Calcular algebras

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To the memory of Richard Timoney

Abstract

A calcular algebra is a subalgebra of $H^\infty(\Omega)$ with norm given by $\|\phi\| = \sup \|\phi(T)\|$ as T ranges over a given class of commutative d-tuples of operators with Taylor spectrum in Ω. We discuss what algebras arise this way, and how they can be represented.

1 Introduction

Let Ω be a bounded open set in $\mathbb{C}^d$. We say that a class $\mathcal{C}$ is *subordinate to* Ω if:

(i) Each element T of $\mathcal{C}$ is a commuting d-tuple of bounded operators on a Hilbert space, with its Taylor spectrum[1] $\sigma(T)$ in Ω.

(ii) For some non-zero Hilbert space $\mathcal{H}$, the set of scalars

$$\{(\lambda^1, \ldots, \lambda^d) : \lambda \in \Omega\} \subseteq \mathcal{C},$$

where we think of λ as a d-tuple of scalar multiples of the identity acting on $\mathcal{H}$. (Note that we use superscripts to denote the coordinates.)

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[1]For a definition of Taylor spectrum of a commuting tuple, see [11].

Given a class $\mathcal{C}$ subordinate to Ω, we define $H^\infty(\mathcal{C})$ to be those holomorphic functions on Ω for which

$$\|\phi\|_{\mathcal{C}} \;=\; \sup\{\|\phi(T)\| : T \in \mathcal{C}\}$$

is finite. It can be shown (see Prop. 2.1 below) that this algebra is always complete, so it is a Banach algebra, which by Property (ii) is always contained contractively in the algebra $H^\infty(\Omega)$ of bounded holomorphic functions on Ω. (We are using H^∞ in two apparently different ways, but identifying Ω with the set of scalars makes the two usages agree). Any Banach algebra of holomorphic functions arising in this way we shall call a *calcular algebra over* Ω.

We shall call the closed unit ball of $H^\infty(\mathcal{C})$ the *Schur class of* $\mathcal{C}$, and denote it by $\mathscr{S}(\mathcal{C})$.

$$\mathscr{S}(\mathcal{C}) \;=\; \{\phi \in \mathrm{Hol}(\Omega) : \|\phi(T)\| \leq 1,\ \forall\ T \in \mathcal{C}\}.$$

Let $\mathscr{S}(\Omega)$ denote the closed unit ball of $H^\infty(\Omega)$.

If $\mathcal{H}$ is a Hilbert space (we shall always assume that Hilbert spaces are not zero-dimensional to avoid trivialities), let $CB(\mathcal{H})^d$ denote the set of commuting d-tuples of elements of $B(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. Given a set S of bounded holomorphic functions on Ω, and a Hilbert space $\mathcal{H}$, one can form the set

$$\mathcal{H}(S) \;=\; \{T \in CB(\mathcal{H})^d :\ \sigma(T) \subseteq \Omega \text{ and } \|\phi(T)\| \leq 1\ \forall\ \phi \in S\}.$$

If $\mathcal{H}$ is a Hilbert space, $\mathcal{C} \subseteq CB(\mathcal{H})^d$ and $S \subseteq \mathscr{S}(\Omega)$, then tautologically one has

$$\mathcal{H}(\mathscr{S}(\mathcal{C})) \quad \supseteq \quad \mathcal{C} \qquad (1.1)$$

$$\mathscr{S}(\mathcal{H}(S)) \quad \supseteq \quad S. \qquad (1.2)$$

Typically these inclusions will be strict. For example, let $d = 1$, and let Ω be the open unit disk $\mathbb{D}$. Let $\mathcal{H}$ be any Hilbert space, and let $\mathcal{C}$ be the set $\{\lambda I : \lambda \in \mathbb{D}\}$. Then $\mathscr{S}(\mathcal{C})$ will equal $\mathscr{S}(\mathbb{D})$, and, by von Neumann's inequality [13], $\mathcal{H}(\mathscr{S}(\mathbb{D}))$ will consist of all contractions on $\mathcal{H}$ whose spectrum is in $\mathbb{D}$. Likewise if S just contains the function z, then $\mathcal{H}(S)$ will be the contractions on $\mathcal{H}$ whose spectrum is in $\mathbb{D}$, and the Schur class of this set will be all of $\mathscr{S}(\mathbb{D})$.

Our first result is that the operations $\mathcal{H}$ and $\mathscr{S}$ stabilize after 3 steps, provided $\mathcal{H}$ is infinite dimensional.

Notation: If T is a commuting d-tuple of bounded operators on a Hilbert space $\mathcal{H}$, we call $\mathcal{H}$ the carrier of T, and write $\mathcal{H} = \mathrm{car}(T)$.

Theorem 1.3. *Let Ω be a bounded open set in $\mathbb{C}^d$, and let $\mathcal{C}$ be any class subordinate to Ω. Let S be a non-empty subset of $\mathscr{S}(\Omega)$. For any Hilbert space $\mathcal{H}$, we have*

$$\mathcal{H}(\mathscr{S}(\mathcal{H}(S))) = \mathcal{H}(S). \tag{1.4}$$

If the dimension of $\mathcal{H}$ is either infinite, or greater than or equal to $\sup\{\dim(\mathrm{car}(T)) : T \in \mathcal{C}\}$*, then*

$$\mathscr{S}(\mathcal{H}(\mathscr{S}(\mathcal{C}))) \ = \ \mathscr{S}(\mathcal{C}) \tag{1.5}$$

Proof: By (1.2), we have

$$\mathcal{H}(\mathscr{S}(\mathcal{H}(S))) \subseteq \mathcal{H}(S). \tag{1.6}$$

Suppose now that $T \in \mathcal{H}(S)$, and ϕ is any function in $\mathscr{S}(\mathcal{H}(S))$. Then $\|\phi(T)\| \leq 1$, so T is in $\mathcal{H}(\mathscr{S}(\mathcal{H}(S)))$, proving (1.4).

By (1.2) again, with $S = \mathscr{S}(\mathcal{C})$, we get

$$\mathscr{S}(\mathcal{C}) \ \subseteq \ \mathscr{S}(\mathcal{H}(\mathscr{S}(\mathcal{C}))). \tag{1.7}$$

Now, assume the dimension of $\mathcal{H}$ is as in the hypothesis. Let $\phi \in \mathscr{S}(\mathcal{H}(\mathscr{S}(\mathcal{C})))$, and let $T \in \mathcal{C}$, with $\mathrm{car}(T) = \mathcal{K}$. We need to show $\|\phi(T)\| \leq 1$. If the dimension of $\mathcal{H}$ is equal to the dimension of $\mathcal{K}$, then T is unitarily equivalent to a d-tuple R on $\mathcal{H}$, and $R \in \mathcal{H}(\mathscr{S}(\mathcal{C}))$ since $\|\psi(R)\| = \|\psi(T)\| \leq 1$ for every ψ in $\mathscr{S}(\mathcal{C})$. Therefore $\|\phi(T)\| = \|\phi(R)\| \leq 1$, and we are done.

If the dimension of $\mathcal{H}$ is larger than the dimension of $\mathcal{K}$, write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\dim(\mathcal{H}_1) = \dim(\mathcal{K})$, and let R_1 on $\mathcal{H}_1$ be unitarily equivalent to T. Choose $\lambda = (\lambda^1, \ldots, \lambda^d)$ in Ω, and let

$$R \ = \ (R_1^1 \oplus \lambda^1 I_{\mathcal{H}_2}, \ldots, R_1^d \oplus \lambda^d I_{\mathcal{H}_2}),$$

Then for any $\psi \in O(\Omega)$, the set of holmorphic functions on Ω, we have $\psi(R) = \psi(R_1) \oplus \psi(\lambda) I_{\mathcal{H}_2}$, so if $\psi \in \mathscr{S}(\mathbb{C})$, we have $\|\psi(R)\| \leq 1$, and therefore $R \in \mathcal{H}(\mathscr{S}(\mathcal{C}))$. Now we get $\|\phi(T)\| \leq \|\phi(R)\| \leq 1$, and again we are done.

Finally we consider the case where $\mathcal{H}$ is infinite dimensional, but the carriers of the elements of $\mathcal{C}$ may have larger dimension. We can assume without loss of generality that $\mathcal{H}$ is separable. We need to find $R \in \mathcal{H}(\mathscr{S}(\mathcal{C}))$ with $\|\phi(R)\| = \|\phi(T)\|$. To do this, it is sufficient to show that there is a separable subspace $\mathcal{K}_1$ of $\mathcal{K}$ that is reducing for $f(T)$ for every $f \in O(\Omega)$ and such that $\|\phi(T)|_{\mathcal{K}_1}\| = \|\phi(T)\|$; for then we can choose R_1 on $\mathcal{H}$ unitarily equivalent to $P_{\mathcal{K}_1} T|_{\mathcal{K}_1}$, where $P_{\mathcal{K}_1}$ is projection onto $\mathcal{K}_1$; the fact that $\mathcal{K}_1$ is reducing means that $\phi(P_{\mathcal{K}_1} T|_{\mathcal{K}_1}) = P_{\mathcal{K}_1}\phi(T)|_{\mathcal{K}_1}$.

Observe that $\{f(T) : f \in O(\Omega)\}$ has a countable dense subset $\mathcal{D}$ in the norm topology of $CB(\mathcal{K})^d$, since $O(\Omega)$ is separable. Let u_j be a sequence of unit vectors in $\mathcal{K}$ such that $\|\phi(T)u_j\| \to \|\phi(T)\|$. Let $\mathcal{K}_1$ be the closed linear span of finite products of elements of $\mathcal{D} \cup \mathcal{D}^*$ applied to finite linear combinations of the vectors u_j. By $\mathcal{D}^*$ we mean

$$\mathcal{D}^* \ = \ \{((T^1)^*, \dots, (T^d)^*) \ : \ (T^1, \dots, T^d) \in \mathcal{D}\}.$$

Then $\mathcal{K}_1$ is a separable subspace of $\mathcal{K}$ on which $\phi(T)$ achieves its norm and that is reducing for every $f(T)$. □

For a given class $\mathcal{C}$, it is of interest to know the smallest dimension of $\mathcal{H}$ that gives equality in (1.5).

Example 1.8 Let $\Omega = \mathbb{D}$, and let $\mathcal{C}$ be all contractions with spectrum in $\mathbb{D}$. Then we can take $\mathcal{H}$ to be one dimensional. Similarly, if $\Omega = \mathbb{D}^2$, and $\mathcal{C}$ is all pairs of commuting contractions with spectrum in $\mathbb{D}^2$, Andô's inequality [4] yields that we can take $\mathcal{H}$ to be one dimensional again.

However, if $d \geq 3$, we let $\Omega = \mathbb{D}^d$, and $\mathcal{C}$ be the class of all d-tuples of commuting contractions with spectrum contained in $\mathbb{D}^d$, then $\mathscr{S}(\mathcal{C})$ is the Schur-Agler class, a proper subset of $\mathscr{S}(\mathbb{D}^d)$ [6, 12]. If $\mathcal{H} = \mathbb{C}^n$, then $\mathcal{H}(\mathscr{S}(\mathcal{C}))$ will be all d-tuples of commuting contractive n-by-n matrices with spectrum in $\mathbb{D}^d$. In [9], it is shown that if $n = 3$, then

$$\mathscr{S}(\mathbb{C}^3(\mathscr{S}(\mathcal{C}))) \ = \ \mathscr{S}(\mathbb{D}^d).$$

It is unknown what the minimal dimension of $\mathcal{H}$ must be in this case to get equality in (1.5), or even whether it must be infinite.

Example 1.9 Let $\mathcal{K}$ be a Hilbert function space on Ω with reproducing kernel k. The multiplier algebra $\mathrm{Mult}(\mathcal{K})$ is always a calcular algebra. Indeed, for each finite set $F = \{\lambda_1, \dots, \lambda_n\} \subset \Omega$, let T_F be the commuting d-tuple $(T_F^1, \dots, T_F^d)$ acting on the n-dimensional subspace of $\mathcal{K}$ spanned by the kernel functions $\{k_{\lambda_j} : 1 \leq j \leq n\}$ defined by

$$T_F^r k_{\lambda_j} \;=\; \overline{\lambda_j^r} k_{\lambda_j} \qquad 1 \leq r \leq d, 1 \leq j \leq n.$$

Define

$$\mathcal{C} = \{T_F^* : F \text{ a finite subset of } \Omega\}.$$

It is straightforward to show that $H^\infty(\mathcal{C}) = \mathrm{Mult}(\mathcal{K})$. Many other examples of calcular algebras are given in [2, Chapter 9].

2 When is a Banach algebra a calcular algebra?

Proposition 2.1. *Let $\mathcal{C}$ be subordinate to Ω. Then $H^\infty(\mathcal{C})$ is a Banach algebra.*

PROOF: We need to prove completeness. Consider a Cauchy sequence $\{\phi_n\}$ in $H^\infty(\mathcal{C})$. Since $\mathcal{C}$ is subordinate to Ω, $\{\phi_n\}$ is a Cauchy sequence in $H^\infty(\Omega)$. Therefore, as $H^\infty(\Omega)$ is complete, there exists $\phi \in H^\infty(\Omega)$ such that

$$\sup_{\lambda \in \Omega} |\phi_n(\lambda) - \phi(\lambda)| \to 0 \;\text{ as }\; n \to \infty. \tag{2.2}$$

We claim that

$$\phi \in H^\infty(\mathcal{C}) \tag{2.3}$$

and

$$\phi_n \to \phi \;\text{ in }\; H^\infty(\mathcal{C}). \tag{2.4}$$

To prove statement (2.3), note that for each $T \in \mathcal{C}$, we have Ω is a neighborhood of $\sigma(T)$. Consequently, continuity of the functional calculus implies that

$$\phi_n(T) \to \phi(T) \qquad \forall\, T \in \mathcal{C}. \tag{2.5}$$

Also, as $\{\phi_n\}$ is a Cauchy sequence in $H^\infty(\mathcal{C})$, there exists a constant M such that

$$\|\phi_n\|_{\mathcal{C}} \leq M \qquad \forall\, n.$$

Therefore, if $T \in \mathcal{C}$,

$$\|\phi(T)\|_{\mathcal{C}} \;=\; \lim_{n\to\infty} \|\phi_n(T)\| \leq \limsup_{n\to\infty} \|\phi_n\|_{\mathcal{C}} \;\leq\; M.$$

Hence,

$$\|\phi\|_{\mathcal{C}} \;=\; \sup_{T\in\mathcal{C}} \|\phi(T)\|_{\mathcal{C}} \;\leq\; M,$$

which proves the membership (2.3).

To prove the limiting relation (2.4), let $\varepsilon > 0$. Choose N such that

$$m, n \geq N \implies \|\phi_n - \phi_m\|_{\mathcal{C}} < \varepsilon.$$

By definition of the norm, this means

$$m, n \geq N \implies \|\phi_n(T) - \phi_m(T)\| < \varepsilon \qquad \forall\, T \in \mathcal{C}.$$

Letting $m \to \infty$ and using statement (2.5) we deduce that

$$n \geq N \implies \|\phi_n(T) - \phi(T)\| < \varepsilon \qquad \forall\, T \in \mathcal{C}.$$

Hence, since $\|\phi_n - \phi\|_{\mathcal{C}} = \sup_{T\in\mathcal{C}} \|\phi_n(T) - \phi(T)\|$,

$$n \geq N \implies \|\phi_n - \phi\|_{\mathcal{C}} \leq \varepsilon.$$

□

Let $\mathcal{A}$ be a unital Banach algebra contractively contained in $H^\infty(\Omega)$. When can it be realized as a calcular algebra? Let S be its unit ball. By Theorem 1.3, $\mathcal{A}$ is a calcular algebra if and only if $\mathscr{S}(\mathcal{H}(S)) = S$, where $\mathcal{H}$ is an infinite dimensional Hilbert space.

This imposes a constraint on $\mathcal{A}$. In particular, there must be an isometric homomorphsim from $\mathcal{A}$ into $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. There is another constraint which stems from the requirement that all the operators in the class have spectrum in the open set Ω.

Proposition 2.6. *If $\mathcal{A}$ is a calcular algebra, then:*

(i) *There is an isometric homomorphism into $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.*

(ii) *If ϕ_n is a bounded sequence in $\mathcal{A}$ that converges uniformly on compact subsets of Ω to a function ψ, then $\psi \in \mathcal{A}$, and $\|\psi\| \leq \liminf \|\phi_n\|$.*

Proof: (i) Suppose $\mathcal{A}$ is $H^\infty(\mathcal{C})$ for some class $\mathcal{C}$ subordinate to an open set Ω. Let $\mathcal{H}$ be any infinite dimensional Hilbert space, and let $\mathcal{C}_1 = \mathcal{H}(\mathscr{S}(\mathcal{C}))$. By Theorem 1.3, we have

$$\mathcal{A} \;=\; H^\infty(\mathcal{C}_1). \tag{2.7}$$

Let $\mathcal{K}$ be the direct sum of cardinality$(\mathcal{C}_1)$ copies of $\mathcal{H}$, with the sum indexed by $\mathcal{C}_1$. Define a map $\pi : \mathcal{A} \to B(\mathcal{K})$ by

$$\pi(\phi) \ = \ \oplus_{T \in \mathcal{C}_1} \phi(T).$$

Then π is a homomorphism, and by (2.7) it is isometric.

(ii) Let ϕ_n be a bounded sequence in $H^\infty(\mathcal{C})$ converging to ψ locally uniformly on Ω. Without loss of generality, we may assume that each ϕ_n is in $\mathscr{S}(\mathcal{C})$. For each T in $\mathcal{C}$, since $\sigma(T) \subseteq \Omega$, it follows from the continuity of the functional calculus that $\psi(T)$ is the limit in norm of $\phi_n(T)$, so ψ is in $\mathscr{S}(\mathcal{C})$. Replacing ϕ_n by a subsequence whose norms converge to $\liminf \|\phi_n\|$ gives the last inequality. □

Remark: If one defines $S = \oplus_{T \in \mathcal{C}_1}(T)$, then one can interpret $\phi(S)$ as $\pi(\phi)$. However, the spectrum of S will be $\overline{\Omega}$, so S is not contained in any class subordinate to Ω. The Taylor functional calculus is defined only for functions holomorphic on a neighborhood of the Taylor spectrum of the d-tuple.

A necessary condition for a Banach algebra to be isometrically isomorphic to an algebra of operators on a Hilbert space is that it satisfies von Neumann's inequality: $\|p(x)\| \leq \|p\|_{H^\infty(\mathbb{D})}$ for any x in the unit ball of the Banach algebra, and any polynomial p. It is not known whether this condition is sufficient.

Calcular algebras come with a sequence of matrix norms. If $[\phi_{ij}]$ is an n-by-n matrix of elements of $H^\infty(\mathcal{C})$, one can define

$$\|[\phi_{ij}]\|_n \ = \ \sup\{\|[\phi_{ij}(T)]\| : T \in \mathcal{C}\},$$

where the norm on the right-hand side is the operator norm on $\mathrm{car}(T) \otimes \mathbb{C}^n$. By a similar argument to Proposition 2.6, one can show that calcular algebras have completely isometric homomorphic embeddings into some $B(\mathcal{K})$.

Algebras that can be completely isometrically realized in this way are characterized by the Blecher-Ruan-Sinclair theorem [5], [10, Cor. 16.7]. This says that the algebra $\mathcal{A}$ must satisfy the Ruan axioms:

$$\begin{array}{rcl} \forall n \in \mathbb{N}, \forall a \in M_n(\mathcal{A}), \forall X, Y \in M_n(\mathbb{C}), \quad \|XaY\|_n & \leq & \|X\| \|a\|_n \|Y\| \\ \forall m, n \in \mathbb{N}, \forall a \in M_m(\mathcal{A}), b \in M_n(\mathcal{A}), \quad \|a \oplus b\|_{m+n} & = & \max \|a\|_m, \|b\|_n, \end{array}$$

and hence be isometrically realizable as an operator space; and the matrix multiplication at each level n must be contractive, *i.e.* if $a = [a_{ij}]$ and $b = [b_{ij}]$

are in $M_n(\mathcal{A})$, then

$$\|[\sum_{k=1}^{n} a_{ik}b_{kj}]\|_n \ \leq \ \|[a_{ij}]\|_n \|[b_{ij}]\|_n.$$

It is straightforward to check that a calcular algebra satisfies the hypotheses of the Blecher-Ruan-Sinclair theorem.

We do not know in general what intrinsic necessary and sufficient conditions on a sub-algebra of $H^\infty(\Omega)$ make it a calcular algebra; we can say something with strong convexity assumptions. Let $\mathcal{P} = \mathbb{C}[z_1, \ldots, z_d]$ denote the polynomials. If f is a function and $r > 0$, define f_r by $f_r(z) = f(rz)$.

Theorem 2.8. *Let $\mathcal{A}$ be a unital Banach algebra that is contractively contained in $H^\infty(\Omega)$, for some bounded open convex set Ω in $\mathbb{C}^d$ that contains 0. Suppose that $\mathcal{P}$ is contained in $\mathcal{A}$ and that for every function $\phi \in \mathcal{A}$, there is a sequence in $\mathcal{P}$ that is bounded in norm by $\|\phi\|$ and converges to ϕ locally uniformly on Ω. Suppose moreover that for every polynomial $p \in \mathcal{P}$, we have $\|p_r\| \leq \|p\|$ for $0 < r < 1$.*

Then $\mathcal{A}$ is a calcular algebra over Ω if and only if the necessary conditions of Proposition 2.6 hold.

PROOF: Suppose both conditions hold, and π embeds $\mathcal{A}$ isometrically in $B(\mathcal{K})$. For each of the coordinate functions $z^j, 1 \leq j \leq d$, define $T^j = \pi(z^j)$. Let $T \in CB(\mathcal{K})^d$ be the tuple $(T^1, \ldots, T^d)$. Then for any polynomial $p \in \mathcal{P}$ we have $\pi(p) = p(T)$. Moreover, if p has no zeroes on $\overline{\Omega}$, then

$$\pi(p\,\frac{1}{p}) \ = \ 1_{\mathcal{K}} \ = \ p(T)\pi(\frac{1}{p}),$$

so

$$\pi(\frac{1}{p}) \ = \ p(T)^{-1}.$$

As p ranges over affine functions whose zero sets are hyperplanes not intersecting $\overline{\Omega}$, we see that $\sigma(T)$ must be contained in $\overline{\Omega}$.

We want the elements of $\mathcal{C}$ to have spectrum in Ω. Let $\mathcal{C} = \{rT : 0 \leq r < 1\}$.

For any polynomial p and any sequence $r_n \uparrow 1$ we have

$$\begin{aligned} \|p\|_{\mathcal{A}} &= \|\pi(p)\| \\ &= \|p(T)\| \\ &= \lim_{n\to\infty} \|p(r_n T)\| \\ &\leq \|p\|_{\mathcal{C}} \\ &= \sup_{0<r<1} \|p_r(T)\| \\ &= \sup_{0<r<1} \|\pi(p_r)\| \\ &\leq \|p\|_{\mathcal{A}}. \end{aligned}$$

So $\mathcal{A}$ and $H^\infty(C)$ assign the same norm to polynomials.

Let ψ be in $\mathcal{A}$ of norm 1. By hypothesis, there is a sequence of polynomials q_n that converges locally uniformly to ψ, with $\|q_n\|_{\mathcal{A}} \leq 1$. Therefore for each $0 \leq r < 1$,

$$\|\psi(rT)\| \;=\; \lim_{n\to\infty} \|q_n(rT)\| \;\leq\; 1.$$

Therefore ψ is in the unit ball of $H^\infty(C)$, and hence $\mathcal{A}$ is contractively contained in $H^\infty(C)$.

Conversely, let $\phi \in \mathscr{S}(C)$. Since Ω is convex, ϕ_r will converge to ϕ locally uniformly on Ω as $r \uparrow 1$. Fix $r < 1$. There is a sequence q_n of polynomials that converges uniformly to ϕ_r on a neighborhood of $\overline{\Omega}$. Therefore $\lim_{n\to\infty} q_n(T) = \phi_r(T)$ is a contraction, and so by Property (ii) we have

$$\|\phi_r\|_{\mathcal{A}} \;\leq\; 1.$$

By a diagonalization argument, we can modify this construction to find polynomials q_n in the unit ball of $\mathcal{A}$ that converge locally uniformly to ϕ, and hence

$$\|\phi\|_{\mathcal{A}} \leq 1.$$

So $H^\infty(C)$ is contractively contained in $\mathcal{A}$, and hence the two algebras are isometrically isomorphic. $\square$

Example 2.9 The disk algebra $A(\mathbb{D})$ cannot be a calcular algebra, since it fails (ii). However, there are subalgebras of the disk algebra that are the multiplier algebra of some Hilbert function spaces on the disk, *e.g.* the space with reproducing kernel

$$k(w,z) \;=\; \sum_{n=0}^{\infty}(n+1)^2 z^n \bar{w}^n.$$

Multiplier algebras for spaces of holomorphic functions are always calcular, as shown in Example 1.9.

Problem 2.10 Find necessary and sufficient conditions for a subalgebra of $H^\infty(\Omega)$ to be a calcular algebra.

3 Realization formulas

In [7] and [8], Dritschel, Marcantognini, and McCullough proved a very general realization formula, building on work of Ambrozie and Timotin in [3], which can be adapted to our current setting.

Let S be a set of functions from a set X to the unit disk $\mathbb{D}$. In this section, we shall make the standing assumption that S restricted to any finite set F generates, as an algebra, all the complex-valued functions on F.

We define K_S to be the set of kernels on X that satisfy

$$K_S \;=\; \{k \mid (1-\psi(z)\bar{\psi}(w))k(z,w) \;\geq\; 0 \quad \forall\, \psi \in S\}.$$

We define $A^\infty(K_S)$ to be

$$A^\infty(K_S) = \{\phi : X \to \mathbb{C} \mid \exists M \geq 0 \text{ s.t. } (M^2 - \phi(z)\bar{\phi}(w))k(z,w) \;\geq\; 0\,\forall\, k \in K_S\},$$

and define $\|\phi\|$ to be the smallest M that works.

Endow S with the topology of pointwise convergence. Let $C_b(S)$ denote the continuous bounded functions on S, which we think of as a C*-algebra. Let $E : X \to C_b(S)$ be the evaluation map $E(z)(\psi) = \psi(z)$, and let $E(w)^*$ mean the complex conjugate of this, the adjoint in the C*-algebra, $E(w)^*(\psi) = \overline{\psi(w)}$.

If ψ is a function from X to $\mathbb{C}$, we say it has a *network realization formula* if there exists a Hilbert space $\mathcal{M}$, a unital *-representation $\rho : C_b(S) \to B(\mathcal{M})$, and a unitary $U : \mathbb{C} \oplus \mathcal{M} \to \mathbb{C} \oplus \mathcal{M}$ that in block matrix form is

$$U \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that

$$\psi(z) \ = \ A + B\rho(E(z))(I - D\rho(E(z)))^{-1}C. \tag{3.1}$$

If $\mathcal{B}$ is a C*-algebra, a positive kernel on a set X with values in $\mathcal{B}^*$, the dual of $\mathcal{B}$, is a function $\Gamma : X \times X \to \mathcal{B}^*$ such that for every finite set $F \subset X$, and every $f : F \to \mathcal{B}$ we have

$$\sum_{z,w \in F} \Gamma(z,w)(f(w)^* f(z)) \geq 0.$$

Here is the Dritschel, Marcantognini, and McCullough theorem.

Theorem 3.2. *Let S be a set of functions from X to $\mathbb{D}$, and let $\phi : X \to \overline{\mathbb{D}}$. The following are equivalent:*

(i) $\phi \in A^\infty(K_S)$ *and* $\|\phi\|_{A^\infty(K_S)} \leq 1$.

(ii) *For each finite set $F \subseteq X$ there exists a positive kernel $\Gamma : F \times F \to C_b(S)^*$ so that, for all $z, w \in F$,*

$$1 - \phi(z)\overline{\phi(w)} \ = \ \Gamma(z,w)(1 - E(z)E(w)^*). \tag{3.3}$$

(iii) *ϕ has a network realization formula.*

Now let us assume that the functions in S are all holomorphic functions on the open set Ω in $\mathbb{C}^d$. By definition, we always have S is contained in the unit ball of $A^\infty(K_S)$, so when $\mathcal{H}$ is infinite dimensional we have $H^\infty(\mathcal{H}(S))$ is contractively contained in $A^\infty(K_S)$ by Theorem 1.3. We shall show in Theorem 3.7 and Proposition 3.5 that the converse holds if S is finite, or if a certain generic assumption holds.

We shall say that T is a *generic matrix d-tuple on* Ω if, for some $n \in \mathbb{N}$, we have that T is a d-tuple of commuting n-by-n matrices that have a common set of n linearly independent eigenvectors with distinct joint eigenvalues, which means there are n linearly independent eigenvectors v_j in $\mathbb{C}^n$ so that

$$T^r v_j \ = \ \lambda_j^r v_j, \qquad 1 \leq r \leq d, \ 1 \leq j \leq n, \tag{3.4}$$

and the n points $\lambda_j = (\lambda_j^1, \ldots, \lambda_j^d)$ are distinct points in Ω. The advantages of working with generic d-tuples were pointed out in [1].

We shall define an algebra $H^\infty_{\rm gen}(S)$ to be the holomorphic functions on Ω for which the norm

$$\|\phi\|_{H^\infty_{\rm gen}(S)} \ := \ \sup\{\|\phi(T)\| \ : \quad T \text{ is a generic matrix } d\text{-tuple on } \Omega, \text{ and } \|\psi(T)\| \leq 1 \ \forall \psi \in S\}.$$

Proposition 3.5. *Let S be a set of holomorphic functions from Ω to $\mathbb{D}$. Then $H^\infty_{\rm gen}(S) = A^\infty(K_S)$ isometrically.*

PROOF: Let ϕ be in the closed unit ball of $A^\infty(K_S)$. Let T be a generic matrix tuple on Ω, with eigenvectors as in (3.4), and assume that $\|\psi(T)\| \leq 1$ for all ψ in S. Let $F = \{\lambda_1, \ldots, \lambda_n\}$. Define a kernel $k(z,w)$ on Ω by setting it to zero unless both z and w are in F, and on F define

$$k(\lambda_i, \lambda_j) \ = \ \langle v_i, v_j \rangle.$$

Then $k \in K_S$, so

$$(1 - \phi(\lambda_i)\overline{\phi(\lambda_j)})\langle v_i, v_j \rangle \ \geq \ 0. \qquad (3.6)$$

Then (3.6) says that $\|\phi(T)\| \leq 1$, so ϕ is in the closed unit ball of $H^\infty_{\rm gen}(S)$.

Conversely, if ϕ is in the closed unit ball of $H^\infty_{\rm gen}(S)$, then for every finite set $F \subset \Omega$, by Theorem 3.2 applied to F, we have that (3.3) holds on F. Hence by the Theorem again, we have ϕ is in the closed unit ball of $A^\infty(K_S)$. □

Theorem 3.7. *Let S be a set of holomorphic functions from Ω to $\mathbb{D}$. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. If S is finite, then $H^\infty(\mathcal{H}(S)) = A^\infty(K_S)$.*

PROOF: By Theorem 1.3, we have $H^\infty(\mathcal{H}(S))$ is contractively contained in $A^\infty(K_S)$. For the converse, let ϕ be in the closed unit ball of $A^\infty(K_S)$, with a network realization formula as above. Let $S = \{\psi_1, \ldots, \psi_n\}$.

Let Λ_j be the elements of $C_b(S)$ defined by $\Lambda_j(\psi_i) = \delta_{ij}$. Since each Λ_j is a projection, we get that $\rho(\Lambda_j) = P_j$ gives n mutually orthogonal projections that sum to the identity on $\mathcal{M}$. Then $\rho(E(z)) = \sum_{j=1}^n \psi_j(z) P_j$.

Expanding (3.1) as a Neumann series in $D\rho(E(z))$, the partial sums ϕ_n will converge locally uniformly on Ω. Therefore if T is in $\mathcal{H}(S)$, since its spectrum is a compact subset of Ω, we get that $\phi(T) = \lim_n \phi_n(T)$. We have $\rho(E(T)) = \sum_{j=1}^n \psi_j(T) \otimes P_j$, and (3.1) extends to

$$\phi(T) \ = \ I_{\mathcal{H}} \otimes A + (I_{\mathcal{H}} \otimes B)\rho(E(T))(I - (I_{\mathcal{H}} \otimes D)\rho(E(T)))^{-1} I_{\mathcal{H}} \otimes C. \quad (3.8)$$

A calculation with (3.8) shows that $I_{\mathcal{H}} - \phi(T)^*\phi(T) \geq 0$, so we conclude $\phi \in \mathscr{S}(\mathcal{H}(S))$. □

Problem 3.9 Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Do $H^\infty(\mathcal{H}(S))$ and $A^\infty(K_S)$ coincide for all non-empty sets S of holomorphic functions from Ω to $\mathbb{D}$?

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