the support of any weak-$*$ limit point of the measures* $\tilde{\mu}_n$ *is contained the closed* r *neighborhood of* $W(T)$*.*

Proof. Any point belonging to the closed support a weak-$*$ limit of the measures $\tilde{\mu}_n$ is a limit of a sub-sequence of points belonging to the respective supports.

Let $\lambda_n \in W(\tilde{T}_n)$. That is, there is a vector $x \in H_n$, of unit length, so that $\lambda_n = \langle \tilde{T}_n x, x\rangle$. Then

$$|\langle T_n x, x\rangle - \langle \tilde{T}_n x, x\rangle| \leq \|T_n - \tilde{T}_n\|,$$

whence

$$\operatorname{dist}(\lambda_n, W(T)) \leq \|T_n - \tilde{T}_n\|.$$

Assume that $\lambda = \lim_k \lambda_{n(k)}$, where $n(k)$ is a subsequence converging to infinity. Then

$$\operatorname{dist}(\lambda, W(T)) \leq \limsup \|T_n - \tilde{T}_n\|.$$

□

A weaker assumption, leading to the same conclusion of the Lemma is

$$\limsup w(T_n - T) \leq r,$$

where $w(A) = \sup\{|\lambda|,\ \lambda \in W(A)\}$ denotes the numerical radius of an operator. Obviously one can refine the conclusion of the above lemma by working with the upper limit set of the sequences of spectra $\sigma(T_n)$, invoking the theory of psudospectra [17]. A notable particular case is described below.

Corollary 2.3. *Assume that operator* T *is normal and* $\lim \|T_n - \tilde{T}_n\| = 0$*. Then the support of any weak-$*$ limit point of the measures* $\tilde{\mu}_n$ *is contained in the convex hull of the spectrum of* T*.*

Proof. The numerical range of a normal operator coincides with the convex hull of the spectrum. □

3. Asymptotic equivalence of finite central truncations

Keeping intact the notations of the previous section we focus on the distance between the finite central truncations T_n and $\tilde{T}_n$. The starting point is the following simple identity.

Lemma 3.1. *Let* $T \in \mathcal{L}(H)$ *be an operator satisfying* $TH_n \subset H_{n+1}$ *for all* $n \geq 0$*. Then*

$$\tilde{T}_n - T_n = A_n^{-1} P_n A (P_{n+1} - P_n) T P_n, \quad n \geq 0.$$

Proof. Let $x, y \in H_n$. Then

$$(\tilde{T}_n x, y) = (Q_n T Q_n x, y) = (T Q_n x, Q_n y) = (Tx, y) = \langle ATx, y\rangle$$

and on the other hand

$$(\tilde{T}_n x, y) = \langle A\tilde{T}_n x, y\rangle = \langle A P_n \tilde{T}_n x, P_n y\rangle = \langle A_n \tilde{T}_n x, y\rangle.$$

In addition,

$$\langle ATx, y\rangle = \langle P_n ATP_n x, y\rangle =$$
$$\langle A_n P_n TP_n x, y\rangle + \langle P_n A(I - P_n)TP_n x, y\rangle.$$

Therefore

$$A_n\tilde{T}_n = A_n T_n + P_n A(I - P_n)TP_n$$

and the conclusion follows from the observation $(I - P_n)TP_n = (P_{n+1} - P_n)TP_n$. □

If the operator T is not necessarily adapted to the chain of subspaces (H_n) we obtain a similar formula:

$$\tilde{T}_n - T_n = A_n^{-1} P_n A(I - P_n)TP_n, \quad n \geq 0. \tag{3.1}$$

The residual term $X_n = A_n^{-1}P_nA(P_{n+1} - P_n)$ naturally appears in the Cholesky type decomposition of the one-step extension of the matrix A_n, to A_{n+1}. Specifically, the matrix A_{n+1} has the following block structure factorization with respect to the orthogonal decomposition $H_{n+1} = H_n \oplus (H_{n+1} \ominus H_n)$.

$$A_{n+1} = \begin{pmatrix} A_n & P_nA(P_{n+1} - P_n) \\ (P_{n+1} - P_n)AP_n & (P_{n+1} - P_n)A(P_{n+1} - P_n) \end{pmatrix} =$$
$$\begin{pmatrix} I & 0 \\ X_n^* & I \end{pmatrix} \begin{pmatrix} A_n & 0 \\ 0 & D_{n+1} \end{pmatrix} \begin{pmatrix} I & X_n \\ 0 & I \end{pmatrix}$$

where $D_{n+1} = (P_{n+1} - P_n)A(P_{n+1} - P_n) - X_n^* A_n X_n$. This matrix factorization, sometimes abridged in numerical analysis by the initials LDU, was instrumental in the classical study of triangular models of Volterra type operators, see [9]; later the same factorization phenomenon provided the basis of the theory of nest algebras [6, 4].

On the more general setting side, we isolate below a basic matrix identity necessary for our study. Assume that the original Hilbert space is decomposed into two orthogonal subspaces:

$$H = H_0 \oplus H_1,$$

with orthogonal projections P_0 and P_1, respectively. The weaker norm is induced by a positive and bounded operator A of the form

$$A = LDL^*,$$

where L^* leaves invariant H_0 and $D > 0$ leaves invariant both H_0 and H_1. Note that D may not be invertible. In matrix form:

$$A = \begin{pmatrix} L_{00} & 0 \\ L_{10} & L_{11} \end{pmatrix} \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} L_{00}^* & L_{10}^* \\ 0 & L_{11}^* \end{pmatrix}.$$

Lemma 3.2. *Assume, with the above conventions, that L_{00} is invertible. Then the projection Q_0 in the A-norm of H onto H_0 is*

$$Q_0 = (L_{00}^*)^{-1} P_0 L^*. \tag{3.2}$$

Proof. We check separately the formula for elements of H_0 and H_1. If $x \in H_0$, then

$$(L_{00}^*)^{-1}P_0L^*x = (L_{00}^*)^{-1}P_0L00^*x = (L_{00}^*)^{-1}L_{00}^*x = x.$$

For $y \in H_1$ and $x \in H_0$ we find

$$\langle A(y - (L_{00}^*)^{-1}P_0L^*y, x\rangle = \langle P_0A(y - (L_{00}^*)^{-1}P_0L^*y, x\rangle =$$
$$\langle P_0LDL^*(y - (L_{00}^*)^{-1}P_0L^*y, x\rangle = \langle L_{00}D_0P_0L^*(y-, x\rangle =$$
$$\langle L_{00}D_0P_0L^*y - L_{00}D_0P_0L^*P_0(L_{00}^*)^{-1}P_0L^*y, x\rangle =$$
$$\langle L_{00}D_0P_0L^*y - L_{00}D_0L_{00}^*(L_{00}^*)^{-1}P_0L^*y, x\rangle = 0.$$

□

A rather general statement derived from these simple observations follows.

Theorem 3.3. *Let H be a separable Hilbert space endowed with a weaker norm and let $(H_n)_{n=0}^{\infty}$ be an increasing chain of finite dimensional subspaces converging to H. Assume that*

$$\lim_n \frac{\dim H_n - \dim H_{n-1}}{\dim H_n} = 0. \tag{3.3}$$

If the central finite truncations (T_n) and $(\tilde{T}_n)$ associated to a linear bounded operator $T \in \mathcal{L}(H)$ which satisfies $T(H_n) \subset H_{n+1}, \quad n \geq 0$, remain close in norm:

$$\sup_n \|T_n - \tilde{T}_n\| < \infty, \tag{3.4}$$

then any weak-$*$ cluster measures $\mu, \tilde{\mu}$ of the counting measures of the spectra of T_n, respectively $\tilde{T}_n$, have the same logarithmic potentials outside the convex hull of their supports.

In practice the spaces H_n are formed by polynomials of degree less than or equal to n on a support of d variables, which is not constrained by an algebraic dependence. Then condition (3.3) is satisfied.

Proof. We simply remark that $\mathrm{rank}(T_n - \tilde{T}_n) \leq \dim H_{n+1} - \dim H_n$, whence

$$\mathrm{trace}|T_n - \tilde{T}_n| \leq (\dim H_{n+1} - \dim H_n)\|T_n - \tilde{T}_n\|, \ n \geq 0.$$

On the other hand, $\sup_n \|\tilde{T}_n\| < \infty$ by the first assumption in the statement. We infer that for every polynomial $p \in \mathbf{C}[z]$,

$$\lim_n \frac{1}{\dim H_n} \mathrm{trace}(p(T_n) - p(\tilde{T}_n)) = 0.$$

That is, any two limit points μ, respectively $\tilde{\mu}$, in the weak-$*$ topology of the counting measures of the eigenvalues of the matrices T_n and $\tilde{T}_n$ have the same complex moments

$$\int z^d d\mu(z) = \int z^d d\tilde{\mu}, \quad d \geq 0,$$

or equivalently, their Cauchy transforms coincide in a neighborhood of infinity:

$$\int \frac{d\mu(z)}{z-\zeta} = \int \frac{d\tilde{\mu}(z)}{z-\zeta}, \quad |\zeta| >> 1.$$

The conclusion follows from the definition of the logarithmic potential and the fact that the two measures have compact support.

□

We are interested in conditions assuring $\lim_{n\to\infty} \|X_n T P_n\| = 0$. One obvious instance being the block-diagonal structure of the operator A, with respect to the chain of subspaces H_n:

$$A = \text{diag } (D_0, D_1, D_2, \ldots). \tag{3.5}$$

A second sufficient condition is

$$\lim_{n\to\infty} \|A_n^{-1} P_n A (P_{n+1} - P_n)\| = 0 \tag{3.6}$$

and a third

$$\sup_n \|A_n^{-1} P_n A (P_{n+1} - P_n)\| < \infty \quad \text{and} \quad \lim_{n\to\infty} \|(I - P_n) T P_n\| = 0. \tag{3.7}$$

Next we show that the latter two sufficient conditions for the asymptotic equivalence of the two sequences of finite central truncations are not affected by a structured perturbation of the operator A. By a *strictly lower triangular operator* with respect to the chain of subspaces $(H_n)_{n=0}^{\infty}$ we mean an element $L \in \mathcal{L}(H)$ satisfying

$$P_n L = P_n L P_{n-1}, \quad n \geq 1,$$

or more intuitively and equivalently

$$L^* P_n = P_{n-1} L P_n, \quad n \geq 1,$$

which in turn means $L^*(H_n) \subset H_{n-1}$.

Theorem 3.4. *Let $(x, y) = \langle Ax, y\rangle$ be a second, weaker inner product structure on a Hilbert space H, implemented by the positive operator $A \in \mathcal{L}(H)$. Denote by P_n the orthogonal projections on H_n, and set $A_n = P_n A P_n$.*

Assume that either

$$\lim_{n\to\infty} \|A_n^{-1} P_n A (P_{n+1} - P_n)\| = 0 \tag{3.8}$$

or

$$\sup_n \|A_n^{-1} P_n A (P_{n+1} - P_n)\| < \infty. \tag{3.9}$$

Consider a lower triangular operator $L \in \mathcal{L}(H)$, and the multiplicative perturbation $B = (I + L) A (I + L^*)$.

If L is compact, then (3.8) implies

$$\lim_{n\to\infty} \|B_n^{-1} P_n B (P_{n+1} - P_n)\| = 0. \tag{3.10}$$

If $I + L$ is invertible, then (3.9) implies

$$\sup_n \|B_n^{-1} P_n B(P_{n+1} - P_n)\| < \infty. \tag{3.11}$$

Proof. If L compact and strictly lower triangular then $I+L$ is invertible. Indeed, in this case $\ker(I + L) = 0$ by a Gauss elimination argument. Then $I + L$ is Fredholm of index zero, hence $I + L$ has closed range and $\ker(I + L^*) = 0$.

Assume that one of the conditions in the statement affecting $A_n^{-1} P_n A(P_{n+1} - P_n)$ holds true. Denote, as customary by now, $L_n = P_n L P_n$ and consider the matrix decomposition

$$I + L_{n+1} = \begin{pmatrix} I + L_n & 0 \\ G_n & F_n \end{pmatrix}$$

where $F_n = (P_{n+1} - P_n)(I + L)(P_{n+1} - P_n)$ and $G_n = (P_{n+1} - P_n)LP_n$. Note that $I + L_n$ is invertible and that $\sup_n \|G_n\| < \infty$ together with $\sup_m \|F_n\| \leq \|I + L\|$.

The LD factorization is in order:

$$I + L_{n+1} = \begin{pmatrix} I + L_n & 0 \\ G_n & F_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ G_n(I + L_n)^{-1} & I \end{pmatrix} \begin{pmatrix} I + L_n & 0 \\ 0 & F_n \end{pmatrix}.$$

Next we multiply $I + L_{n+1}$ by the left factor of A_{n+1}:

$$(I + L_{n+1}) \begin{pmatrix} I & 0 \\ X_n^* & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ G_n(I + L_n)^{-1} & I \end{pmatrix} \begin{pmatrix} I + L_n & 0 \\ F_n X_n^* & F_n \end{pmatrix} =$$

$$\begin{pmatrix} I & 0 \\ G_n X_n^*(I + L_n)^{-1} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ F_n X_n^*(I + L_n)^{-1} & I \end{pmatrix} \begin{pmatrix} I + L_n & 0 \\ 0 & F_n \end{pmatrix}.$$

All in all

$$P_{n+1} B P_{n+1} = P_{n+1}(I + L)P_{n+1} A P_{n+1}(I + L^*)P_{n+1} =$$

$$\begin{pmatrix} I & 0 \\ (G_n + F_n X_n^*)(I + L_n)^{-1} & I \end{pmatrix} \begin{pmatrix} B_n & 0 \\ 0 & F_n D_{n+1} F_n^* \end{pmatrix} \times$$

$$\times \begin{pmatrix} I & (I + L_n^*)^{-1}(G_n^* + X_n F_n^*) \\ 0 & I \end{pmatrix}.$$

Consequently a closed form expression for the perturbed key term follows:

$$B_n^{-1} P_n B(P_{n+1} - P_n) = (I + L_n^*)^{-1}(G_n^* + X_n F_n^*).$$

If L is compact, then $G_n = (P_{n+1} - P_n)LP_n$ tends to zero in norm. It remains to remark that in either case $\sup_n \|(I + L_n^*)^{-1}\| < \infty$. This can be inferred from the fact that the inverse of $I + L$ is also lower triangular, hence $(I + L_n)^{-1} = P_n(I + L)^{-1} P_n$ for all $n \geq 0$. □

On the abstract side, we can now construct a large class of examples of finite central truncations with the same operator norm asymptotics.

Corollary 3.5. *Let $T \in \mathcal{L}(H)$ be a linear bounded operator possessing a block Hessenberg matrix with respect to the complete and increasing chain of finite dimensional subspaces $(H_n)_{n=0}^{\infty}$. Let $D \in \mathcal{L}(H)$ be a positive compact block-diagonal operator and let $L \in \mathcal{L}(H)$ be any compact, strictly lower triangular operator, both with respect to the same chain $(H_n)_{n=0}^{\infty}$. Define a pre-hilbert space structure on H by*

$$(x, y) = \langle (I + L)D(I + L^*)x, y\rangle, \quad x, y \in H.$$

Then the finite central truncations T_n and $\tilde{T}_n$ defined by orthogonal projections onto H_n in the two norms satisfy

$$\lim_n \|T_n - \tilde{T}_n\| = 0.$$

In view of Theorem 3.3 one can relax the structure of the weak norm, with control of the logarithmic potential of the limit of counting measures, provided the dimensions in the chain (H_n) have a moderate growth. Specifically, we state mutates mutandis the following observation.

Corollary 3.6. *Let $T \in \mathcal{L}(H)$ be a linear bounded operator possessing a block Hessenberg matrix with respect to the complete and increasing chain of finite dimensional subspaces $(H_n)_{n=0}^{\infty}$. Let $D \in \mathcal{L}(H)$ be a positive compact block-diagonal operator and let $L \in \mathcal{L}(H)$ be any bounded strictly lower triangular operator, both with respect to the same chain $(H_n)_{n=0}^{\infty}$. Define a pre-hilbert space structure on H by*

$$(x, y) = \langle (I + L)D(I + L^*)x, y\rangle, \quad x, y \in H.$$

Then the finite central truncations T_n and $\tilde{T}_n$ defined by orthogonal projections onto H_n in the two norms satisfy

$$\sup_n \|T_n - \tilde{T}_n\| < \infty.$$

The classical factorization theory of integral operators[8, 9] offers a second, useful criteria of producing admissible weak norm, as above, via small additive perturbations of a diagonal operator.

Theorem 3.7. *Let $(H_n)_{n=0}^{\infty}$ be a chain of finite dimensional spaces in a Hilbert space H with $\dim H_n = n, n \geq 0$, and let $D > 0$ be a positive, diagonal operator with respect to the chain (H_n).*

Assume that $F \in \mathcal{L}(H)$ is a Hilbert-Schmidt operator on H, which is D-symmetric (i.e. $DF = F^*D$). If $D + DF > 0$, then there is a bounded diagonal operator D' and a strictly lower triangular operator $L \in \mathcal{L}(H)$ with the property

$$D + DF = (I + L)D'(I + L^*).$$

Proof. According to Krein's boundedness criterion [14], the operator F is bounded in the D-norm, and has the same spectrum with respect to H and the D-norm.

In particular, the spectrum of F is real, and -1 is not an eigenvalue, otherwise $D(I+F)$ would not be positive. By the invariance of Fredholm index under compact perturbations, $I+F$ is an invertible operator. According to [7] Theorem XXII.4.2, or see [9] Chapter IV for an earlier version, a factorization of the type

$$I+F=(I+L)E(I+U)$$

exists, where E is a diagonal bounded operator with respect to the chain (H_n), L is a strictly lower triangular, Hilbert-Schmidt operator on H and U is a strictly upper triangular, Hilbert-Schmidt operator on H. Note that $I+L$ and $I+U$ are also invertible, by the same index invariance. Then $I+F^*=(I+U^*)E^*(I+L^*)$ is a similar factorization, and they are related by the intertwining formula

$$D(I+F)=(I+F^*)D,$$

that is

$$D(I+L)E(I+U)=(I+U^*)E^*(I+L^*)D.$$

Therefore

$$(I+U^*)^{-1}D(I+L)E=E^*(I+L^*)D(I+U)^{-1}.$$

But the left hand side of the latter identity is lower triangular, while the righthand side is upper triangular, hence both are diagonal. That is

$$(I+U^*)^{-1}D(I+L)E=DE=E^*D=E^*(I+L^*)D(I+U)^{-1}.$$

This proves that $E=E^*$ and

$$D(I+L)=(I+U^*)D.$$

In conclusion

$$D(I+F)=D(I+L)E(I+U)=(I+U^*)DE(I+U)$$

and the theorem is proved. □

Since the operator F in the statement of the Theorem is D-bounded, there exists a linear bounded operator $S\in\mathcal{L}(H)$, so that

$$\sqrt{D}F=S\sqrt{D}.$$

Then

$$DF=\sqrt{D}S\sqrt{D}$$

is self-adjoint, and the spectrum of S coincides with the spectrum of F, with respect to either space. The perturbation in the statement of the theorem becomes

$$D+DF=D+\sqrt{D}S\sqrt{D},$$

with S a Hilbert-Schmidt operator on H.

4. Krylov subspaces

An important scenario for the phenomena described in the previous section is offered by the cyclic subspaces of a fixed operator $T \in \mathcal{L}(H)$. Specifically, we fix a non-zero vector $\xi \in H$ and define the finite dimensional subspaces

$$H_n = \text{span}\{\xi, T\xi, \ldots, T^{n-1}\xi\}.$$

We assume that $\dim H_n = n$, that is there is no degeneracy in the T-cyclic subspaces defined by the vector ξ. We also can assume without loss of generality that the union of spaces H_n is dense in H. The second, weaker inner product structure is induced by the positive, non-invertible operator A.

In this case the "orthogonal polynomials" associated to the cain of subspaces (H_n) and the two norms play a central role:

$$\phi_n \in H_{n+1} \ominus H_n, \quad \|\phi_n\| = 1, \quad n \geq 0,$$

$$\psi_n \in H_{n+1} \ominus_A H_n, \quad \langle A\psi_n, \psi_n \rangle = 1, \quad n \geq 0.$$

Indeed, we can write

$$\phi_n = \kappa_n T^n \xi + \kappa_{n,n-1} T^{n-1}\xi + \ldots,$$

$$\psi_n = \gamma_n T^n \xi + \gamma_{n,n-1} T^{n-1}\xi + \ldots.$$

The choice of positive leading coefficients

$$\frac{1}{\kappa_n} = \inf_{h \in H_n} \|T^n \xi - h\|,$$

$$\frac{1}{\gamma_n} = \inf_{h \in H_n} \|\sqrt{A}(T^n \xi - h)\|$$

leaves no room for ambiguity.

The orthogonal projections introduced in the previous section are:

$$P_{n+1} = \phi_n \langle \cdot, \phi_n \rangle + \phi_{n-1} \langle \cdot, \phi_{n-1} \rangle + \ldots + \phi_0 \langle \cdot, \phi_0 \rangle,$$

respectively

$$Q_{n+1} = \psi_n (\cdot, \psi_n) + \psi_{n-1} (\cdot, \psi_{n-1}) + \ldots + \psi_0 (\cdot, \psi_0).$$

We translate the computations of the general framework in these terms.

Proposition 4.1. *The difference between the two finite central truncations of the operator T along the Krylov subspaces* span$\{\xi, T\xi, \ldots, T^{n-1}\xi\}$ *is of rank one and has the expression*

$$\tilde{T}_n - T_n = \frac{\kappa_{n-1}}{\kappa_n} (\phi_n - \frac{\kappa_n}{\gamma_n} \psi_n) \langle \cdot, \phi_{n-1} \rangle.$$

Proof. Start with the rank one projection

$$P_{n+1} - P_n = \phi_n \langle \cdot, \phi_n \rangle$$

and consider a vector $h \in H_n$:

$$h = h_{n-1}\phi_{n-1} + h_{n-2}\phi_{n-2} + \dots,$$

where $h_k = \langle h, \phi_k \rangle$. Then

$$(P_{n+1} - P_n)Th = h_{n-1}(P_{n+1} - P_n)T\phi_{n-1} = h_{n-1}\phi_n\langle T\phi_{n-1}, \phi_n\rangle =$$
$$h_{n-1}\phi_n\langle \kappa_{n-1}T^n\xi + \dots, \phi_n\rangle = h_{n-1}\phi_n\frac{\kappa_{n-1}}{\kappa_n},$$

due to the orthogonality property of ϕ_n.

Further on, let $u = A_n^{-1}P_nA\phi_n$, that is $u \in H_{n-1}$ and

$$Au - A\phi_n \perp H_{n-1},$$

which implies $\phi_n - u = \lambda\psi_n$. The comparison of the leading coefficients yields $\lambda\gamma_n = \kappa_n$, whence

$$A_n^{-1}P_nA\phi_n = \phi_n - \frac{\kappa_n}{\gamma_n}\psi_n.$$

In conclusion

$$A_n^{-1}P_nA(P_{n+1} - P_n)TP_n = \frac{\kappa_{n-1}}{\kappa_n}(\phi_n - \frac{\kappa_n}{\gamma_n}\psi_n)\langle \cdot, \phi_{n-1}\rangle.$$

□

Corollary 4.2. *In the conditions of the Proposition,*

$$\text{trace}|\tilde{T}_n - T_n| = \|\tilde{T}_n - T_n\| = \frac{\kappa_{n-1}}{\kappa_n}\sqrt{\frac{\kappa_n^2}{\gamma_n^2}\|\psi_n\|^2 - 1}.$$

Proof. Indeed,

$$\langle \phi_n, \frac{\kappa_n}{\lambda_n}\psi_n\rangle = 1$$

therefore

$$\|\phi_n - \frac{\kappa_n}{\lambda_n}\psi_n\|^2 = \frac{\kappa_n^2}{\gamma_n^2}\|\psi_n\|^2 - 1.$$

Second, the norm and trace norm coincide on any rank one operator. □

The "orthogonal polynomials" point of view allows a simple interpretation of the LDU decomposition of the operator A. To this aim, we write

$$\phi_n = c_{n,n}\psi_n + c_{n,n-1}\psi_{n-1} + \dots + c_{n,0}\psi_0, \quad n \geq 0, \tag{4.1}$$

so that

$$\langle A\phi_n, \phi_k\rangle = \sum_{j\leq n}\sum_{\ell\leq k}\langle A\psi_j, \psi_\ell\rangle c_{n,j}\overline{c_{k,\ell}} = \sum_{j\leq \min(n,k)} c_{n,j}\overline{c_{k,j}}.$$

Thus the lower triangular matrix $C = (c_{n,j})$ satisfies, at all stages $n \geq 0$:

$$A_n = P_nAP_n = P_nCP_nC^*P_n = P_nCC^*P_n.$$

That is C represents in the orthonormal basis (ϕ_n) a linear bounded operator, also denoted by C. Let $C_n = P_n C P_n$. The diagonal elements are identifiable from the linear decomposition above:

$$c_{nn} = \frac{\kappa_n}{\gamma_n}.$$

Also,

$$\langle A\phi_n, \phi_n\rangle = c_{n,n}^2 + |c_{n,n-1}|^2 + \ldots + |c_{n,0}|^2,$$

and

$$c_{n,n-1}\psi_{n-1} + \ldots + c_{n,0}\psi_0 = \phi_n - c_{n,n}\psi_n = A_n^{-1} P_n A\phi_n.$$

Let $D = \mathrm{diag}(\frac{\kappa_n^2}{\gamma_n^2})$,, so that there exists a strictly lower diagonal matrix L_n with the property

$$C_n = (I + L_n)\sqrt{D_n}, \quad n \geq 0.$$

Specifically

$$L_{nj} = \frac{c_{nj}}{c_{jj}}, \quad j < n.$$

Note that one can recover the entries of the diagonal matrix D via determinantal formulae:

$$\det A_{n+1} = \det D_{n+1} = c_{00}c_{11}\ldots c_{nn} = [\frac{\kappa_0\kappa_1\ldots\kappa_n}{\gamma_0\gamma_1\ldots\gamma_n}]^2, \tag{4.2}$$

and

$$\frac{\kappa_n^2}{\gamma_n^2} = \frac{\det A_{n+1}}{\det A_n}. \tag{4.3}$$

In this way we recover the LDU decomposition

$$A_n = C_n C_n^* = (I + L_n) D_n (I + L_n^*).$$

According to Corollary 3.5, a sufficient condition for the asymptotic equivalence of the two sequences of finite central truncations is the finiteness of the Hilbert-Schmidt norm of the infinite matrix L. We state this observation as a partial conclusion of the computations above.

Proposition 4.3. *Let $T \in \mathcal{L}(H)$ be a linear bounded operator with cyclic vector ξ and let A be a positive, non-invertible linear operator on H. The ascending chain of Krylov subspaces of T carries the orthonormal basis $(\phi_n)_{n=0}^\infty$ and the A-orthonormal basis $(\psi_n)_{n=0}^\infty$. If the transition matrix (c_{nj}) between the two bases (4.1) satisfies*

$$\sum_n \sum_{j<n} |\frac{c_{nj}}{c_{jj}}|^2 < \infty, \tag{4.4}$$

then the finite central truncations of T are asymptotically equivalent

$$\lim_{n\to\infty} \|T_n - \tilde{T}_n\| = 0.$$

Similarly, if we start with the inverse of the transition matrix:

$$\psi_n = b_{nn}\phi_n + b_{n,n-1}\phi_{n-1} + \ldots + b_{n0}\phi_0 \tag{4.5}$$

we end up with a matrix decomposition

$$B_n A_n B_n^* = I, \quad n \geq 0.$$

Since B_n is a lower triangular matrix, or directly, we infer

$$B_n C_n = B_n(I + L_n)\sqrt{D_n} = I,$$

whence

$$B_n = \sqrt{D_n}^{-1}(I + L_n)^{-1} = \sqrt{D_n}^{-1}(I + M_n)$$

where M_n is another strictly lower triangular matrix. Note that L is Hilbert-Schmidt if and only if M is:

$$M_n = (I + L_n)^{-1} - I = (I + L_n)^{-1}L_n$$

plus the convergence of $(I + L_n)^{-1}$ in norm to $(I + L)^{-1}$, and vice-versa. Note that

$$b_{nn} = \frac{\gamma_n}{\kappa_n} = c_{nn}^{-1}.$$

We conclude that the condition in the statement of the Theorem is equivalent to

$$\sum_n \sum_{j<n} |\frac{b_{nj}}{b_{nn}}|^2 < \infty.$$

Along the same lines, we remark that

$$A_n^{-1}P_n A\phi_n = \phi_n - \frac{\psi_n}{b_{nn}} = -\sum_{j<n} \frac{b_{nj}}{b_{nn}}\phi_j,$$

and, for every $n \geq 0$,

$$\|A_n^{-1}P_n A\phi_n\|^2 = \sum_{j<n} |\frac{b_{nj}}{b_{nn}}|^2.$$

Corollary 4.4. *Let, in the condition of the Theorem,* (b_{nj}) *denote the base change matrix (4.5) of the A-orhtonormal basis* (ψ_n) *into the orthonormal basis* (ϕ_n)*.*

If

$$\lim_n \sum_{j<n} |\frac{b_{nj}}{b_{nn}}|^2 = 0,$$

then

$$\lim_{n\to\infty} \|T_n - \tilde{T}_n\| = 0.$$

Proof. To complete the proof we only need to remark that the extra factor $\frac{\kappa_{n-1}}{\kappa_n}$ in the difference $T_n - \tilde{T}_n$ is uniformly bounded:

$$\frac{\kappa_{n-1}}{\kappa_n} = \langle T\phi_{n-1}, \phi_n \rangle \leq \|T\|.$$

□

Next we relax the decay assumption (4.8) and state a similar result to the Theorem above, with a weaker conclusion.

Proposition 4.5. *Let $T \in \mathcal{L}(H)$ be a linear bounded operator with cyclic vector ξ and let A be a positive, non-invertible linear operator on H. The ascending chain of Krylov subspaces of T carries the orthonormal basis $(\phi_n)_{n=0}^{\infty}$ and the A-orthonormal basis $(\psi_n)_{n=0}^{\infty}$. If the transition matrix (c_{nj}) between the two bases (4.1) satisfies*

$$\sup_j \Big(\sum_{n>j} \frac{|c_{nj}|}{c_{jj}}\Big) \sup_n \Big(\sum_{j<n} \frac{|c_{nj}|}{c_{jj}}\Big) < 1, \tag{4.6}$$

then the finite central truncations of T remain at finite distance

$$\sup_n \|T_n - \tilde{T}_n\| < \infty.$$

Proof. In order to use Theorem 3.4 we have to check that the operator $I + L$ is invertible on $\ell^2(\mathbf{N})$, where L has matrix entries $L_{nj} = \frac{c_{nj}}{c_{jj}}$ for $j < n$ and zero otherwise. We use Schur's boundedness criterion:

$$\|L\xi\|^2 = \sum_n |\sum_{j<n} L_{nj}\xi_j|^2 \leq \sum_n (\sum_{j<n} |L_{jn}|)(\sum_{j<n} |L_{jn}||\xi_j|^2) \leq$$

$$M \sum_{j<n} |L_{jn}||\xi_j|^2 \leq MN\|\xi\|^2,$$

where $M = \sup_n(\sum_{j<n} \frac{|c_{nj}|}{c_{jj}})$ and $N = \sup_j(\sum_{n>j} \frac{|c_{nj}|}{c_{jj}})$. □

As not always a complete hold of the orthogonal polynomial coefficients is available, a sufficient criterion for the asymptotic equivalence of finite central truncations is stated below. Note that this affects solely the weaker norm A and not the operator T.

Proposition 4.6. *Let $(H_n)_{n=0}^{\infty}$ be an ascending chain of subspaces of a complex Hilbert space H, satisfying* $\dim H_n = n$ *for all $n \geq 0$. Let P_n denote the orthogonal projection on $H_n, n \geq 0$, respectively, and denote by $(\phi_n)_{n=0}^{\infty}$ the associated orthonormal basis: $H_n = \mathrm{span}\{\phi_0, \phi_1, \ldots, \phi_{n-1}\}$.*

Assume $A \in \mathcal{L}(H)$ is a positive, bounded linear operator with matrix entries $a_{k\ell} = \langle A\phi_k, \phi_\ell \rangle,\ k, \ell \geq 0$. If

$$\sup_{k<\ell} \frac{a_{\ell\ell}}{a_{kk}} < \infty \tag{4.7}$$

and

$$\sum_{k<\ell} \frac{|a_{k\ell}|^2}{a_{kk}a_{\ell\ell}} < \infty \tag{4.8}$$

then the central truncations $A_n = P_nAP_n$ *satisfy*

$$\lim_n A_n^{-1}P_nA(P_{n+1}-P_n) = 0.$$

Proof. Since a is a positive operator, the diagonal entries a_{nn} are all positive. Denote by $D = \mathrm{diag}(\sqrt{a_{00}}, \sqrt{a_{11}}, \sqrt{a_{22}}, \ldots)$ the square root of the diagonal of A. We can factor

$$A = D(I+S)D$$

with the matrix B of entries

$$s_{k\ell} = \frac{a_{k\ell}}{\sqrt{a_{kk}}\sqrt{a_{\ell\ell}}}, k \neq \ell,$$

and zero on the diagonal.

Assumption 4.8 is equivalent to B being Hilbert-Schmidt. According to the main factorization theorem (see for instance [9] Chapter IV) there exists a Hilbert-Schmidt operator L which is strictly lower triangular with respect to the chain (H_n), with the property

$$I + S = (I+L)E(I+L^*),$$

where E is a diagonal operator. As remarked in Section 2, one finds in this case

$$\lim_n \|(I_n+S_n)^{-1}P_n(I+S)(P_{n+1}-P_n)\| = 0.$$

To infer a similar conclusion for the operator A we start by the identities:

$$A_n = D_n(I_n+S_n)D_n,$$

and

$$D(P_{n+1}-P_n) = \sqrt{a_{nn}}(P_{n+1}-P_n),\ n \geq 0.$$

Whence

$$A_n^{-1}P_nA(P_{n+1}-P_n) = D_n^{-1}(I_n+S_n)^{-1}D_n^{-1}D_nP_n(I+S)(P_{n+1}-P_n)\sqrt{a_{nn}} =$$

$$\mathrm{diag}(\frac{a_{nn}}{a_{00}}, \ldots, \frac{a_{nn}}{a_{(n-1)(n-1)}})^{1/2}(I_n+S_n)^{-1}P_n(I+S)(P_{n+1}-P_n).$$

The boundedness assumption in the statement takes over and finishes the proof. □

We can reformulate the above result in terms of bases of vectors. Recall that $(\phi_n)_{n=0}^{\infty}$ is the orthonormal basis subjacent the chain $(H_n)_{n=0}^{\infty}$, while $(\psi_n)_{n=0}^{\infty}$ is the ON basis in the weaker A-norm, with respect to the same chain. The second condition in the statement of the Proposition simply states that the matrix formed by the cosine values with respect to the A-norm

$$\cos(\phi_k, \phi_\ell)_A = \frac{\langle A\phi_k, \phi_\ell \rangle}{\|\phi_k\|_A \|\phi_\ell\|_A}, \quad k, \ell \geq 0,$$

is a Hilbert-Schmidt perturbation of the identity matrix. In other terms, the vectors $\frac{\phi_k}{\|\phi_k\|_A}$ form a Bari basis of the weaker norm Hilbert space K. A Bari basis, in the terminology of Krein, is a Riesz basis which is quadratically close to an orthonormal basis, see [3, 8].

Regardless to say that, conform to Theorem 3.3, in all scenarios considered in this section the counting measures of the finite central truncations in the two norms will have gravi-equivalent cluster points in the weak-$*$ topology, that is limiting measures with the same logarithmic potentials at infinity.

5. Examples

5.1. **Hardy space operators.** Let $H^2(r\mathbb{D})$ denote the Hardy space of the disk centered at $z = 0$ and of radius $r > 0$. That is the space of Fourier series with non-negative coefficients

$$f(re^{i\theta}) = \sum_{k=0}^{\infty} c_k r^k e^{ik\theta},$$

which are square summable:

$$\|f\|_r^2 = \sum_{k=0}^{\infty} |c_k|^2 r^{2k} < \infty.$$

Then $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is an analytic function in the open disk $r\mathbb{D}$, with non-tangential boundary values $f(re^{i\theta})$.

It is relevant for our study to remark that the monomials $z^k, k \geq 0$, are simultaneously orthogonal on all disks $r\mathbb{D}$. In addition, the norms $\|f\|_r^2$ are non-decreasing as functions of $r > 0$. Let $H_n = \text{span}\{1, z, \ldots, z^{n-1}\}$ denote the subspace of polynomials of degree less than n.

Fix a values $0 < r < 1$ and consider the Hilbert space $H^2(\mathbb{D})$ and the weaker norm $\|\cdot\|_r$ on it, giving rise by completion to the Hilbert space $H^2(r\mathbb{D})$. The positive, compact and diagonal operator

$$D = \text{diag}(1, r^2, r^4, r^6, \ldots)$$

links, in the spirit of the present note, the two norms:

$$\langle Df, f \rangle = \langle f, f \rangle_r.$$

Let $T : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D})$ denote an *arbitrary* linear and bounded operator, and let T_n, $\tilde{T}_n$ denote its finite central truncations with respect to the chain (H_n), in the norms $\|\cdot\|_1$, respectively $\|\cdot\|_r$. Corollary 3.5 implies then $T_n = \tilde{T}_n$. This shows that the asymptotic values of the spectra of the truncations T_n have very little to do with the spectrum of T. Indeed, consider an analytic Toeplitz operator, that is the multiplier $T = M_F$ with a bounded analytic function, defined in a neighborhood of the closed unit disk. Then the operator

$$T : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D})$$

is subnormal and has spectrum equal to $F(\overline{\mathbb{D}})$, while

$$T : H^2(r\mathbb{D}) \longrightarrow H^2(r\mathbb{D})$$

has spectrum equal to $F(r\overline{\mathbb{D}})$.

Originally, this invariance phenomenon was discovered on Toeplitz matrices by Schmidt and Spitzer [16], and led to a fascinating, yet far from being complete, analysis of the respective eigenvalue distribution, see [5].

Remark that instead of a second Hardy space $H^2(r\mathbb{D})$ we can take any Hilbert space of analytic functions on the disk $r\mathbb{D}$ which admits the monomials as an orthogonal basis. As for instance a Bergman type space with a rotationally invariant weight. Next we exploit Theorem 4.8 in the same context.

Let $L(z, \overline{w})$ be a positive semi-definite function, analytic in $z, |z| < \frac{1}{r}$ and anti-analytic in $w, |w| < \frac{1}{r}$, continuous on the closed bidisk $\frac{1}{r}\overline{\mathbb{D}} \times \frac{1}{r}\overline{\mathbb{D}}$. We define the positive semi-definite, Hilbert-Schmidt operator $S \in \mathcal{L}(H^2(\mathbb{D}))$ by

$$(Sf)(z) = \int_{\mathbb{T}} L(z, \overline{w}) f(w) \frac{dw}{iw}, \quad f \in H^2(\mathbb{D}).$$

With the choice of the diagonal operator D as before, we want to assure the continuity of $F = \sqrt{D}^{-1} S \sqrt{D}$, that is of the integral operator

$$(Ff)(z) = \int_{\mathbb{T}} L(\frac{z}{r}, \overline{w}) f(rw) \frac{dw}{iw} = \int_{\mathbb{T}} L(\frac{z}{r}, \frac{\overline{u}}{r}) f(u) \frac{du}{iu}.$$

But F is Hilbert-Schmidt on the Hardy space due to the continuity assumption on the kernel L.

Let $(Af)(z) = f(rz) + (Sf)(z), \quad f \in H^2(\mathbb{D})$, and consider the weaker norm $\|f\|_A^2 = \langle Af, f \rangle$. Theorem 4.8 implies then that for an arbitrary operator $T \in \mathcal{L}(H^2(\mathbb{D}))$ the finite central truncations with respect to the chain of polynomial spaces (H_n) and the norms $\| \cdot \|, \| \cdot \|_A$ are asymptotically equivalent. Note that the norm $\| \cdot \|_A$ is equivalent to $\| \cdot \|_r$, but the monomials are no longer orthogonal in the inner product induced by the operator A.

A different example on Hardy space, this time exploiting Theorem 3.3, can be constructed as follows. For a function $f \in H^2(\mathbb{D})$ we denote by $\hat{f}(n)$ its Fourier transform, that is ($\hat{f}(n)$ are the coefficients of the Taylor expansion of f at $z = 0$:

$$f(z) = \hat{f}(0) + \hat{f}(1)z + \hat{f}(2)z^2 + \dots.$$

Consider a bounded analytic function in the disk $h \in H^\infty(\mathbb{D})$ subject to the normalization $\|h\|_\infty = \sup_{|z|<1} |h(z)| < 1$. We define the Toeplitz operator

$$(Rf)(z) = f(z) + \frac{1}{2\pi}\int_{\mathbf{T}} \frac{\overline{h(\zeta)}f(\zeta)}{\zeta - z}\frac{d\zeta}{i\zeta}, \quad f \in H^2,\ |z| < 1.$$

The matrix associated to R in the orthonormal basis (z^n) is of the form $I + L^*$, where L is strictly lower-triangular and strictly contractive $\|L\| < 1$. Let (γ_n) denote an arbitrary sequence of positive numbers, converging to zero. We define the weaker norm on Hardy space by

$$\langle Af, f\rangle = \sum_{n=0}^{\infty} \gamma_n |\widehat{Rf}(n)|^2, \quad f \in H^2.$$

Then Theorem 3.3 asserts that any Hessenberg matrix

$$T = \begin{pmatrix} t_{00} & t_{01} & t_{02} & t_{03} & \dots \\ t_{10} & t_{11} & t_{12} & t_{13} & \\ 0 & t_{21} & t_{22} & t_{23} & \dots \\ 0 & 0 & t_{32} & t_{33} & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix},$$

bounded as an operator in the ℓ^2-norm, has the central finite truncations (T_n) and $(\tilde{T}_n)$ at uniform distance:

$$\sup_n \|T_n - \tilde{T}_n\| < \infty.$$

Moreover, if the sub diagonal of T converges to zero: $\lim_n t_{n+1,n} = 0$, then

$$\lim_n \|T_n - \tilde{T}_n\| = 0.$$

In both cases, the counting measures of the spectra of T_n, respectively $\tilde{T}_n$ will have limit measures with the same logarithmic poet tail at infinity.

5.2. **Weighted ℓ^2-spaces.** We start with a positive, bounded weight $w : \mathbb{Z}^n \longrightarrow \mathbb{R}$. The norm

$$\|f\|_w^2 = \sum_{\alpha \in \mathbb{Z}^n} w(\alpha)|f_\alpha|^2, \quad f = (f_\alpha),$$

is continuous on $\ell^2(\mathbb{Z}^n)$, and in general weaker than the standard norm

$$\|f\|^2 = \sum_{\alpha \in \mathbb{Z}^n} |f_\alpha|^2, \quad f = (f_\alpha).$$

Denoting by $(e_\alpha)_{\alpha \in \mathbb{N}^n}$ the standard orthonormal basis on $\ell^2(\mathbb{Z}^n)$, we find that the norm $\|\cdot\|_w$ is implemented by the diagonal operator $D = \operatorname{diag}(\sqrt{w(\alpha)}$. Given an order on $\mathbb{Z}^n$, for instance derived from the lexicographical order of the positive orthant, we define the finite dimensional subspaces

$$H_\beta = \operatorname{span}\{e_\alpha,\ \alpha < \beta\}.$$

Given a linear bounded operator T on $\ell^2(\mathbb{Z}^n)$, we discover the rather tautological fact that the finite central truncations of T with respect to the chain of subspaces $H_\alpha,\ \alpha \in \mathbb{Z}^n$, in the two norms $\|\cdot\|, \|\cdot\|_w$ are identical. This observation can be proved directly, as $f_\alpha = \frac{e_\alpha}{\sqrt{w(\alpha)}}$ is the orthonormal system of vectors in the weaker norm:

$$(x, f_\alpha)f_\alpha = \frac{\langle Dx, e_\alpha\rangle e_\alpha}{w(\alpha)} = \frac{\langle x, De_\alpha\rangle e_\alpha}{w(\alpha)} = \langle x, e_\alpha\rangle e_\alpha.$$

As an application we consider the chain of Sobolev spaces on the torus $\mathbb{T}^n$. The Fourier system $\exp(\alpha\cdot),\ \ \alpha \in \mathbb{Z}^n$ is orthogonal on every Sobolev space. Let $P(x, D)$ denote a pseudo-differential operator of order zero, on $\mathbb{T}^n$. According to the observation above, the finite central truncations of $P(x, D)$ with respect to the Fourier orthogonal system is independent of the order of the Sobolev space norm in which is computed. Specifically, the linear bounded operators

$$P(x, D) : H^s(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n)$$

have the same finite central truncations along the chain of Fourier modes of increasing higher frequency, *independent* of the order $s \in \mathbb{R}$.

5.3. **Jacobi matrices.** Let μ be a positive measure supported on a bounded interval $[0, a]$ of the real axis. The operator of multiplication by the variable $A = M_x$ is positive, bounded and injective as soon as μ does not have a point mass at zero. We will take $H = L^2(\mu)$ and for the A-norm we consider the space $K = L^2(x\mu)$. The filtration given by polynomials and their degrees

$$H_n = \{p \in \mathbf{C}[z];\ \deg p \le n-1\},\ n \ge 0,$$

produces systems of orthonormal polynomials $(\phi_n) \subset L^2(\mu)$ and $(\psi_n) \subset L^2(x\mu)$. The positive operator $A = M_x$ of multiplication by the variable has a three diagonal matrix representation with respect to the basis of μ-orthogonal polynomials (ϕ_j):

$$xp_j(x) = b_{j+1}p_{j+1}(x) + a_j p_j(x) + b_j p_{j-1}(x),\quad j \ge 0.$$

Note that in this case the orthogonal projections P_n onto the subspaces H_n in the chain satisfy

$$P_nA(I - P_n) = P_nA(P_{n+1} - P_n),\quad n \ge 0.$$

Let $T \in \mathcal{L}(L^2(\mu))$ be *any* linear bounded operator. In view of the proof of Lemma 3.1 we infer

$$T_n - \tilde{T}_n = A_n^{-1}P_nA(I - P_n)TP_n = A_n^{-1}P_nA(P_{n+1} - P_n)TP_n$$

hence

$$\lim_n \|T_n - \tilde{T}_n\| = 0$$

as soon as

$$\lim_n \|A_n^{-1}P_nA(P_{n+1} - P_n)\| = 0.$$

Summing-up we have obtained the following result.

Proposition 5.1. *Let*

$$A = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \dots \\ \overline{a_1} & b_1 & a_2 & 0 & \dots \\ 0 & \overline{a_2} & b_2 & a_3 & \dots \\ 0 & 0 & \overline{a_3} & b_3 & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

be a positive Jacobi matrix subject to the conditions

$$b_{n+1} \le b_n, \quad n \ge 0$$

and

$$\sum_{k=1}^{\infty} \frac{|a_k|^2}{b_{k-1} b_k} < \infty.$$

Then any linear bounded operator $T : \ell^2(\mathbf{N}) \longrightarrow \ell^2(\mathbf{N})$ has asymptotically equivalent truncations with respect to the main diagonal, in the ℓ^2, respectively A-norms.

The most interesting case is a compact and positive Jacobi matrix A, and this is directly linked to Stieltjes famous memoir devoted to continued fractions and their convergence domains. Under these assumptions the original measure μ is discrete, with only $x = 0$ as accumulation point of its point masses $\lambda_k > 0$. Specifically

$$\|f\|^2 = \int |f|^2 d\mu = \sum_{k=1}^{\infty} \gamma_k |f(\lambda_k)|^2,$$

is the original norm, while the weaker one is

$$\|f\|_A^2 = \sum_{k=1}^{\infty} \gamma_k \lambda_k^2 |f(\lambda_k)|^2.$$

The weights (γ_k) above are positive and constrained by the two assumptions in the Proposition. For details, examples and ample references on compact Jacobi matrices see [18].

5.4. **Integral kernels. I (Discrete chains).** Let μ be a positive measure with compact support in $\mathbf{C}$. We consider the Lebesgue space $L^2(\mu)$ and the closure of complex polynomials $H = P^2(\mu)$ in its norm. The chain of subspaces defined by the degree filtration

$$H_n = \{p \in \mathbf{C}[z]; \quad \deg p \le n\}, \ n \ge 0,$$

is dense in H, and we assume it is non-stationary, that is the measure μ has infinite support.

Let $K(z,w) = \sum_{j=0}^{\infty} P_j(z)\overline{P_j(w)}$ be a positive semi-definite kernel, obtained as an infinite sum of non-negative rank-one kernels with polynomial components P_j. We impose the Hilbert-Schmidt condition

$$\sum_{j=0}^{\infty} \|P_j\|^2 < \infty.$$

In this way the integral operator

$$(Af)(z) = \int K(z,w)f(z)d\mu(z), \quad f \in H,$$

is compact and non-negative. We also assume that the system (P_j) is complete in H, that is the range of A is dense in H, hence its kernel is trivial.

We denote as before by $(\phi_n)_{n=0}^{\infty}$ the sequence of orthonormal polynomials in the metric of H. The matrix associated to A in this basis is

$$a_{j\ell} = \langle A\phi_j, \phi_\ell\rangle = \sum_k \langle \phi_j, P_k\rangle\langle P_k, \phi_\ell\rangle.$$

The diagonal entries are

$$a_{jj} = \sum_{k=0}^{\infty} |\langle \phi_j, P_k\rangle|^2 = \|P_k\|^2.$$

The two assumptions in Theorem 4.6 become

$$\sup_{j<\ell} \frac{\|P_\ell\|}{\|P_j\|} < \infty, \tag{5.1}$$

and

$$\sum_{j<\ell} \langle \frac{\phi_j}{\|P_j\|}, P_k\rangle\langle P_k, \frac{\phi_\ell}{\|P_\ell\|}\rangle < \infty. \tag{5.2}$$

Assume in addition that

$$\deg P_k = k, \quad k \geq 0.$$

Then

$$\langle \phi_j, P_k\rangle = 0$$

as soon as $j > k$. Whence we derive the sufficient inequality

$$\sum_j \sum_{k>j} \frac{\|P_k\|^2}{\|P_j\|^2} < \infty \tag{5.3}$$

for both conditions (5.1) and (5.2) to hold.

In conclusion, in the presence of a Hilbert-Schmidt integral operator A with kernel $K(z,w) = \sum_k P_k(z)\overline{P_k(w)}$ with $\deg P_k \leq k,\ k \geq 0$, Theorem 4.6 shows that every linear bounded operator $T : P^2(\mu) \longrightarrow P^2(\mu)$ subject to the Hessenberg matrix condition

$$\deg(Tp) \leq \deg(p) + 1, \quad p \in \mathbf{C}[z],$$

has asymptotically equivalent finite central truncations in the original norm and with respect to the A-norm, provided the series (5.3) converges.

5.5. **Integral kernels. II (Continuous chains).** From the extensive literature on Volterra type operators we extract an illustrative case. The ground Hilbert space H is $L^2[0,\infty)$ with respect to Lebesgue measure. For every $t \geq 0$ the space of functions in H with support contained in $[0,t]$ is a closed subspace called H_t. The orthogonal projection onto H_t is denoted P_t. Let $\gamma \in L^1[0,\infty)$ and consider the associated Wiener-Hopf operator:

$$(Lf)(x) = f(x) + \int_0^x \gamma(x-t)f(t)dt, \quad f \in H.$$

We know from Gelfand's theory that L is invertible on H if and only if its symbol does not vanish:

$$1 + \int_0^\infty e^{i\xi t}\gamma(t)dt \neq 0, \quad \xi \in \mathbf{R}, \tag{5.4}$$

see for instance Theorem XXX.2.6 in [7]. The triangular nature of L is also reflected by $P_t L(I-P_t) = 0, \quad t \geq 0$. Let $\delta \in L^\infty[0,\infty)$ be a function which does not vanish almost everywhere, and define the multiplication operator

$$(Df)(x) = |\delta(x)|^2 f(x), \quad f \in H.$$

Under these conditions the linear operator $A = LDL^*$ is bounded and positive, but possibly not invertible. Specifically, the A-norm is given in closed form by the expression:

$$\|f\|_A^2 = \int_0^\infty |\delta(x)f(x) + \delta(x)\int_x^\infty \overline{\gamma}(t-x)f(t)dt|^2 dx.$$

Let Q_t stand for the orthogonal projection of H onto $H_t = L^2[0,t]$, in the weaker, A-norm. In view of Lemma (3.2), the map Q_t is bounded in the original norm, for all $t \geq 0$.

Given an arbitrary linear bounded operator T on H one defines the truncations on H_s as $T_s = P_s T P_s$ and $\tilde{T}_s = Q_s T Q_s$. Under the non-vanishing symbol condition (5.4) we derive from Lemma 3.1:

$$\|T_s - \tilde{T}_s\| \leq C \sup_{s\geq 0} \|P_s T(I-P_s)\|,$$

where the constant C depends only on the kernel γ.

5.6. **Doubly orthogonal systems.** An immediate consequence of our first computations is that a doubly orthogonal basis in the original metric and the weaker one will give identical finite central truncations for *any* operator. Under such a scenario the positive operator $A \in \mathcal{L}(H)$ implementing the weak norm is diagonalized by this system of vectors. Obviously not every positive operator has pure point spectrum, hence can be diagonalized. Two relevant functional models for such a scenario are briefly recorded below.

5.6.1. *Natural Hilbert spaces of potential theory.* A classical framework involving simultaneous spectral analysis in two different Hilbert spaces is offered by potential theory. The note [13] based on Krein's foundational work [14] is a recent illustration for this theme. Rather than discussing spectral permanence in the presence of two norms, we indicate some possible applications of the present article to the rapidly growing field of spectral analysis of layer potentials, see [1].

Let Γ be a closed surface in $\mathbf{R}^d, d \geq 2$, sufficiently regular (for instance of Lipschitz type). Two Hilbert spaces stand out in the study of Dirichlet's problem with data on Γ via layer potentials. First is the Lebesgue space $L^2(\Gamma, d\sigma)$ with respect to area measure on Γ, and second is the energy space $\mathfrak{H}$ consisting of pairs $h = (h_i, h_e)$ of harmonic functions on Ω_i (the interior of Γ), respectively Ω_e (the exterior of Γ) possessing finite energy

$$[h]^2 = \int_{\Omega_i} |\nabla(h_i)|^2 dx + \int_{\Omega_e} |\nabla(h_e)|^2 dx < \infty.$$

We denote by $-E(x)$ the fundamental solution of Laplace operator: $\Delta E = -\delta$. The two norms are related by the *single layer potential*

$$S_f(z) = \int_\Gamma E(z-y) f(y) d\sigma(y), \quad z \notin \Gamma, \ f \in L^2(\Gamma),$$

and by the operator obtained by passing to boundary values on Γ:

$$(Sf)(x) = \int_\Gamma E(x-y) f(y) d\sigma(y), \quad x \in \Gamma, \ f \in L^2(\Gamma).$$

It turns out that the singularity appearing in the latter integral operator is removable and the result, modulo a scaling and exclusion of constant functions in dimension $d = 2$, is a positive compact operator $S : L^2(\Gamma) \longrightarrow L^2(\Gamma)$. The identity connecting the two norms is very simple

$$[S_f]^2 = \langle Sf, f \rangle, \quad f \in L^2(\Gamma).$$

In this way the continuous inclusion $L^2(\Gamma) \subset \mathfrak{H}$ fits into the framework developed in the present article.

But it is the *double layer potential*

$$D_f(z) = \int_\Gamma \frac{\partial}{\partial n_y} E(z-y) f(y) d\sigma(y)$$

and its boundary induced operator historically responsible for the solvability of the Dirichlet problem. The latter is a compact perturbation of the identity only for smooth Γ, its essential spectrum being in the general case highly relevant for an array of applications, see for details [1].

Since Galerkin approximation is ubiquitous in all numerical experiments, a conclusion of formula (3.1) is in order.

Lemma 5.2. *Let T be a linear bounded operator acting on $L^2(\Gamma, d\sigma)$, where Γ is a closed, Lipschitz surface in Euclidean space. The finite central truncations of T along the system of eigenvectors $(\phi_j)_{j=0}^{\infty}$ of the single layer potential operator attached to Γ coincide in the L^2, respectively energy space norms.*

Proposition 5.3. *Let (ϕ_j) denote the doubly orthogonal system of functions in the spaces $L^2(\Gamma)$ and $\mathfrak{H}$. Assume that $K : \mathfrak{H} \longrightarrow L^2(\Gamma)$ is a compact operator with the property that $I + K : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ is invertible.*

The two sequences of finite central truncations of a linear bounded operator $T \in \mathcal{L}(L^2)$, with respect to the chain of finite dimensional subspaces generated by the vectors $(I + K)\phi_j$, $j \geq 0$, satisfy:

$$\lim_n \|T_n - \tilde{T}_n\|_{\mathcal{L}(L^2,\mathfrak{H})} = 0.$$

Proof. The opertator $I + K$ is also invertible from $\mathfrak{H}$ to $\mathfrak{H}$, as it is Fredholm of zero index and possesses dense range. Note also that that the restriction of K to $L^2(\Gamma)$ is a compact operator from $L^2(\Gamma)$ to itself. For consistency, we return to the standard notation of this article and denote the weaker norm by

$$\langle x, y\rangle_{\mathfrak{H}} = \langle Ax, y\rangle_{2,\Gamma}, \quad x, y \in L^2(\Gamma).$$

Denote $R = I+K$ and let H_n denote the linear span of the vectors $\phi_0, \phi_1, \dots, \phi_{n-1}$. We compute in closed form the orthogonal projection $\hat{P}_n$ of L^2 onto the finite dimensional subspace $\hat{H}_n = RH_n$:

$$\hat{P}_n = RP_n(P_nR^*RP_n)^{-1}P_nR^*.$$

Indeed, $\hat{P}_n^* = \hat{P}_n = \hat{P}_n^2$ and the range of $\hat{P}_n$ is $\hat{H}_n$.

Since K is compact, $\lim_n(P_nK - KP_n) = 0$ in the operator norm, hence

$$\lim_n \|\hat{P}_n - P_n\| = \lim_n \|RP_n(P_nR^*RP_n)^{-1}P_nR^* - P_nR(R^*R)^{-1}R^*P_n\| =$$

$$\lim_n \|P_n(R(R^*R)^{-1}R^* - I)P_n\| = 0.$$

We infer that for any bounded linear operator $T \in \mathcal{L}(L^2)$ one has

$$\lim_n \|P_nTP_n - \tilde{P}_nT\tilde{P}_n\| = 0.$$

On the other hand, according to the Lemma, the orthogonal projections Q_n onto H_n in the $\mathfrak{H}$-norm coincide with P_n. By repeating the argument above, and using the fact that $K \in \mathcal{L}(\mathfrak{H})$ is compact, we find that the orthogonal projections $\hat{Q}_n$ onto RH_n satisfy in the $\mathfrak{H}$ metric:

$$\lim_n \|\hat{Q}_n - Q_n\|_{\mathcal{L}(\mathfrak{H})} = 0$$

and a fortiori

$$\lim_n \|\hat{Q}_n - Q_n\|_{\mathcal{L}(L^2,\mathfrak{H})} = 0. \tag{5.5}$$

For a given linear bounded transform $T \in \mathcal{L}(L^2)$ we note the estimates:

$$\|(\hat{P}_n - \hat{Q}_n)T|_{\hat{H}_n}\|_{\mathcal{L}(L^2,\mathfrak{H})} \leq \|A\|^{1/2}\|(\hat{P}_n - P_n)T|_{\hat{H}_n}\|_{\mathcal{L}(L^2)} + \|(Q_n - \hat{Q}_n)T|_{\hat{H}_n}\|_{\mathcal{L}(L^2,\mathfrak{H})},$$

whence

$$\lim_n \|(\hat{P}_n - \hat{Q}_n)T|_{\hat{H}_n}\|_{\mathcal{L}(L^2,\mathfrak{H})} = 0,$$

as desired.

□

We would like to offer a second proof of the asymptotic equivalence (5.5), and for this we will enter into the structure of the compact operator $K : \mathfrak{H} \longrightarrow L^2(\Gamma)$. More precisely, the boundedness of K implies

$$K = C\sqrt{A},$$

where $C \in \mathcal{L}(L^2)$, while the compactness of K implies that C is compact as an endomorphism of L^2. Let us denote $K^\sharp$ the adjoint with respect to the inner product of $\mathfrak{H}$. The identity

$$\langle AKx, y\rangle = \langle Ax, K^\sharp y\rangle$$

implies

$$\sqrt{A}K^\sharp = C^* A.$$

We claim that

$$\lim_n \|\sqrt{A}(\tilde{Q}_n - Q_n)\|_{\mathcal{L}(L^2)} = 0$$

which is the same as (5.5). Indeed, notice first that the chain of projections $P_n = Q_n$ commute with the positive operator A:

$$P_n A = AP_n = P_n A P_n.$$

The projection $\hat{Q}_n$ has the closed form:

$$\hat{Q}_n = RP_n[P_n R^\sharp R P_n]^{-1} P_n R^\sharp,$$

or

$$\hat{Q}_n = (I + C\sqrt{A})P_n[P_n(I + K^\sharp)(I + C\sqrt{A})P_n]^{-1}P_n(I + K^\sharp).$$

Remark that

$$\sqrt{A}(I + C\sqrt{A}) = (I + \sqrt{A}C)\sqrt{A},$$

and

$$\sqrt{A}_n P_n(I + K^\sharp) = P_n\sqrt{A}(I + K^\sharp) = P_n(I + C^*\sqrt{A})\sqrt{A}.$$

Consequently

$$\sqrt{A}\hat{Q}_n = (I + \sqrt{A}C)P_n\sqrt{A}_n[\sqrt{A}_n[P_n(I + K^\sharp)(I + C\sqrt{A})P_n]^{-1}\sqrt{A_n}P_n(I + K^\sharp) =$$

$$(I + \sqrt{A}C)P_n\sqrt{A}_n[(I + C^*\sqrt{A})(I + \sqrt{A}C)P_n\sqrt{A}_n]^{-1}P_n(I + C^*\sqrt{A})\sqrt{A} =$$

$$(I + \sqrt{A}C)P_n[(I + C^*\sqrt{A})(I + \sqrt{A}C)P_n]^{-1}P_n(I + C^*\sqrt{A})\sqrt{A}.$$

Finally the compactness of C takes over and we find

$$\lim_n \|\sqrt{A}(\hat{Q}_n - P_n)\|_{\mathcal{L}(L^2)} = 0.$$

With the aid of the eigenfunctions ϕ_j of the single layer potential operator S and its spectrum

$$S\phi_j = \lambda_j \phi_j, \quad j \geq 0,$$

one can illustrate the corollary above by an explicit matrix condition. Quite specifically, assume for all $j \geq 0$ that $\|\phi_j\|_{2,\Gamma} = 1$, so that $(\frac{\phi_j}{\sqrt{\lambda_j}})$ is an orthonormal basis of the energy space $\mathfrak{H}$. Let

$$a_{jk} = \langle K\phi_j, \phi_k \rangle, \quad j, k \geq 0,$$

denote the matrix entries of a linear transformation K. If

$$\sum_{j,k=0}^{\infty} \frac{|a_{jk}|^2}{\lambda_j} < \infty$$

then the operator K is Hilbert-Schmid as a linear transformation from $\mathfrak{H}$ to L^2.

A classical example of doubly orthogonal system in $L^2(\Gamma)$ and energy space $\mathfrak{H}$ is offered by the unit sphere $\Gamma = S^{d-1}$ in $\mathbf{R}^d$, $d \geq 3$. Indeed, all spherical harmonics h diagonalize the single layer potential operator:

$$Sh = \frac{h}{2 \deg h + d - 2}.$$

Of course the multiplicity of the eigenvalue $\frac{1}{2 \deg h + d - 2}$ depends on the number of linearly independent spherical harmonic polynomials of a prescribed degree.

5.6.2. *Analytic extension by series expansion.* A similar scenario for doubly orthogonal systems (this time of analytic functions) is offered by the so-called embedding or restriction operators. While this is a rich area of continuous research, closely related to sampling and interpolation of analytic functions, we choose a simple, yet representative, example.

Let Ω be an open set in the complex plane and let μ denote a positive measure compactly supported on Ω, so that the restriction map from Bergman's space of square integral analytic function with respect to area measure

$$R : L^2_a(\Omega) \longrightarrow L^2(\mu), \quad Rf = f|_{\mathrm{supp}\mu},$$

is well defined and continuous. Specifically that means that there is a positive constant C with the property

$$\|f\|_{2,\mu} \leq C\|f\|_{2,\Omega}, \quad f \in L^2_a(\Omega).$$

Moreover, Montel Theorem implies that the restriction operator R is compact and moreover the eigenvalues of the modulus R^*R, denoted (λ_n), decay exponentially, see [12] for the precise statement.

Assume the support of the measure μ infinite, so that R is also an injective map and let $f_n \in L^2_a(\Omega)$ denote the orthonormal basis of Bergman space formed by the eigenvectors of the positive and compact operator $A = R^*R$:

$$R^*Rf_n = \lambda_n f_n, \quad n \geq 0.$$

In other words

$$\int_\Omega f_n \overline{g} dA = \frac{1}{\lambda_n} \int f_n \overline{g} d\mu, \quad n \geq 0, \ g \in L^2_a(\Omega).$$

In particular we infer that $(\frac{f_n}{\sqrt{\lambda_n}})$ is an orthonormal basis of a closed subspace of $L^2(\mu)$. We adopt the ad-hoc and ambiguous notations $L^2_a(\mu)$ rather than $L^2_a(\mu, \Omega)$, for this subspace of $L^2(\mu)$, the closure of the range of R. One immediate and remarkable consequence of these elementary observations is the characterization of all elements of $L^2(\mu)$ which analytically extend to Ω and are square summable there: namely these are functions $h \in L^2(\mu)$ satisfying

$$h = \sum_{n=0}^{\infty} c_n \frac{f_n}{\sqrt{\lambda_n}},$$

subject to the additional decay condition for the coefficients:

$$\sum_{n=0}^{\infty} \frac{|c_n|^2}{\lambda_n} < \infty.$$

This global analytic extension phenomenon, by means of more sophisticated sums than power series has far reaching consequences, see for instance [10]. For the topics of the present article two conclusions are in order.

To relate the above construct to our general setting, we deal in this case with the Bergman space $L^2(\Omega)$ and the weaker norm of $L^2_a(\mu)$ induced by the positive and compact operator $A = R^*R$. Let $T \in \mathcal{L}(L^2(\Omega))$ be an arbitrary linear and bounded transformation of Bergman space. We can regard $T : \mathcal{D} \subset L^2_2(\mu) \longrightarrow L^2_a(\mu)$ as a densely defined operator, with closed graph, in the weaker norm. The first observation is that the double orthogonal system of functions (f_n) in $L^2_a(\Omega)$, respectively $L^2(\mu)$ gives rise to identical finite central truncations of T with respect to the two norms. Exactly as the basis of monomials behave on concentric disks for Toeplitz or more general operators.

Second, we isolate a class of perturbations of the doubly orthogonal system of functions (f_n) which do not alter too much the finite central truncations of a given operator. The proof of Proposition 5.3 applies line by line with the following result.

Proposition 5.4. *Let Ω be a planar domain and μ a positive measure compactly supported by an infinite subset of Ω. Denote by (f_n) the system of doubly orthogonal functions, in Bergman space $L^2_a(\Omega)$ and $L^2(\mu)$. Let $K \in \mathcal{L}(L^2(\mu), L^2_a(\Omega))$ be a compact operator so that $I + K$ is invertible on Bergman space. Denote $g_n =$*

$f_n + K f_n,\ n \geq 0$. *For every linear bounded transformation* $T \in \mathcal{L}(L^2_a(\Omega))$ *the finite central truncations* $T_n, \tilde{T}_n$ *along the subspaces generated by* $\{g_0, g_1, ..., g_{n-1}\},\ n \geq 1$, *in the Bergman space norm and respectively the weak norm* $L^2(\mu)$ *satisfy:*

$$\lim_n \|T_n - \tilde{T}_n\|_{\mathcal{L}(L^2(\mu), L^2_a(\Omega))} = 0.$$

The conclusion of the proposition becomes effective for a linear operator T which is bounded from $L^2(\mu)$ to $L^2_a(\Omega)$. Indeed, in this case there exists a positive constant M with the property:

$$\|Tf\|_{2,\Omega} \leq M \|f\|_{2,\mu}, \quad f \in L^2(\mu).$$

Within the notation of the proof of Proposition 5.3:

$$\|\sqrt{A}(\hat{Q}_n - \hat{P}_n)Tg\|_{2,\Omega} \leq M \|\sqrt{A}(\hat{Q}_n - \hat{P}_n)\|_{\mathcal{L}(L^2_a(\Omega))} \|\sqrt{A}g\|_{2,\Omega}, \quad g \in \hat{H}_n,$$

hence

$$\lim_n \|T_n - \tilde{T}_n\|_{\mathcal{L}(L^2(\mu))} = 0,$$

with the counting measure asymptotics consequences we outlined at the beginning of this article. Of course this is no surprise as the above "regularizing" assumption turns T into a compact operator on $L^2(\mu)$. However, the proof of Proposition 5.3 uses only the assumption $AP_n = P_n A = P_n A P_n,\ n \geq 1$, and this scenario covers way more general situations. We do not expand here the details.

A classical example of double orthogonality is provided by an ellipse and the measure $\sqrt{1-x^2}dx$ supported on the interval between the two foci, located at ± 1. In this situation Chebyshev polynomials of the second kind $(U_n)_{n=0}^{\infty}$ are doubly orthogonal, see for instance [10]. Note that all confocal ellipses possess the same system of simultaneous orthogonal polynomials. A converse was proved by Szegö and Walsh, see [10].

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