Elmore delay in the fractional order domain

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***Abstract*—Interconnect design has recently become one of the important factors that affect the circuit delay and performance especially in the deep submicron technology. The modelling of interconnects is typically based on using Elmore definitions of the delay time and rise time. So, a general formula for Elmore delay time and rise time in the fractional order domain are presented in this work. It is found from the new formulas of the delay time and rise time that these timing values could be controlled or tuned by the fractional orders. Hence, the fractional order can compensate for the components value. Furthermore, a case study of shunt compensation circuit is studied here to show the impact of the fractional orders on the delay time. The impact of the component values along with the fractional order on the new timing definition is studied using MATLAB analysis.**

Keywords—Fractional clculus; Elmore delay time; Elmore rise time; transmission line; interconnect

I. INTRODUCTION

Advancement in VLSI technology offers very high scale of integrated circuits in a system-on-chip. Indeed, with the continuous scaling of technology, the interconnect delay becomes the dominant source of delay [1, 2]. Hence, the interconnect performance becomes a vital factor in determining the circuit performance in both timing and power. One way to model the interconnect response is to use accurate interconnect models and signal propagation characterization to simulate the response of interconnects [1, 3] . This is done by using RLC models in the tree model to simulate the different nodes response [4, 5, 6]. Although this model gives good results, its accuracy degrades with scaling down the technology and its complexity increases as well.

On the other hand, some research for modeling interconnects based on the fractional calculus has started recently [7, 1]. Fractional calculus is used to characterize systems with infinite memory whereas the integer order calculus uses a finite memory to characterize systems [8]. Hence, fractional calculus is used to model real world phenomena in many disciplines like agriculture [9] and chaotic systems [10]. Furthermore, many fundamentals of the conventional circuit theory like filters and oscillators and stability techniques have been generalized to the fractional order domain [11, 12, 13, 14].The Riemann – Liouville definition of a fractional derivative of order α is given by [8]:

$$D^{\alpha}f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, & m-1<\alpha<m, \\ \frac{d^m}{dt^m}f(t) & \alpha = m. \end{cases} \quad (1)$$

where $0<\alpha<1$, from the Laplace transform of (1), the impedance of a fractional order element is represented by:

$$Z(s) = k_o s^{\alpha} \quad (2)$$

where k_o is a constant and α is the fractional order. Consequently, fractional order models for interconnects based on using fractional order elements rather than using traditional elements are proposed in [7, 1]. Actually, these fractional order models addressed the non-linear response of the transmission line by including the impact of the fractional orders. Hence, the modeling work for the fractional order is more accurate than the modeling in the integer order domain.

Most of the transmission line modelling work is based on Elmore definition of delay time and rise time. In [15] Elmore has introduced a general approach for calculating the propagation delay of a linear system given its transfer function. The popularity of the Elmore delay is mainly due to the existence of a simple tractable formula for the delay that has recursive properties, making the calculation of the circuit delays highly efficient even in large circuits [16].Yet, the formulas of [15] are based on the integer order calculus in calculating the slope of the rising and falling signals. Hence, this could be the reason of the inaccurate fractional order model for interconnects. So, the definitions of Elmore for both the delay time and the rise time should be generalized to the fractional order domain. The purpose of this work is to generalize Elmore definitions of the delay and rise times into the fractional order domain. This should modify the rise time and delay time to include the impact of the fractional orders on the interconnect timing. Furthermore, the fractional orders can be used to optimize the system delay-rise time for a specific times. Hence, in the second section the fractional order analysis of Elmore delay-rise time is presented. Then, a case study of shunt compensated circuit is discussed in section III and finally the conclusion in section IV.

II. PORPOSED ANALYSIS

The normalized system function of a stable system with finite number of lumped fractional order elements of different orders is given by:

$$H(s) = \frac{1+a_1s^{\alpha_1}+a_2s^{\alpha_2}+\cdots+a_ns^{\alpha_n}}{1+b_1s^{\beta_1}+b_2s^{\beta_2}+\cdots+b_ms^{\beta_m}} \quad (3)$$

where the coefficients a_i, b_i, α_i and β_i are real numbers and $\beta_m > \alpha_n$. Hence, the normalized transient response of this fractional order system to a unit step function $u(t)$is obtained from the fractional order inverse Laplace transform [8]. On the other hand, Elmore definition of the delay time $(T_D)$is the time required for the response to reach half of its final value as depicted in Fig. 1. Furthermore, the rise time (T_R) is the time needed to reach the point where is the reciprocal of the slope of the tangent drawn to the response curve at its half-value point as illustrated in Fig. 1. Then, both the delay time and the rise time are given by:

$$T_D = \int_0^\infty t v'(t) dt \tag{4.a}$$

$$T_R = \left\{2\pi\left[\int_0^\infty t^2 v'(t) dt \ T_D^2\right]\right\}^{0.5} \tag{4.b}$$

where $v'(t)$is the derivative of the transient voltage. The definitions of (4) are assuming integer order integration and hence they are based on using traditional integer order elements$\{R, L, C\}$. Also, the impact of the fractional order α of the fractional order elements is not considered in the slope calculations for the rise time although it is expected to have a great impact on the response slope [11]. Thus, using formulas of (4) to determine the timing response of a system which utilizes a fractional order elements is not correct as these formulas ignore the impact of the fractional order.

On the other hand, the system function $H(s)$ and the transient response $e'(t)$ are related by the direct Laplace transform as given in (5):

$$H(s) = \int_0^M E_\alpha(-s^\alpha t^\alpha) e'(t) dt^\alpha \tag{5}$$

where $E_\alpha(Z)$ is the Mittag-Luffer function which is given by

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)} \tag{6}$$

Hence, by expanding the Laplace integral (5) in a power series in s, which will be a valid expansion of $H(s)$ for values of s lying within the circle of convergence $|s| = |s_1|$· Consequently, the transfer function of (3) could be expressed as follows:

$$H(s) = 1 - \int_0^\infty \frac{(st)^\alpha}{\Gamma(\alpha+1)} e'(t) dt^\alpha + \int_0^\infty \frac{(st)^{2\alpha}}{\Gamma(2\alpha+1)} e'(t) dt^\alpha + \cdots \tag{7}$$

Indeed, from (7) new definitions for the delay and rise time are obtained as follows:

$$T_D = \int_0^\infty t^\alpha e'(t) dt^\alpha \tag{8.a}$$

$$T_R = \left(2\pi\left\{\int_0^\infty t^{2\alpha} e'(t) dt^\alpha - T_D^2\right\}\right)^{0.5} \tag{8.b}$$

Actually, the definitions of (8) are the general expression for Elmore timing definitions of (4). It is important to note here that the definitions of (8) are dependent on the fractional order α. Hence, the impact of the fractional order delay on the system timing is included in case of using the formulas of (8). Also, by making $\alpha = 1$, the formulas of (8) become the same as the formulas of (4) which confirm the analysis.

To simplify the analysis, assume the fractional orders of (3) are dependent on the real value α where $\alpha_i = i\alpha, \beta_j = j\alpha$ and $i = 1 \dots n$ and $j = 1 \dots m$. Hence the transfer function of (3) can be rewritten as follows:

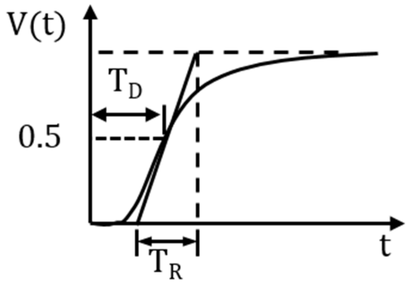


Fig. 1 Curve shows the original definition of Elmore rise and delay time.

$$H(s) = \frac{1 + a_1 s^\alpha + a_2 s^{2\alpha} + \cdots + a_n s^{n\alpha}}{1 + b_1 s^\alpha + b_2 s^{2\alpha} + \cdots + b_m s^{m\alpha}} \tag{9}$$

By expanding the transfer function of (9) and then approximating to the second order components, the delay and rise time definitions of (8) are given by:

$$T_D = \Gamma(\alpha + 1)(b_1 - a_1) \tag{10.a}$$

$$\frac{T_R^2}{2\pi} = \Gamma(2\alpha + 1)(a_2 - b_2 - b_1(a_1 - b_1)) - T_D^2 \tag{10.b}$$

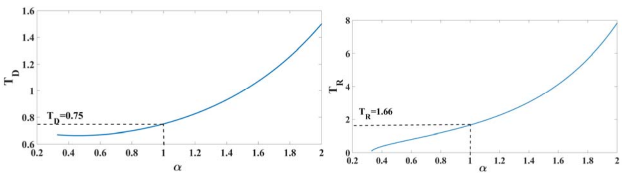
where Γ is the gamma function given by [8]:

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz \tag{11}$$

From (10), the delay and rise time are functions of the fractional order α. By putting the fractional order $\alpha = 1$, the delay definitions of (10) become equal the delay and rise expressions of [15]. Hence, the timing expressions of (10) are the general form for Elmore delay definitions in the fractional order domain.

Furthermore, the impact of the fractional order α on the rise and delay time is illustrated in Fig. 2 (a, b) respectively. For $\alpha < 1$, the change in the delay time is very small with respect to α. On the other hand, the change in the rise time is large with the fractional order α. Hence, the rise time of the pulse can be controlled independently without affecting the delay time of the pulse as illustrated in Fig. 2. This increases the design degree of freedom and flexibility. Moreover, the behavior of Elmore time formula is acting as a capacitance effect for $\alpha > 1$ and as an inductive response for $\alpha < 1$. This is because the delay and rise time increase as the value of α increases which means the capacitance effect increase on the system timing performance. The rise time and delay time does not have values for $\alpha < 0.33$ as the system is unstable and hence no valid solution for T_R, and T_D for this range at $a_1 = b_2 = 0.25$, $a_2 = 0$, and $b_1 = 1$.

Moreover, the impact of the change in b_1 at different values of the fractional order α is depicted in Fig. 3.



(a) (b)

Fig. 2 (a) Change of the delay time with respect to the fractional order α when$a_1 = b_2 = 0.25$, $a_2 = 0$, and $b_1 = 1$, (b) Change of the rise time with respect to the fractional order α when $a_1 = b_2 = 0.25$, $a_2 = 0$, and $b_1 = 1$.

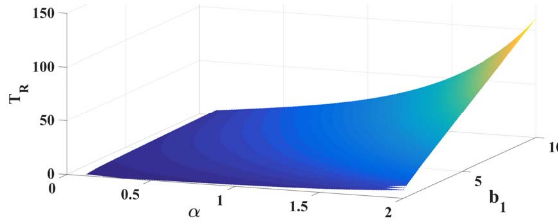


Fig. 3 Impact of α and b_1 on the rise time (T_R) at $a_1 = b_2 = 0.25$, $a_2 = 0$,

Interesting to note here that, for small values of b_1, the effect of both α and b_1 is very small. On the other hand, as the value of b_1 increases, the rise time increases strongly with the large values of $(\alpha > 1.5)$. Yet, for small value of the fractional order α, the effect of b_1 on the rise time is very small as illustrated in Fig. 3. Hence, the effect of b_1 on the rise time could be compensated by changing the fractional order α. This actually adds a very important feature to Elmore delay which is robustness without affecting the circuit components by changing the fractional order.

III. CASE STUDY

The fractional order model of Elmore delay is applied to a shunt model shown in Fig. 4 to prove the reliability of the proposed model. The transfer function for this circuit is given by:

$$G(s) = \frac{1}{R}\frac{1+s^{\alpha}L/R}{1+s^{\alpha}RC+LCs^{2\alpha}} \quad (12)$$

So, from (11) the parameters a_1, b_1, b_2 are given as follows:

$$a_1 = \frac{L}{R}, \quad b_1 = RC, \quad b_2 = LC \quad (13)$$

Hence from (10) and (13) Elmore rise and delay times are given by:

$$T_D = \frac{\Gamma(\alpha+1)}{R}(R^2C - L) \quad (14.a)$$

$$T_R^2 = 2\pi\{\Gamma(2\alpha+1)(-2LC + R^2C^2) - T_D^2\} \quad (14.b)$$

From (14) for $\alpha = 1$, $R = C = 1$ and $L = 1/4$, which is corresponding to the critical shunt compensation, $T_D = 3/4$ and $T_R = 1.66$ which is the same as the values calculated in [15]. So, the definitions of (14) are the general representation for the shunt compensation delay in the fractional order domain.

From (14), the relation between the rise time and delay time is non-linear with the circuit parameters and the fractional order α. The change in the delay time with respect to the capacitance is very high at large values of the fractional order α. On the other hand, for $\alpha < 1$ the effect of the capacitor C on the delay time is very small as illustrated in the MATLAB analysis of Fig 5(a). Furthermore, the impact of the fractional order α and the inductance L on the delay time is illustrated in Fig. 5(b). Delay time increases with the increase in α with small values of inductance L. Moreover, for the same value of α, the delay time decreases with the increase in the inductance. From Fig. 5, the delay time does not have a valid value for all values of the inductance or capacitance with different order. This means, the shunt compensation circuit is not stable for these points. Actually, the fractional order α adds another degree of freedom to the

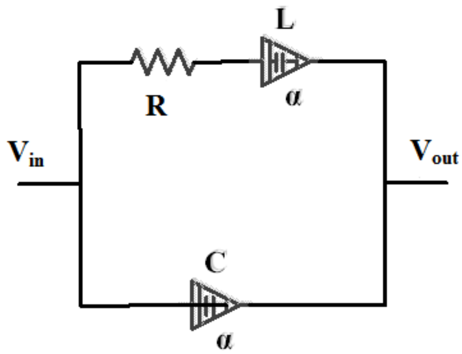


Fig. 4 Shunt compensation with fractional order inductor and capacitor with same fractional order α

delay formula which could be used to compensate for the effect of the components to achieve a specific delay time. In other words, the system could be designed for a specific delay time with minim change in the components value.

The change of the rise time and delay time with respect to C, L, and the fractional order α is very similar as illustrated in Fig. 6 (a, b) respectively. For the same value of capacitance, the rise time increases with the increase in the fractional order α. Moreover, the rise time increases with the increase in the fractional order for same value of the capacitance. Hence, to achieve a small value of the rise time, small value of the capacitance with small order should be used. Yet, the rise time does not have a valid value for all combinations of the fractional order α and C as shown in Fig. 6(a). So, the system could be designed to fulfill a specific rise time by tuning the fractional order α.

By comparing Fig. 6(a, b), the impact of the inductance on the rise time is higher than the impact of the capacitance on the rise time for the same value of the fractional order. This means, the rise time could have also two levels of tuning by changing the inductance and capacitance for same order. On the other hand, for same value of the fractional order, the impact of the capacitance and the inductance on the delay time is very similar as shown in Fig. 5 (a, b). So, the new definition of Elmore delay time and rise time increase the design degree of freedom by including the effect of the fractional order.

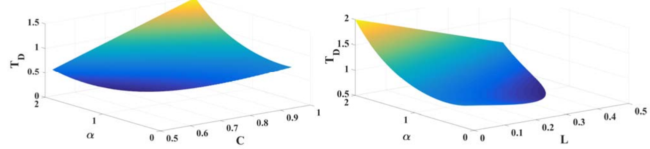
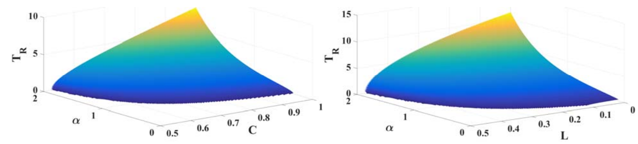


Fig. 5 (a) change of the delay time with respect to α and C at $R = 1\,\Omega, L = 0.25H$, (b) change of the delay time with respect to α and L at $R = 1\,\Omega, C = 1F$



(a) (b)

Fig. 6 (a) change of the rise time with respect to α and C at $R = 1\,\Omega, L = 0.25H$, (b) change of the rise time with respect to α and L at $R = 1\,\Omega, C = 1F$

Furthermore, the fractional order could be used to compensate for the effect of the inductance or the capacitance on the rise time or delay time.

IV. CIRCUIT SIMULATION

To verify the analysis, circuit simulation for the circuit illustrated in Fig. 4 using Advanced Design System (ADS) is presented in this section. The fractional order elements are emulated using the ladder network of [17]. The circuit is simulated using same values of the resistance, capacitance and inductance but with different fractional orders to show the impact of the fractional order on the delay and rise time. When, the order of the capacitor is less than the ideal case, the circuit response has more delay as shown in Fig. 7. While for simulation using the traditional capacitor, the response is same as the idea response. Hence, the using fractional order capacitor in the circuit makes the circuit response very similar to the actual interconnect response. Furthermore, using fractional order inductance in the model improves the circuit rise and delay time as illustrated in Fig. 7. Hence, the fractional order inductance could be integrated with the interconnect to improve its response.

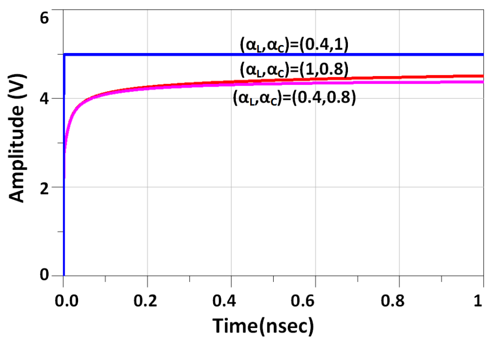


Fig. 7 Circuit simulation for the circuit of Fig 4 for same values of $R = 10\Omega$, $L = 3.7\mu H$, and $C = 10nF$ but different orders

V. CONCLUSION

In this work, a new definition for Elmore delay time and rise time in the fractional order domain is introduced. The proposed formula takes the effect of the fractional order into account when calculating the rise and delay times. By including the fractional order in the timing formulas another degree of freedom is added which increases the design flexibility and widens the optimization space. Finally, a case study of a shunt compensation with two fractional order elements of the same order is studied. From the MATLAB analysis, the shunt compensation timing could be optimized for a specific values by changing only the fractional order without affecting the components value.

The future work will involve comparing the analytical data with the characterisation of actual physical transmission lines which we sent for fabrication recently.

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