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Discrete-Time Adaptive Control of Uncertain Sampled-Data Systems with Uncertain Input Delay: A Reduction

K. Abidi[1] *H.J. Soo*[2] *I. Postlethwaite*[3]

[1] *Electrical Power Engineering Programme, Newcastle University in Singapore, Singapore*
[2] *No affiliation, Singapore*
[3] *School of Engineering, Newcastle University, Newcastle upon Tyne NE1 7RU, U.K*
* *E-mail: khalid.abidi@ncl.ac.uk*

Abstract: This paper proposes a discrete-time adaptive control approach for uncertain single-input and single-output (SISO) linear time-invariant sampled-data systems with uncertain, constant input time delay that has a known upper-bound, without explicitly estimating the time delay. To cope with the unknown time delay a reduction approach similar to that proposed by *Artstein* in 1982 is used which results in a delay-free system that simplifies the control law design. In addition, the proposed control approach is capable of coping with bounded exogenous disturbances. A rigorous stability analysis shows that the proposed control approach drives the system output to a bound around the reference signal asymptotically, in the presence of an exogenous disturbance. Moreover, simulation results are shown to verify the approach.

1 Introduction

Processes with delayed control action (i.e. input time delay) are encountered in many applications: in chemical engineering, where process dynamics are approximated as first/second order systems with dead time; in robotics, where the processing of large volumes of sensory data can introduce computational delays [1]; and in bilateral teleoperation, where communications delays can destabilise force feedback loops and pose a hazard to the remote operator [2]. In order to guarantee closed-loop performance and stability, feedback control laws of such plants must take into account this time delay [3, 4].

In the presence of input time delay, the control action at any given moment does not influence the plant state immediately; it only takes effect when it ‘reaches’ the plant after the delay has elapsed, some time into the future. In a discrete-time plant with dynamics $x_{k+1} = f(x_k, u_{k-d}, k)$, the effect of the current control input u_k is given by $x_{k+d+1} = f(x_{k+d}, u_k, k + d)$. This suggests that a control law should predict the future plant state in order to compensate for it. Examples of predictor-based control laws for linear and nonlinear plants are [5] and [6] respectively.

A rather different approach to time delay compensation is taken in Artstein’s model reduction [7]. The idea is to express the original dynamics with time delay as an equivalent delay-free dynamics, by employing a suitable substitution. Thus, control law design and other analysis tasks for time delay systems are ‘reduced’ to the delay-free problem. Artstein’s reduction is applicable to linear, possibly time varying systems with distributed input delays (this includes lumped delays as a special case), provided certain conditions are satisfied (which in many applications, they are).

To illustrate the method with an example adapted from [7, Example 5.1], consider the linear system with input delay $\dot{x}(t) = Ax(t) + Bu(t - d)$. The appropriate state substitution is

$$\eta(t) = x(t) + \int_{t-d}^{t} e^{A(t-d-\tau)} Bu(\tau)d\tau \qquad (1)$$

which yields an equivalent delay-free system in $\eta(t)$ given by

$$\dot{\eta}(t) = A\eta(t) + e^{-Ad}Bu(t) \qquad (2)$$

The control law for the latter is a state-feedback law $u(t) = K\eta(t)$ where K is chosen to stabilise $(A, e^{-Ad}B)$. Substituting in the definition of $\eta(t)$ gives the control law in terms of $x(t)$

$$u(t) = Kx(t) + K\int_{t-d}^{t} e^{A(t-d-\tau)} Bu(\tau)d\tau \qquad (3)$$

Uncertainties in the plant parameters, including the duration of the time delay, can be dealt with using robust control laws, or with adaptive control. *In [8], a discrete-time robust approach utilizing Artstein’s reduction paves the way for the use of robust state-feedback control design techniques on the transformed delay-free dynamics. Another example in the continuous-time case is delay-independent truncated predictor feedback, [9].* This is a state-feedback control law, where the gains are computed using a Lyapunov equation based method. However, if the plant is unstable the amount of delay that the method can handle is limited, [10]. *In the case of output feedback, predictor-based rejection control is an interesting solution, [11, 12]. A filtered Smith predictor is used to predict the future output of the plant. This is fed to an extended state observer which not only estimates the plant state but also the ‘total disturbance’ comprising the exogenous disturbance as well as any modelling errors present in the filtered Smith predictor, [13].*

For time delays which are not well-characterised, adaptive control may be more appropriate. Early adaptive control approaches for time delay systems did not address uncertainty in the time delay [14] [15]. The reason is that adaptive laws rely on the plant representation being linear in the uncertain parameters, whereas the time delay appears inside the argument of the control input. One solution is to express the plant dynamics in terms of the entire input history over the delay interval (given by the function $u(x, t)$ where x parameterises a point on the interval), and to model the delay as a transport PDE [16]. Thus, the time delay can be estimated along with other plant parameters, and used to compute a predictor-based control law [17]. However, the resulting adaptive laws for both time delay and parameter estimation are complicated.

Most continuous-time control approaches will, in practice, be implemented on sampled-data systems. To discretise the control law (3), the integral term can be approximated by numerical quadrature. It is known that certain approximations can destabilise the system [18, 19]. Thus, having a discrete-time control law in the first place would simplify implementation.

The discrete-time adaptive posicast control approach (APC) [20, 21] is a model-reference adaptive control approach that achieves reference trajectory tracking on a plant with an unknown, constant, upper-bounded time delay. However, the model-tracking error does not vanish asymptotically, and its bound depends on the mismatch between the delay upper-bound assumed by the control approach and the true delay. The adaptive law also contains parameters that may be difficult to tune in practice.

This paper proposes a discrete-time adaptive reference-tracking control approach for a *SISO*, linear time-invariant system with an unknown, constant input time delay that has a known upper-bound. The approach taken here may be considered an application of Artstein's model reduction to the discrete-time case, with the modifications needed to support stable adaptive laws.* The key ideas are outlined as follows:

• To accommodate uncertainty in the time delay, the plant dynamics is expressed in a manner that is 'agnostic' to the specific value of the time delay. This enables plant parameters to be estimated using recursive least squares, even without knowledge of the time delay, thus dispensing with the need to explicitly estimate the time delay.
• To perform the model reduction, recall that in the non-adaptive case, by introducing a substitution in a new variable η, the plant dynamics can equivalently be expressed as a delay-free dynamics in η. In the adaptive case, the plant parameters in the definition of η will be estimated by adaptation laws. However, due to the fact that the adaptive parameters are time-varying, the dynamics in η is no longer delay-free. A further reduction is required by introducing a second substitution $\hat{\eta}$, which leads to a delay-free dynamics in terms of η and $\hat{\eta}$. Using this dynamics, the control law is easily obtained.

A stability analysis shows that with the proposed control approach, the plant output tracks the reference signal to within a bound asymptotically, in the presence of a disturbance.

The organization of this paper is as follows: In Section 2, the problem definition is given. In Section 3, the main result which includes the adaptive control law design and stability analysis is presented. In Section 4, a simulation example that verfies the approach is shown followed by the concluding remarks in Section 5.

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm. For notational convenience, the mathematical expression "f_k" represents the value of the signal f at the k^{th} sampling instant.

2 PROBLEM DEFINITION

Consider the n^{th} order continuous-time SISO system with input delay given as

$$\frac{d^n}{dt^n}y + \sum_{i=0}^{n-1} a_{n-i}\frac{d^i}{dt^i}y = \sum_{j=0}^{m'} b_{m'-j}\frac{d^j}{dt^j}u(t-\tau_{\mathrm{d}}) + \omega(t) \qquad (4)$$

where $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control input, $a_1, \ldots, a_n, b_0, \ldots, b_{m'} \in \mathbb{R}$ are constant uncertain parameters, $m' \in \mathbb{Z}$ is the order of the highest derivative of the control input, $\tau_{\mathrm{d}} \in \mathbb{R}$ is the constant uncertain time-delay and $\omega(t) \in \mathbb{R}$ is an unmeasurable *bounded* exogenous disturbance.

Now consider that the system (4) is sampled at a uniform time interval T (where in general the time delay τ_{d} may not be an integer multiple of T) such that it is described by the sampled-data model given as

$$\begin{aligned} y_k = \phi_1 y_{k-1} + \cdots + \phi_n y_{k-n} + \gamma_0 u_{k-d-1} + \cdots \\ + \gamma_m u_{k-d-m-1} + \nu_{k-1} \end{aligned} \qquad (5)$$

where $k \in \mathbb{Z}^+$ corresponds to the k^{th} time-step, $\phi_1, \ldots, \phi_n, \gamma_0, \ldots, \gamma_m \in \mathbb{R}$ are constant uncertain parameters, $d \in \mathbb{Z}^+$ is the uncertain constant delay in time-steps that satisfies $dT \le \tau_{\mathrm{d}} \le (d+1)T$ and $|\nu_k| \le \nu_{\max} \in \mathbb{R}$ corresponds to the sampling of the exogenous disturbance which has a known bound $\nu_{\max}$. The system (4) and the sampled-data system (5) satisfy the following assumptions:

Assumption 1. *The relative degree of the system* (4) *is at least 1.*

Assumption 2. *The delay τ_{d} is bounded as $\tau_{\mathrm{d}} \le \tau_{\mathrm{p}}$ and τ_{p} satisfies $pT \le \tau_{\mathrm{p}} \le (p+1)T$ where p is the upper-bound on the delay in time-steps.*

Assumption 3. *The sign of ϕ_n is known a priori and there exists a $|\phi_{n,\mathrm{m}}| > 0$ such that $|\phi_n| \ge |\phi_{n,\mathrm{m}}|$.*

Remark 1. *Note that for a system of order n, the parameter $\phi_n \neq 0$. Furthermore, since in practical applications a nominal model is known, it is reasonable to assume a bound on some of the parameters similar to that in Assumption 3, [22].*

The control problem is to find a bounded control input u_k in sampled-time which will drive the output, $y(t)$, to track a reference signal, $r(t)$ (or drive the sampled output, y_k, to track the sampled reference, r_k) asymptotically, while keeping all system signals bounded. The reference signal is assumed to have a constant steady state value, i.e., $\lim_{t\to\infty} r(t) = \lim_{k\to\infty} r_k = r_0$.

3 MAIN RESULT

In this section an adaptive estimator and the adaptive law design is presented followed by the control law design. The control law is computed from a delay free dynamics derived from the adaptive estimator using a reduction approach inspired by [7]. Finally, the section concludes with a rigorous stability analysis of the system to verify the validity of the approach.

3.1 Adaptive Estimator Design

Consider the sampled-data system (5) expressed in the form

$$\begin{aligned} y_k &= \sum_{j=1}^{n} \phi_j y_{k-j} + \sum_{i=0}^{p_{\mathrm{m}}} \psi_i u_{k-i-1} + \nu_{k-1} \\ &= \boldsymbol{\theta}^\top \boldsymbol{\zeta}_{k-1} + \nu_{k-1} \end{aligned} \qquad (6)$$

where $p_{\mathrm{m}} = p + m$, $\boldsymbol{\theta}^\top \triangleq \left[\phi_1 \;\vdots\; \cdots \;\vdots\; \phi_n \;\vdots\; \psi_0 \;\vdots\; \cdots \;\vdots\; \psi_{p_{\mathrm{m}}}\right] \in \mathbb{R}^{p_{\mathrm{m}}+n+1}$ is the vector of uncertain parameters and $\boldsymbol{\zeta}_k^\top \triangleq \left[y_k \;\vdots\; \cdots \;\vdots\; y_{k-n+1} \;\vdots\; u_k \;\vdots\; \cdots \;\vdots\; u_{k-p_{\mathrm{m}}}\right] \in \mathbb{R}^{p_{\mathrm{m}}+n+1}$ is the augmented signal vector that contains the output and control input history. The parameters $\psi_i \in \mathbb{R}$ are defined as

$$\psi_i = \begin{cases} \gamma_{i-d} & d \le i \le d+m \\ 0 & \text{otherwise} \end{cases} \qquad i \in [0, p_{\mathrm{m}}]. \qquad (7)$$

Now consider the adaptive estimator given as

$$\begin{aligned} \hat{y}_k &= \sum_{j=1}^{n} \hat{\phi}_{j,k-1} y_{k-j} + \sum_{i=0}^{p_{\mathrm{m}}} \hat{\psi}_{i,k-1} u_{k-i-1} \\ &= \hat{\boldsymbol{\theta}}_{k-1}^\top \boldsymbol{\zeta}_{k-1} \end{aligned} \qquad (8)$$

where $\hat{y}_k$ is the estimate of the output signal y_k and $\hat{\boldsymbol{\theta}}_k^\top \triangleq \left[\hat{\phi}_{1,k} \;\vdots\; \cdots \;\vdots\; \hat{\phi}_{n,k} \;\vdots\; \hat{\psi}_{0,k} \;\vdots\; \cdots \;\vdots\; \hat{\psi}_{p_{\mathrm{m}},k}\right] \in \mathbb{R}^{p_{\mathrm{m}}+n+1}$ is the estimate of the parameter vector $\boldsymbol{\theta}$ respectively. The purpose of the adaptive estimator (8) *is to facilitate in the computation of the control law which would otherwise be difficult due to the uncertain parameters in the system* (6).

An earlier version of this work which also applied only to scalar systems was presented in [23]. For a different approach, see also [24].

Proceeding with the adaptive law design, if the output estimation error is defined as $\tilde{y}_k \triangleq y_k - \hat{y}_k$ and the augmented parameter estimation error vector is defined as $\tilde{\boldsymbol{\theta}}_k \triangleq \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k$ then the output estimation error dynamics is obtained as

$$\tilde{y}_k = \tilde{\boldsymbol{\theta}}_{k-1}^{\top} \boldsymbol{\zeta}_{k-1} + \nu_{k-1}. \tag{9}$$

From (9) the adaptive law is derived as

$$\hat{\boldsymbol{\theta}}_k = \begin{cases} \mathcal{L}\left[\hat{\boldsymbol{\theta}}_{k-1} + \alpha_{k-1}\rho_{k-1}P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k\right] & \forall k \in (k_0, \infty) \\ \hat{\boldsymbol{\theta}}_{k_0} & \forall k \in [0, k_0] \end{cases} \tag{10}$$

where $k_0 \geq 0$ is the initial time-step, $\mathcal{L}[\cdot]$ is an operator, $\alpha_k > 0$ is a positive coefficient that guarantees system controllability, $\rho_k \in [0, 1]$ is a deadzone coefficient, $P_k \in \mathbb{R}^{(n+p_{\mathrm{m}}+1)\times(n+p_{\mathrm{m}}+1)}$ is the symmetric positive-definite covariance matrix and is defined as

$$P_k = \begin{cases} P_{k-1} - \dfrac{\alpha_{k-1}\rho_{k-1}P_{\mathrm{x},k-1}}{1+\alpha_{k-1}\rho_{k-1}\xi_{k-1}} & \forall k \in (k_0, \infty) \\ P_{k_0} & \forall k \in [0, k_0] \end{cases} \tag{11}$$

with $P_{\mathrm{x},k} \triangleq P_k\boldsymbol{\zeta}_k\boldsymbol{\zeta}_k^{\top}P_k$ and $\xi_k \triangleq \boldsymbol{\zeta}_k^{\top}P_k\boldsymbol{\zeta}_k$. Finally, the deadzone coefficient ρ_k is defined as

$$\rho_{k-1} = \begin{cases} 1 - \dfrac{\nu_{\max}}{|\tilde{y}_k|} & \text{if } |\tilde{y}_k| \geq \nu_{\max} \\ 0 & \text{if } |\tilde{y}_k| < \nu_{\max} \end{cases}. \tag{12}$$

The purpose of the operator $\mathcal{L}[\cdot]$ is to guarantee that $|\hat{\phi}_{n,k}| \geq |\phi_{n,\mathrm{m}}|$. The definition of the operator $\mathcal{L}[\cdot]$ is given as

$$\mathcal{L}[\hat{\boldsymbol{\theta}}_k] = \begin{cases} \hat{\boldsymbol{\theta}}_k & |\hat{\phi}_{n,k}| \in (|\phi_{n,\mathrm{m}}|, \infty) \\ \left[\ \hat{\boldsymbol{\phi}}_k^{\top} \ \vdots\ \phi_{n,\mathrm{m}} \ \vdots\ \hat{\boldsymbol{\psi}}_k^{\top}\ \right]^{\top} & \hat{\phi}_{n,k} \in [-|\phi_{n,\mathrm{m}}|, |\phi_{n,\mathrm{m}}|] \end{cases} \tag{13}$$

where $\hat{\boldsymbol{\phi}}_k = \left[\ \hat{\phi}_{1,k} \ \vdots\ \cdots \ \vdots\ \hat{\phi}_{n-1,k}\ \right] \in \mathbb{R}^{n-1}$ and $\hat{\boldsymbol{\psi}}_k = \left[\ \hat{\psi}_{0,k} \ \vdots\ \cdots \ \vdots\ \hat{\psi}_{p_{\mathrm{m}},k}\ \right]^{\top} \in \mathbb{R}^{p_{\mathrm{m}}+1}$, respectively. *Thus, if (10) yields a value for $\hat{\phi}_{n,k}$ such that $|\hat{\phi}_{n,k}| < |\hat{\phi}_{n,\mathrm{m}}|$, then $\mathcal{L}[\cdot]$ will saturate $\hat{\phi}_{n,k}$ at $\phi_{n,\mathrm{m}}$.*

3.2 *Control Law Design*

Consider once more the adaptive estimator (8). As was previously stated, a reduction approach will be utilized to derive a delay free dynamics from the adaptive estimator (8) that will simplify the computation of the control law.

To proceed with the reduction approach, consider the adaptive estimator (8) written in an augmented form as

$$\begin{aligned} &\left[\ \hat{y}_k \ \vdots\ y_{k-1} \ \vdots\ \cdots \ \vdots\ y_{k-n+1}\ \right]^{\top} \\ &\quad = \hat{\Phi}_{k-1}\left[\ y_{k-1} \ \vdots\ \cdots \ \vdots\ y_{k-n}\ \right]^{\top} + \sum_{i=0}^{p_{\mathrm{m}}} \hat{\boldsymbol{\psi}}_{i,k-1}u_{k-i-1} \end{aligned} \tag{14}$$

where the matrix $\hat{\Phi}_k \in \mathbb{R}^{n\times n}$ and the vector $\hat{\boldsymbol{\psi}}_{i,k} \in \mathbb{R}^n$ are given as

$$\hat{\Phi}_k = \begin{bmatrix} \hat{\phi}_{1,k} & \cdots & \cdots & \cdots & \hat{\phi}_{n,k} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{\psi}}_{i,k} = \begin{bmatrix} \hat{\psi}_{i,k} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

repectively. Next, the vectors $\boldsymbol{\eta}_k, \hat{\boldsymbol{\eta}}_k \in \mathbb{R}^n$ are introduced and defined as

$$\hat{\boldsymbol{\eta}}_k \triangleq \left[\ \hat{y}_k \ \vdots\ y_{k-1} \ \vdots\ \cdots \ \vdots\ y_{k-n+1}\ \right]^{\top} + \sum_{i=1}^{p_{\mathrm{m}}} \hat{\boldsymbol{\beta}}_{i,k-1}u_{k-i} \tag{15}$$

and

$$\boldsymbol{\eta}_{k-1} \triangleq \left[\ y_{k-1} \ \vdots\ \cdots \ \vdots\ y_{k-n}\ \right]^{\top} + \sum_{i=1}^{p_{\mathrm{m}}} \hat{\boldsymbol{\beta}}_{i,k-1}u_{k-i-1} \tag{16}$$

where $\hat{\boldsymbol{\beta}}_{i,k} \in \mathbb{R}^n$. The vectors (15) and (16) are basically the sum of the output and the weighted control input history. The terms in (15) and (16) are rearranged such that the augmented output vectors are expressed as

$$\left[\ \hat{y}_k \ \vdots\ y_{k-1} \ \vdots\ \cdots \ \vdots\ y_{k-n+1}\ \right]^{\top} = \hat{\boldsymbol{\eta}}_k - \sum_{i=1}^{p_{\mathrm{m}}} \hat{\boldsymbol{\beta}}_{i,k-1}u_{k-i} \tag{17}$$

and

$$\left[\ y_{k-1} \ \vdots\ \cdots \ \vdots\ y_{k-n}\ \right]^{\top} = \boldsymbol{\eta}_{k-1} - \sum_{i=1}^{p_{\mathrm{m}}} \hat{\boldsymbol{\beta}}_{i,k-1}u_{k-i-1}. \tag{18}$$

Substitution of (17) and (18) in (14) as well as adding and subtracting the term $\hat{\boldsymbol{\beta}}_{0,k-1}u_{k-1}$ on the right-hand-side of the resulting expression give a system of the form

$$\begin{aligned} \hat{\boldsymbol{\eta}}_k &= \hat{\Phi}_{k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}u_{k-1} - \hat{\Phi}_{k-1}\sum_{i=1}^{p_{\mathrm{m}}}\hat{\boldsymbol{\beta}}_{i,k-1}u_{k-i-1} \\ &\quad - \hat{\boldsymbol{\beta}}_{0,k-1}u_{k-1} + \sum_{i=0}^{p_{\mathrm{m}}}\hat{\boldsymbol{\psi}}_{i,k-1}u_{k-i-1} + \sum_{i=1}^{p_{\mathrm{m}}}\hat{\boldsymbol{\beta}}_{i,k-1}u_{k-i} \\ &= \hat{\Phi}_{k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k}u_{k-1} - \left(\hat{\boldsymbol{\beta}}_{0,k-1} - \hat{\boldsymbol{\beta}}_{1,k-1}\right)u_{k-1} \\ &\quad - \sum_{i=1}^{p_{\mathrm{m}}-1}\left(\hat{\Phi}_{k-1}\hat{\boldsymbol{\beta}}_{i,k-1} - \hat{\boldsymbol{\beta}}_{i+1,k-1}\right)u_{k-i-1} - \hat{\Phi}_{k-1} \\ &\quad \times \hat{\boldsymbol{\beta}}_{p_{\mathrm{m}},k-1}u_{k-p_{\mathrm{m}}-1} + \sum_{i=0}^{p_{\mathrm{m}}}\hat{\boldsymbol{\psi}}_{i,k-1}u_{k-i-1}. \end{aligned} \tag{19}$$

The parameters $\hat{\boldsymbol{\beta}}_{i,k} \forall i \in [0, p_{\mathrm{m}}]$ are computed from the matrix $\hat{\Phi}_k$ and the vectors $\hat{\boldsymbol{\psi}}_{i,k} \forall i \in [0, p_{\mathrm{m}}]$ as

$$\hat{\boldsymbol{\beta}}_{i,k} = \begin{cases} \displaystyle\sum_{j=0}^{p_{\mathrm{m}}} \hat{\Phi}_k^{-j}\hat{\boldsymbol{\psi}}_{j,k} & i = 0 \\ \displaystyle\sum_{j=i}^{p_{\mathrm{m}}} \hat{\Phi}_k^{i-j-1}\hat{\boldsymbol{\psi}}_{j,k} & i \in [1, p_{\mathrm{m}}] \end{cases} \tag{20}$$

and that results in the simplification of the system (19) into the delay free dynamics of the form

$$\hat{\boldsymbol{\eta}}_k = \hat{\Phi}_{k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}u_{k-1}. \tag{21}$$

The control law can now be designed using the system (21) with the condition that $\hat{\Phi}_k, \hat{\boldsymbol{\beta}}_{0,k}$ is a controllable pair. The controllability of the system (21) is addressed in **Lemma 1**.

Remark 2. *In (20), the inverse of the matrix $\hat{\Phi}_k$ is required and, therefore, $\hat{\Phi}_k$ must be a non-singular matrix. From the definition of $\hat{\Phi}_k$, the determinant $|\hat{\Phi}_k| = -\hat{\phi}_{n,k}$ and, since, $|\hat{\phi}_{n,k}| \geq |\hat{\phi}_{n,\mathrm{m}}| > 0$ the matrix $\hat{\Phi}_k$ is non-singular.*

Remark 3. *Note that in the system (21), $\hat{\boldsymbol{\eta}}_k$ is a function of $\boldsymbol{\eta}_{k-1}$. It will be shown in **Lemma 4** that if a feedback gain $\boldsymbol{\varphi}_{\mathrm{x},k-1} \in \mathbb{R}^n$ is selected such that the matrix $\hat{\Phi}_{k-1} - \hat{\boldsymbol{\beta}}_{0,k-1}\boldsymbol{\varphi}_{\mathrm{x},k-1}^\top$ is Hurwitz, then $\boldsymbol{\eta}_k$ is uniformly bounded and, consequently, $\hat{\boldsymbol{\eta}}_k$ is uniformly bounded.*

Lemma 1. *For a proper selection of the initial adaptive law parameters and the coefficient α_k, the pair $\hat{\Phi}_k, \hat{\boldsymbol{\beta}}_{0,k}$ is controllable.*

Proof: Consider the pair $\hat{\Phi}_k, \hat{\boldsymbol{\beta}}_{0,k}$ and the fact that controllability requires that the controllability matrix $W_{\mathrm{c},k} \triangleq \left[\hat{\boldsymbol{\beta}}_{0,k} \;\vdots\; \hat{\Phi}_k\hat{\boldsymbol{\beta}}_{0,k} \;\vdots\; \cdots \;\vdots\; \hat{\Phi}_k^{n-1}\hat{\boldsymbol{\beta}}_{0,k}\right] \in \mathbb{R}^{n\times n}$ be non-singular. To express $W_{\mathrm{c},k}$ explicitly in terms of the adaptive parameters, $\hat{\boldsymbol{\beta}}_{0,k}$ in (20) is given as

$$\hat{\boldsymbol{\beta}}_{0,k} = \hat{\boldsymbol{\psi}}_{0,k} + \hat{\Phi}_k^{-1}\hat{\boldsymbol{\psi}}_{1,k} + \cdots + \hat{\Phi}_k^{-p_{\mathrm{m}}}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k}. \tag{22}$$

Substitution of (22) in the definition of the controllability matrix $W_{\mathrm{c},k}$, it is obtained that

$$W_{\mathrm{c},k} = \sum_{i=0}^{p_{\mathrm{m}}} \hat{\Phi}_k^{-i}\left[\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\; \hat{\Phi}_k\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\; \cdots \;\vdots\; \hat{\Phi}_k^{n-1}\hat{\boldsymbol{\psi}}_{i,k}\right] \tag{23}$$

which is now explicitly in terms of the adaptive parameters. Since, (23) relies on the inverse of $\hat{\Phi}_k$ it is convenient to define $W_{\Phi,k} \triangleq \hat{\Phi}_k^{p_{\mathrm{m}}} W_{\mathrm{c},k}$ such that the premultiplication of both sides of (23) with $\hat{\Phi}_k^{p_{\mathrm{m}}}$ results in

$$W_{\Phi,k} = \sum_{i=0}^{p_{\mathrm{m}}} \hat{\Phi}_k^{p_{\mathrm{m}}-i}\left[\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\; \hat{\Phi}_k\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\; \cdots \;\vdots\; \hat{\Phi}_k^{n-1}\hat{\boldsymbol{\psi}}_{i,k}\right]. \tag{24}$$

Consider now the adaptive law (10) when $|\hat{\phi}_{n,k}| \in (|\phi_{n,\mathrm{m}}|, \infty)$. The adaptive law for each parameter can be written as

$$\begin{aligned}
\hat{\phi}_{1,k} &= \hat{\phi}_{1,k-1} + \alpha_{k-1}\hat{\varphi}_{1,k-1}\\
&\;\vdots \qquad\quad \vdots\\
\hat{\phi}_{n,k} &= \hat{\phi}_{n,k-1} + \alpha_{k-1}\hat{\varphi}_{n,k-1}\\
\hat{\psi}_{0,k} &= \hat{\psi}_{0,k-1} + \alpha_{k-1}\hat{\varphi}_{n+1,k-1}\\
&\;\vdots \qquad\quad \vdots\\
\hat{\psi}_{p_{\mathrm{m}},k} &= \hat{\psi}_{p_{\mathrm{m}},k-1} + \alpha_{k-1}\hat{\varphi}_{n+p_{\mathrm{m}}+1,k-1}
\end{aligned} \tag{25}$$

where $\hat{\varphi}_{i,k-1} = \rho_{k-1}\mathbf{s}_i P_k \boldsymbol{\zeta}_{k-1}\tilde{y}_k \; \forall i \in [1, n+p_{\mathrm{m}}+1]$ and $\mathbf{s}_i$ being the i^{th} row of an identity matrix of size $n + p_{\mathrm{m}} + 1$. Then $\hat{\Phi}_k$ is written as

$$\begin{aligned}
\hat{\Phi}_k &= \hat{\Phi}_{k-1} + \alpha_{k-1}\begin{bmatrix}\hat{\varphi}_{1,k-1} & \cdots & \hat{\varphi}_{n,k-1}\\ 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{bmatrix}\\
&= \hat{\Phi}_{k-1} + \alpha_{k-1}\hat{\Phi}_{\varphi,k-1}
\end{aligned} \tag{26}$$

and $\hat{\boldsymbol{\psi}}_{i,k} \; \forall i \in [0, p_{\mathrm{m}}]$ is similarly written as

$$\begin{aligned}
\hat{\boldsymbol{\psi}}_{i,k} &= \hat{\boldsymbol{\psi}}_{i,k-1} + \alpha_{k-1}\begin{bmatrix}\hat{\varphi}_{n+i+1,k-1}\\ 0\\ \vdots\\ 0\end{bmatrix}\\
&= \hat{\boldsymbol{\psi}}_{i,k-1} + \alpha_{k-1}\hat{\boldsymbol{\varphi}}_{n+i+1,k-1}.
\end{aligned} \tag{27}$$

Substitution of (26) and (27) in (24), results in an expression for $W_{\Phi,k}$ given as

$$\begin{aligned}
&W_{\Phi,k}\\
&= \sum_{i=0}^{p_{\mathrm{m}}}\left(\hat{\Phi}_{k-1}^{p_{\mathrm{m}}-i} + \alpha_{k-1}\hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\right)\Big[\Big(\hat{\boldsymbol{\psi}}_{i,k-1} + \alpha_{k-1}\\
&\quad \times \hat{\boldsymbol{\varphi}}_{n+i+1,k-1}\Big) \;\vdots\; \left(\hat{\Phi}_{k-1} + \alpha_{k-1}\hat{\Theta}_{1,k-1}\right)\Big(\hat{\boldsymbol{\psi}}_{i,k-1}\\
&\quad + \alpha_{k-1}\hat{\boldsymbol{\varphi}}_{n+i+1,k-1}\Big) \;\vdots\; \cdots \;\vdots\; \left(\hat{\Phi}_{k-1}^{n-1} + \alpha_{k-1}\hat{\Theta}_{n-1,k-1}\right)\\
&\quad \times \left(\hat{\boldsymbol{\psi}}_{i,k-1} + \alpha_{k-1}\hat{\boldsymbol{\varphi}}_{n+i+1,k-1}\right)\Big]\\
&= \sum_{i=0}^{p_{\mathrm{m}}}\hat{\Phi}_{k-1}^{p_{\mathrm{m}}-i}\left[\hat{\boldsymbol{\psi}}_{i,k-1} \;\vdots\; \hat{\Phi}_{k-1}\hat{\boldsymbol{\psi}}_{i,k-1} \;\vdots\; \cdots \;\vdots\; \hat{\Phi}_{k-1}^{n-1}\hat{\boldsymbol{\psi}}_{i,k-1}\right]\\
&\quad + \alpha_{k-1}\sum_{i=0}^{p_{\mathrm{m}}}\Big[\hat{\Phi}_{k-1}^{p_{\mathrm{m}}-i}\hat{\boldsymbol{\varphi}}_{n+i+1,k-1} + \hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\\
&\quad \hat{\Phi}_{k-1}\boldsymbol{\varphi}_{n+i+1,k-1} + \hat{\Theta}_{1,k-1}\hat{\boldsymbol{\psi}}_{i,k} + \hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\hat{\Phi}_k\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\\
&\quad \cdots \;\vdots\; \hat{\Phi}_{k-1}\boldsymbol{\varphi}_{n+i+1,k-1}^{n-1} + \hat{\Theta}_{n-1,k-1}\hat{\boldsymbol{\psi}}_{i,k} + \hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\\
&\quad \times \hat{\Phi}_k^{n-1}\hat{\boldsymbol{\psi}}_{i,k}\Big]
\end{aligned} \tag{28}$$

where $\hat{\Theta}_{i,k-1} \triangleq \alpha_{k-1}^{-1}\left(\hat{\Phi}_k^i - \hat{\Phi}_{k-1}^i\right)$. Note that the first term on the right-hand-side of (28) is a single time-step delayed (24). Therefore, (28) is simplified as

$$W_{\Phi,k} = W_{\Phi,k-1} + \alpha_{k-1}\Omega_k \tag{29}$$

where

$$\begin{aligned}
\Omega_k = \sum_{i=0}^{p_{\mathrm{m}}}\Big[&\hat{\Phi}_{k-1}^{p_{\mathrm{m}}-i}\hat{\boldsymbol{\varphi}}_{n+i+1,k-1} + \hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\; \hat{\Phi}_{k-1}\\
&\times\boldsymbol{\varphi}_{n+i+1,k-1} + \hat{\Theta}_{1,k-1}\hat{\boldsymbol{\psi}}_{i,k} + \hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\hat{\Phi}_k\hat{\boldsymbol{\psi}}_{i,k} \;\vdots\\
&\cdots \;\vdots\; \hat{\Phi}_{k-1}\boldsymbol{\varphi}_{n+i+1,k-1}^{n-1} + \hat{\Theta}_{n-1,k-1}\hat{\boldsymbol{\psi}}_{i,k} + \hat{\Theta}_{p_{\mathrm{m}}-i,k-1}\\
&\times \hat{\Phi}_k^{n-1}\hat{\boldsymbol{\psi}}_{i,k}\Big].
\end{aligned} \tag{30}$$

Consider now the expression (29) when $k = k_0 + 1$ and suppose that the initial adaptive parameters are selected such that W_{Φ,k_0} is a non-singular matrix then it is obtained that

$$W_{\Phi,k_0+1} = W_{\Phi,k_0}\left(I + \alpha_{k_0}W_{\Phi,k_0}^{-1}\Omega_{k_0+1}\right) \tag{31}$$

where W_{Φ,k_0+1} is a non-singular matrix if and only if $\alpha_{k_0}^{-1} \neq \lambda\left[-W_{\Phi,k_0}^{-1}\Omega_{k_0+1}\right]$, where $\lambda[\cdot]$ is the set of eigenvalues. Then, in general, $W_{\Phi,k}$ is non-singular if the initial value W_{Φ,k_0} is non-singular and $\alpha_{k-1}^{-1} \neq \lambda\left[-W_{\Phi,k-1}^{-1}\Omega_k\right]$.

Furthermore, since $\hat{\Phi}_k$ is a non-singular matrix, if $W_{\Phi,k}$ is a non-singular matrix then $W_{\mathrm{c},k}$ is also a non-singular matrix and the pair $\hat{\Phi}_k, \hat{\boldsymbol{\beta}}_{0,k}$ is controllable. □

Remark 4. *Note that the choice of the coefficient α_k has been included for the sake of technical completeness and that $\alpha_k = 1$ will work in most cases since it is highly unlikely for 1 to be exactly an eigenvalue of $-W_{\Phi,k-1}^{-1}\Omega_k$, [25]. Moreover, since $-W_{\Phi,k-1}^{-1}\Omega_k$ has a finite number of eigenvalues, α_k can be selected from a sufficiently large set of pre-defined values as long as the inverse of one of those pre-defined values is not an eigenvalue of $-W_{\Phi,k-1}^{-1}\Omega_k$, [21].*

*Considering that the controllability of the system (21) is established, as shown in **Lemma 1***, the control law is proposed as

$$u_k = -\boldsymbol{\varphi}_{\mathrm{x},k}^{\top}\boldsymbol{\eta}_k + \varphi_{\mathrm{r},k} r_k \tag{32}$$

where the feedback gain vector $\boldsymbol{\varphi}_{\mathrm{x},k}$ can be computed using a Pole Placement or any optimal control approaches. The gain $\varphi_{\mathrm{r},k} \in \mathbb{R}$ is computed such that the steady state value of the output y_k converges with the steady state value of the reference signal $\lim_{k\to\infty} r_k = r_0$ and is given as

$$\varphi_{\mathrm{r},k}^{-1} = \mathbf{c}^{\top}\left[\left(I + \sum_{i=1}^{p_{\mathrm{m}}} \hat{\boldsymbol{\beta}}_{i,k}\boldsymbol{\varphi}_{\mathrm{x},k}^{\top}\right)\left(I - \hat{\Phi}_{\mathrm{m},k}\right)^{-1} \times \hat{\boldsymbol{\beta}}_{0,k} - \sum_{i=1}^{p_{\mathrm{m}}} \hat{\boldsymbol{\beta}}_{i,k}\right] \tag{33}$$

where $\mathbf{c}^{\top} \triangleq \left[1 \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right] \in \mathbb{R}^n$ and $\hat{\Phi}_{\mathrm{m},k} \triangleq \hat{\Phi}_k - \hat{\boldsymbol{\beta}}_{0,k}\boldsymbol{\varphi}_{\mathrm{x},k}^{\top}$. The derivation of the expression (33) will be presented in the proof of **Theorem 1**.

Remark 5. *If a Pole Placement approach is used to compute $\boldsymbol{\varphi}_{\mathrm{x},k}$ then the control law (32) is obtained as*

$$u_k = -\left[0 \,\vdots\, \cdots \,\vdots\, 0 \,\vdots\, 1\right] W_{\mathrm{c},k}^{-1}\left(\prod_{j=1}^{n}\left(\mu_j I - \hat{\Phi}_k\right)\right)\boldsymbol{\eta}_k + \varphi_{\mathrm{r}} r_k \tag{34}$$

where $\mu_1, \ldots, \mu_n$ are the desired closed-loop eigenvalues.

Remark 6. *Note that in (32), the inverse of $\varphi_{\mathrm{r},k}^{-1}$ is required and, therefore, $\varphi_{\mathrm{r},k}^{-1}$ must be nonsingular. As some of the terms are dependent on the desired eigenvalues $\mu_1, \ldots, \mu_n$ it is possible to tune the desired eigenvalues to avoid the remote possibility of a singular $\varphi_{\mathrm{r},k}$.*

3.3 Stability Analysis

In this section, it is shown that the parameter adaptation produces bounded and convergent parameter estimates (**Lemma 2** and **Lemma 3**), that the adaptive system model converges in input-output behaviour to the true system (**Lemma 4**), and that the proposed adaptive control law drives the system state to track the reference r_k (**Theorem 1**).

Lemma 2. *For the system (9) with the adaptive laws (10) and (11) it is true that*

$$\lim_{k\to\infty} \frac{\alpha_{k-1}\rho_{k-1}}{1+\alpha_{k-1}\rho_{k-1}\xi_{k-1}}\tilde{y}_k^2 = 0 \tag{35}$$

Furthermore, it is also true that the parameter estimate $\hat{\boldsymbol{\theta}}_k$ is bounded, hence, the parameter estimation error $\tilde{\boldsymbol{\theta}}_k$ is also bounded.

*Proof of **Lemma 2** is presented in section 6.1 of the Appendix.*

Lemma 3. *Using the results in **Lemma 2**, it is true that*

$$\lim_{k\to\infty}\left\|\Delta\hat{\boldsymbol{\theta}}_k\right\| = \lim_{k\to\infty}\left\|\hat{\boldsymbol{\theta}}_k - \hat{\boldsymbol{\theta}}_{k-1}\right\| = 0 \tag{36}$$

and, therefore, $\lim_{k\to\infty}\left\|\Delta\hat{\Phi}_k\right\| = \lim_{k\to\infty}\left\|\hat{\Phi}_k - \hat{\Phi}_{k-1}\right\| = 0$, $\lim_{k\to\infty}\left\|\Delta\hat{\Phi}_k^{-1}\right\| = \lim_{k\to\infty}\left\|\hat{\Phi}_k^{-1} - \hat{\Phi}_{k-1}^{-1}\right\| = 0$ *and* $\lim_{k\to\infty}\left\|\Delta\hat{\boldsymbol{\beta}}_{i,k}\right\| = \lim_{k\to\infty}\left\|\hat{\boldsymbol{\beta}}_{i,k} - \hat{\boldsymbol{\beta}}_{i,k-1}\right\| = 0 \;\forall i \in [0, p_{\mathrm{m}}]$.

*Proof of **Lemma 3** is presented in section 6.2 of the Appendix.*

Lemma 4. *Using the results in **Lemma 2** and **Lemma 3**, the vector $\boldsymbol{\eta}_k$ defined in (16) satisfies the stable dynamics*

$$\boldsymbol{\eta}_k = \hat{\Phi}_{\mathrm{m},k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}\varphi_{\mathrm{r},k-1}r_{k-1} + \left[\tilde{y}_k \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^{\top}. \tag{37}$$

and, as a result, is bounded as

$$\|\boldsymbol{\eta}_k\| \le c_0 + c_1 \max_{i\in[0,k]} |\tilde{y}_{k-i}| \tag{38}$$

for some positive constants c_0,c_1. Furthermore, the output estimation error converges $\tilde{y}_k$ to a bound of $\nu_{\max}$ asymptotically, i.e.

$$\lim_{k\to\infty} |\tilde{y}_k| \le \nu_{\max}. \tag{39}$$

*Proof of **Lemma 4** is presented in section 6.3 of the Appendix.*

Remark 7. *Since $|\tilde{y}_k|$ is uniformly bounded then, from (38), $\|\boldsymbol{\eta}_k\|$ is uniformly bounded. Furthermore, from (21), (32) and the fact that the adaptive parameters are bounded then $\|\hat{\boldsymbol{\eta}}_k\|$ is also uniformly bounded.*

Theorem 1. *The output of the closed-loop system approaches a bound of ϵ around the steady state value of the reference asymptotically, i.e. $\lim_{k\to\infty}|y_k - r_0| \le \epsilon$.*

Proof: *Consider **Lemma 4** and the stable dynamics (38) given by*

$$\boldsymbol{\eta}_k = \hat{\Phi}_{\mathrm{m},k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}\varphi_{\mathrm{r},k-1}r_{k-1} + \left[\tilde{y}_k \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^{\top}. \tag{40}$$

From **Lemma 2** and **Lemma 3**, it is shown that the adaptive parameters are bounded and converge at steady state. Therefore, there exists $\hat{\Phi}_{\mathrm{m,ss}}$, $\hat{\boldsymbol{\beta}}_{0,\mathrm{ss}}$ and $\varphi_{\mathrm{r,ss}}$ such that $\hat{\Phi}_{\mathrm{m,ss}} = \lim_{k\to\infty}\hat{\Phi}_{\mathrm{m},k}$, $\hat{\boldsymbol{\beta}}_{0,\mathrm{ss}} = \lim_{k\to\infty}\hat{\boldsymbol{\beta}}_{0,k}$ and $\varphi_{\mathrm{r,ss}} = \lim_{k\to\infty}\varphi_{\mathrm{r},k}$. Then the dynamics (40) is written as

$$\boldsymbol{\eta}_k = \hat{\Phi}_{\mathrm{m,ss}}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,\mathrm{ss}}\varphi_{\mathrm{r,ss}}r_0 + \boldsymbol{\delta}_k \tag{41}$$

where

$$\boldsymbol{\delta}_k = \left(\hat{\Phi}_{\mathrm{m},k-1} - \hat{\Phi}_{\mathrm{m,ss}}\right)\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}\varphi_{\mathrm{r},k-1}r_{k-1} - \hat{\boldsymbol{\beta}}_{0,\mathrm{ss}}\varphi_{\mathrm{r,ss}}r_0 + \left[\tilde{y}_k \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^{\top} \tag{42}$$

and since all the terms on the right-hand-side of (42) are bounded then $\lim_{k\to\infty}\|\boldsymbol{\delta}_k\| \le \nu_{\max}$. The solution of (41) is given as

$$\boldsymbol{\eta}_k = \hat{\Phi}_{\mathrm{m,ss}}^{k}\boldsymbol{\eta}_0 + \sum_{i=0}^{k-1}\hat{\Phi}_{\mathrm{m,ss}}^{i}\hat{\boldsymbol{\beta}}_{0,\mathrm{ss}}\varphi_{\mathrm{r,ss}}r_0 + \sum_{i=0}^{k-1}\hat{\Phi}_{\mathrm{m,ss}}^{i}\boldsymbol{\delta}_{k-i} \tag{43}$$

where it is assumed that the initial time step is $k = 0$ and $\boldsymbol{\eta}_0$ is the initial value of the vector $\boldsymbol{\eta}_k$. At steady state $\lim_{k\to\infty}\boldsymbol{\eta}_k$ is given as

$$\lim_{k\to\infty}\boldsymbol{\eta}_k = \left(I - \hat{\Phi}_{\mathrm{m,ss}}\right)^{-1}\hat{\boldsymbol{\beta}}_{0,\mathrm{ss}}\varphi_{\mathrm{r,ss}}r_0 + \bar{\boldsymbol{\delta}}_k \tag{44}$$

where $\bar{\boldsymbol{\delta}}_k = \lim_{k\to\infty}\sum_{i=0}^{k-1}\hat{\Phi}_{\mathrm{m,ss}}^{i}\boldsymbol{\delta}_{k-i}$ is bounded since $\boldsymbol{\delta}_k$ is bounded and $\hat{\Phi}_{\mathrm{m,ss}}$ has stable eigenvalues. Now, consider the definition of $\boldsymbol{\eta}_k$ given as

$$\boldsymbol{\eta}_k = \left[y_k \;\vdots\; \cdots \;\vdots\; y_{k-n+1} \right]^\top + \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,k} u_{k-i} \tag{45}$$

premultiplying (45) with $\mathbf{c}^\top$ and subtracting r_0 from both sides gives

$$y_k - r_0 = \mathbf{c}^\top \left(\boldsymbol{\eta}_k - \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,k} u_{k-i} \right) - r_0. \tag{46}$$

The steady state tracking error $\lim_{k\to\infty}(y_k - r_0)$ is given as

$$\begin{aligned}\lim_{k\to\infty}(y_k - r_0) &= \mathbf{c}^\top \lim_{k\to\infty} \left(\boldsymbol{\eta}_k - \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,k} u_{k-i} \right) - r_0 \qquad (47)\\ &= \mathbf{c}^\top \left(\lim_{k\to\infty} \boldsymbol{\eta}_k - \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \lim_{k\to\infty} u_{k-i} \right) - r_0.\end{aligned}$$

From (32) it is obtained that

$$\lim_{k\to\infty} u_k = -\boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \lim_{k\to\infty} \boldsymbol{\eta}_k + \varphi_{\mathrm{r,ss}} r_0. \tag{48}$$

Substitution of (44) in (48), results in the steady state of the control input as

$$\begin{aligned}\lim_{k\to\infty} u_k = &-\boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \left(I - \hat{\Phi}_{\mathrm{m,ss}} \right)^{-1} \hat{\boldsymbol{\beta}}_{0,\mathrm{ss}} \varphi_{\mathrm{r,ss}} r_0 \\ &+ \varphi_{\mathrm{r,ss}} r_0 - \boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \bar{\boldsymbol{\delta}}_k. \end{aligned} \tag{49}$$

Finally, the substitution of (44) and (49) in (47) and simplifying gives

$$\begin{aligned}\lim_{k\to\infty}(y_k - r_0) &= \mathbf{c}^\top \left(\lim_{k\to\infty} \boldsymbol{\eta}_k - \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \lim_{k\to\infty} u_{k-i} \right) - r_0 \\ &= \mathbf{c}^\top \left[\left(I + \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \right) \left(I - \hat{\Phi}_{\mathrm{m,ss}} \right)^{-1} \right. \\ &\quad \left. \times \hat{\boldsymbol{\beta}}_{1,\mathrm{ss}} - \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \right] \varphi_{\mathrm{r,ss}} r_0 - r_0 + \mathbf{c}^\top \left[\bar{\boldsymbol{\delta}}_k \right. \\ &\quad \left. + \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \bar{\boldsymbol{\delta}}_{k-i} \right]. \end{aligned} \tag{50}$$

From (33), the inverse of the steady state value $\varphi_{\mathrm{r,ss}}$ is given as

$$\begin{aligned}\varphi_{\mathrm{r},ss}^{-1} = \mathbf{c}^\top &\left[\left(I + \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \right) \left(I - \hat{\Phi}_{\mathrm{m,ss}} \right)^{-1} \hat{\boldsymbol{\beta}}_{0,\mathrm{ss}} \right. \\ &\left. - \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \right] \end{aligned} \tag{51}$$

then (50) is simplified further to give

$$\lim_{k\to\infty}(y_k - r_0) = \mathbf{c}^\top \left[\bar{\boldsymbol{\delta}}_k + \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \bar{\boldsymbol{\delta}}_{k-i} \right]. \tag{52}$$

Since the right-hand-side of (51) is a finite series of bounded terms, there exists a positive constant ϵ such that

$$\left| \mathbf{c}^\top \left[\bar{\boldsymbol{\delta}}_k + \sum_{i=1}^{p_\mathrm{m}} \hat{\boldsymbol{\beta}}_{i,\mathrm{ss}} \boldsymbol{\varphi}_{\mathrm{x,ss}}^\top \bar{\boldsymbol{\delta}}_{k-i} \right] \right| \leq \epsilon \tag{53}$$

and

$$\lim_{k\to\infty} |y_k - r_0| \leq \epsilon. \tag{54}$$

□

4 SIMULATION EXAMPLE

Consider an unstable 3rd order system given by

$$\begin{aligned}\dddot{y} = \ddot{y} - 400\dot{y} - 140y + 0.25\ddot{u}(t-\tau_\mathrm{d}) - 0.5\dot{u}(t-\tau_\mathrm{d}) \\ + 300u(t-\tau_\mathrm{d}) + 0.5\sin\left(\frac{1}{4}\pi t\right)\end{aligned} \tag{55}$$

where the system delay is set as $\tau_\mathrm{d} = 0$ and $\tau_\mathrm{d} = 0.3$s. The system is sampled with a sampling-interval of $T = 0.1$s such that *the sampled-data system is given as*

$$\begin{aligned}y_k = 0.075y_{k-1} - 0.285y_{k-2} + 1.1y_{k-3} + 0.052u_{k-d-1} \\ + 0.113u_{k-d-2} + 0.058u_{k-d-3}\end{aligned} \tag{56}$$

where $n = 3$ and $m = 2$. The initial condition of the system is set at $y(0) = 0$ while the reference $r(t)$ is given as

$$r(t) = \begin{cases} 0 & t \in [0,1] \\ 1 & t \in (1,4] \\ 1.5 & t \in (4,7] \\ 2 & t \in (7,4] \\ 0.5 & t \in (10,\infty) \end{cases} \tag{57}$$

and the desired closed-loop poles for the resulting sampled-data system are $\mu_1 = \mu_2 = \mu_3 = 0.15$. To investigate the performance of the closed-loop system with respect to mismatch between the delay upper-bound τ_p and the true delay τ_d, the closed-loop system is simulated with $\tau_\mathrm{d} = 0$s while the upper bound is set as $\tau_\mathrm{p} = 0$s and then simulated with $\tau_\mathrm{d} = 0.3$s and $\tau_\mathrm{p} = 0.5$s.

Assuming $\pm 20\%$ parametric uncertainty, the initial adaptive parameters $\hat{\phi}_{1,0}$, $\hat{\phi}_{2,0}$, and $\hat{\phi}_{3,0}$ are set as $\left[\hat{\phi}_{1,0} \;\vdots\; \hat{\phi}_{2,0} \;\vdots\; \hat{\phi}_{3,0}\right] = \left[0.06 \;\vdots\; -0.2 \;\vdots\; 1.3\right]$, respectively. To set the remaining parameters and considering that d is uncertain, inspired by (7), the initial values of $\hat{\psi}_{i,0}$ can be given as

$$\hat{\psi}_{i,0} = \begin{cases} \hat{\gamma}_{i-p,0} & p \leq i \leq p_\mathrm{m} \\ 0 & \textit{otherwise} \end{cases} \qquad i \in [0, p_\mathrm{m}] \tag{58}$$

where $\hat{\gamma}_{0,0} = 0.07$, $\hat{\gamma}_{1,0} = 0.09$ and $\hat{\gamma}_{2,0} = 0.07$ are the initializations of the estimates for the parameters γ_0, γ_1 and γ_2, respectively.

To show the performance of the controller under no delay conditions, i.e., when $\tau_\mathrm{d} = 0s$ and $\tau_\mathrm{p} = 0s$, the covariance matrix is initialised as $P_0 = 10^4 I_{p+6\times p+6}$ and the coefficient that ensures controllability is set as $\alpha_k = 1$. The initial parameter vector given as $\hat{\boldsymbol{\theta}}_0^\top = \left[\hat{\phi}_{1,0} \;\vdots\; \hat{\phi}_{2,0} \;\vdots\; \hat{\phi}_{3,0} \;\vdots\; \hat{\gamma}_{0,0} \;\vdots\; \hat{\gamma}_{1,0} \;\vdots\; \hat{\gamma}_{2,0}\right] = \left[0.06 \;\vdots\; -0.2 \;\vdots\; 1.3 \;\vdots\; 0.07 \;\vdots\; 0.09 \;\vdots\; 0.07\right]$. For the case when the delay is *$\tau_\mathrm{d} = 0.3s$ and the mismatched upper-bound is $\tau_\mathrm{p} = 0.5s$* the *initial* parameter vector is revised, *using* (58), to $\hat{\boldsymbol{\theta}}_0^\top = \left[0.06 \;\vdots\; -0.2 \;\vdots\; 1.3 \;\vdots\; 0 \;\vdots\; \cdots \;\vdots\; 0 \;\vdots\; 0.07 \;\vdots\; 0.09 \;\vdots\; 0.07\right]$ while all other parameters remain unchanged. In Fig. 1-5 the results are shown for the convergence of the output $y(t)$ with the reference $r(t)$, the convergence of the output estimate $\hat{y}(t)$ with the output $y(t)$, the control input profile and the convergence of selected adaptive parameters. As expected, the output estimate $\hat{y}(t)$ converges to the actual output $y(t)$ and, as a result, $y(t)$ converges to the reference signal asymptotically. It can also be seen from the results that when their is a mismatch between τ_d and τ_p, then the settling-time is longer. Note that in Fig. 1 the delay has been corrected so that the ouput tracking results at different delay values can be compared. *In Fig. 6, the eigenvalues of $-W_{\Phi,k-1}^{-1}\Omega_k$ when $\tau_\mathrm{d} = 0.3s$ and $\tau_\mathrm{p} = 0.5s$ are plotted on the complex-plane. It can be seen that some eigenvalues are close to, but, don't lie exactly on the point 1 and, therefore, $\alpha_k^{-1} = 1$ is not an eigenvalue of $-W_{\Phi,k-1}^{-1}\Omega_k$ and the pair $\hat{\Phi}_k, \hat{\boldsymbol{\beta}}_{0,k}$ is controllable. In Fig. 7, the sensitivity of the transient tracking performance is shown with respect to the percentage of parametric uncertainty.*

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As it can be seen, the maximum transient tracking error increases with the increase in the percentage of uncertainty.

Finally, the system is simulated with a longer system delay of $\tau_{\rm d} = 0.5$s while the upper bound is set as $\tau_{\rm d} = 0.5$s. *The adaptive law parameters are, once more, initialized assuming* 20% *parametric uncertainty while the values of* P_0 *and* α_k *are the same as the previous cases. The system is simulated and the upper bound is increased from* $\tau_{\rm d} = 0.5$s to $\tau_{\rm p} = 1$s. The results are shown in Fig. 8-10. The results for this case show a similar performance to the shorter delay case which is to be expected. Similar to Fig. 1, the delay is corrected in Fig. 8 so that the different output tracking results can be compared.

Remark 8. *Note that the initial value of the matrix* $W_{\Phi,0}$ *is computed from the initial adaptive parameter vector* $\hat{\boldsymbol{\theta}}_0$ *using (24) and if* $W_{\Phi,0}$ *is a singular matrix then the initial adaptive parameters can be tuned slighly to achieve a non-singular* $W_{\Phi,0}$ *and guarantee controllability of the pair* $\hat{\Phi}_k, \hat{\boldsymbol{\beta}}_{0,k}$ *as shown in* ***Lemma 1***.

5 CONCLUSIONS

In this paper, a discrete-time adaptive control approach was designed for a SISO system with an unknown, constant time-delay with a known upper-bound. A reduction approach was used, inspired by [7], that resulted in a delay-free system which simplified the control law design. A rigorous stability proof was presented that shows that the adaptive control law drives the system output to track the reference signal within a bound asymptotically, in the presence of disturbance. Finally, numerical simulations were shown that illustrate the ability of the adaptive control law to cope with mismatches between the delay upper-bound and the true delay in the system.

The present work can be expanded upon in a number of ways: generalisation to MIMO plants, incorporation of disturbance observers to reduce the effect of disturbances, and extension to cope with time-varying delays.

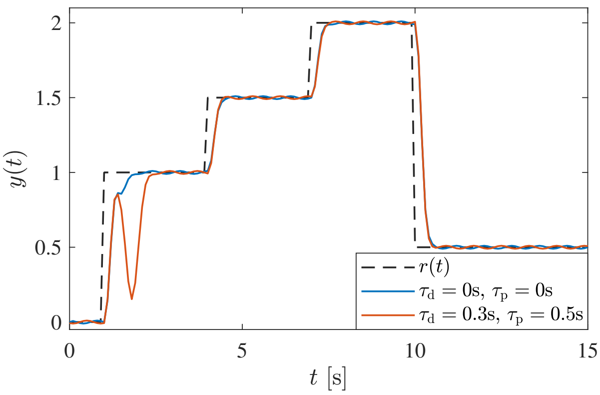

Fig. 1: Tracking of $y(t)$.

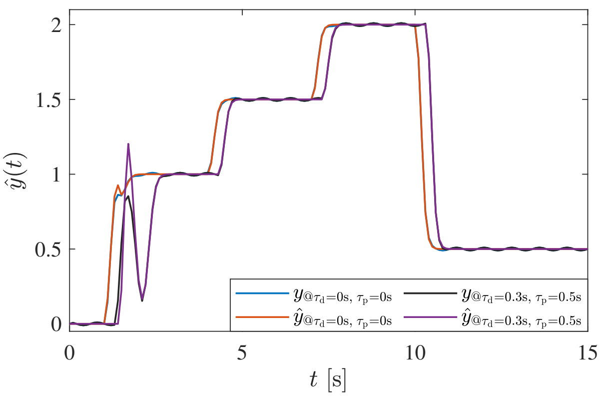

Fig. 2: Output estimate $\hat{y}(t)$ of the system.

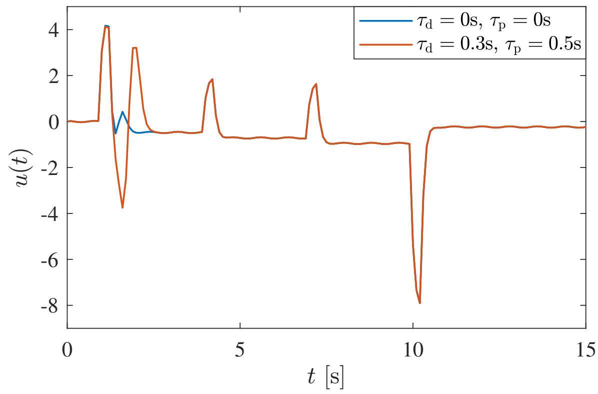

Fig. 3: Control input $u(t)$ of the system.

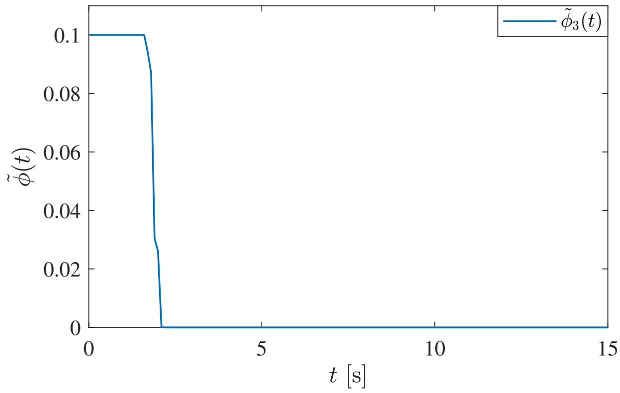

Fig. 4: Convergence of $\hat{\phi}_3$ when $\tau_{\rm d} = 0.3$s and $\tau_{\rm p} = 0.5$s.

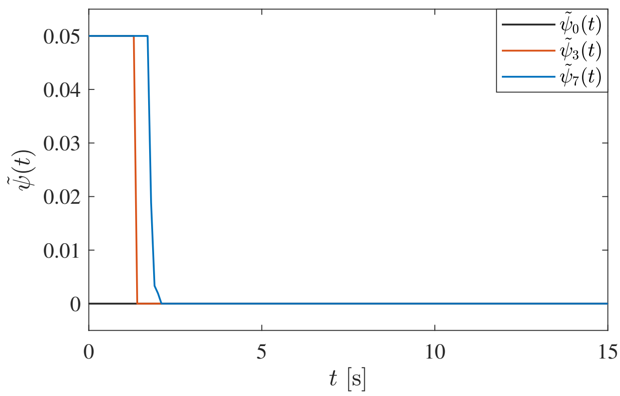

Fig. 5: Convergence of $\hat{\psi}_0$, $\hat{\psi}_3$ and $\hat{\psi}_7$ for $\tau_{\rm d} = 0.3$s and $\tau_{\rm p} = 0.5$s.

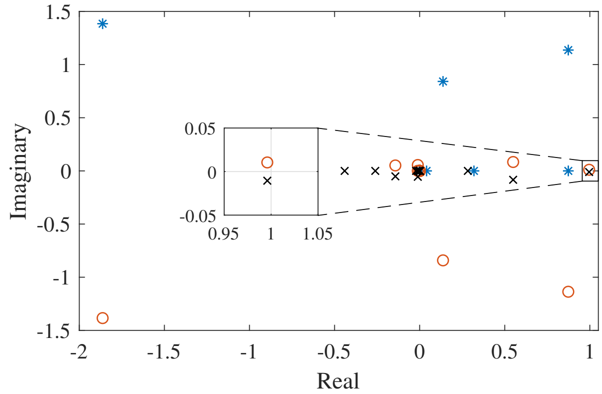

Fig. 6: Eigenvalues of $-W^{-1}_{\Phi,k-1}\Omega_k$ for $\tau_{\rm d} = 0.3$s and $\tau_{\rm p} = 0.5$s.

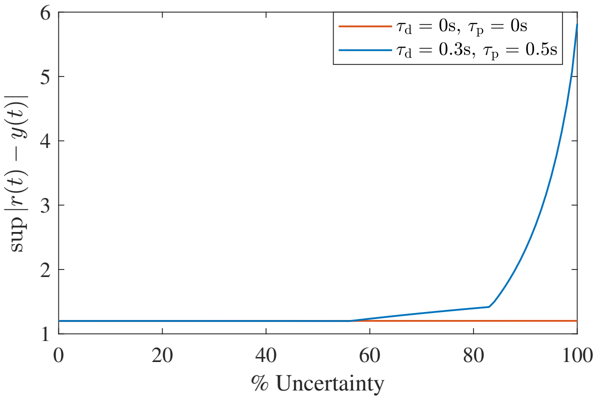

Fig. 7: Maximum tracking error relative to % Uncertainty.

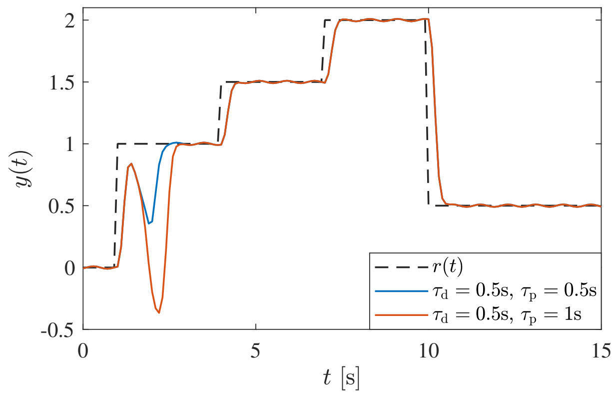

Fig. 8: Tracking of $y(t)$ for a longer time-delay.

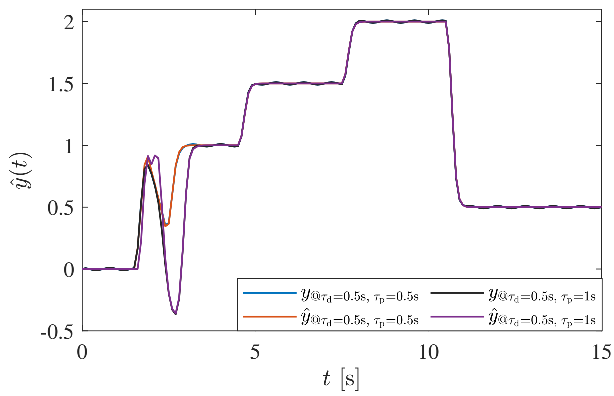

Fig. 9: Output estimate $\hat{y}(t)$ for a longer time-delay.

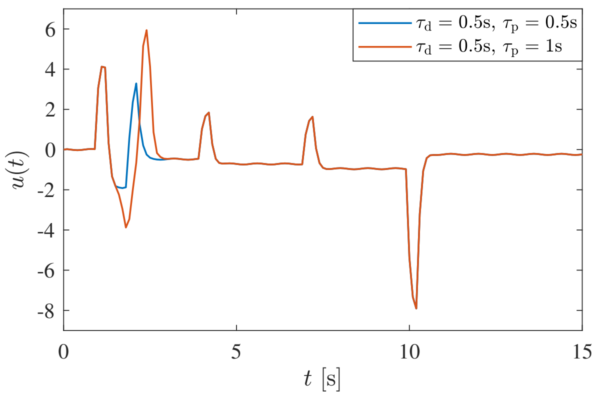

Fig. 10: Control input $u(t)$ for a longer time-delay.

6 Appendix

6.1 Proof Lemma 2

Proof: Consider the positive function

$$V_k = \tilde{\boldsymbol{\theta}}_k^\top P_k^{-1} \tilde{\boldsymbol{\theta}}_k. \tag{59}$$

The backward difference ΔV_k is given as

$$\Delta V_k = V_k - V_{k-1} = \left[\tilde{\boldsymbol{\theta}}_k^\top P_k^{-1} \tilde{\boldsymbol{\theta}}_k - \tilde{\boldsymbol{\theta}}_{k-1}^\top P_{k-1}^{-1} \tilde{\boldsymbol{\theta}}_{k-1}\right]. \tag{60}$$

Before proceeding further, using the result in [22, 23] it is obtained that

$$\left(\boldsymbol{\theta} - \mathcal{L}\left[\hat{\boldsymbol{\theta}}_k\right]\right)^\top \left(\boldsymbol{\theta} - \mathcal{L}\left[\hat{\boldsymbol{\theta}}_k\right]\right) \leq \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k\right)^\top \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k\right) \tag{61}$$

and that if P_k^{-1} is positive-definite then the inequality (61) is simplified as

$$\begin{aligned}\left(\boldsymbol{\theta} - \mathcal{L}\left[\hat{\boldsymbol{\theta}}_k\right]\right)^\top P_k^{-1} \left(\boldsymbol{\theta} - \mathcal{L}\left[\hat{\boldsymbol{\theta}}_k\right]\right)& \\ \leq \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k\right)^\top P_k^{-1} \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_k\right).&\end{aligned} \tag{62}$$

Continuing with the computation of ΔV_k, substitution of (62) and the adaptation law (10) in (60) results in

$$\begin{aligned}\Delta V_k &= V_k - V_{k-1} \\ &\leq \left(\tilde{\boldsymbol{\theta}}_{k-1} - \alpha_{k-1}\rho_{k-1}P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k\right)^\top P_k^{-1}\Big(\tilde{\boldsymbol{\theta}}_{k-1} - \alpha_{k-1} \\ &\quad \times \rho_{k-1}P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k\Big) - \tilde{\boldsymbol{\theta}}_{k-1}^\top P_{k-1}^{-1}\tilde{\boldsymbol{\theta}}_{k-1} \\ &\leq \tilde{\boldsymbol{\theta}}_{k-1}^\top P_k^{-1}\tilde{\boldsymbol{\theta}}_{k-1} - \tilde{\boldsymbol{\theta}}_{k-1}^\top P_{k-1}^{-1}\tilde{\boldsymbol{\theta}}_{k-1} - \tilde{\boldsymbol{\theta}}_{k-1}^\top P_k^{-1} \\ &\quad \times \alpha_{k-1}\rho_{k-1}P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k - \left(\alpha_{k-1}\rho_{k-1}P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k\right)^\top P_k^{-1} \\ &\quad \times \tilde{\boldsymbol{\theta}}_{k-1} + \alpha_{k-1}^2\rho_{k-1}^2\left(P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k\right)^\top P_k^{-1}\left(P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k\right) \\ &\leq \tilde{\boldsymbol{\theta}}_{k-1}^\top\left(P_k^{-1} - P_{k-1}^{-1}\right)\tilde{\boldsymbol{\theta}}_{k-1} - 2\alpha_{k-1}\rho_{k-1}\tilde{\boldsymbol{\theta}}_{k-1}^\top\boldsymbol{\zeta}_{k-1}\tilde{y}_k \\ &\quad + \alpha_{k-1}^2\rho_{k-1}^2\boldsymbol{\zeta}_{k-1}^\top P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k^2\end{aligned} \tag{63}$$

and using the fact that $P_k^{-1} = P_{k-1}^{-1} + \alpha_{k-1}\rho_{k-1}\boldsymbol{\zeta}_{k-1}\boldsymbol{\zeta}_{k-1}^\top$, (63) is simplified to the form

$$\begin{aligned}\Delta V_k &\leq \alpha_{k-1}\rho_{k-1}\tilde{\boldsymbol{\theta}}_{k-1}^\top\boldsymbol{\zeta}_{k-1}\boldsymbol{\zeta}_{k-1}^\top\tilde{\boldsymbol{\theta}}_{k-1} - 2\alpha_{k-1}\rho_{k-1} \\ &\quad \times \tilde{\boldsymbol{\theta}}_{k-1}^\top\boldsymbol{\zeta}_{k-1}\tilde{y}_k + \alpha_{k-1}^2\rho_{k-1}^2\boldsymbol{\zeta}_{k-1}^\top P_k\boldsymbol{\zeta}_{k-1}\tilde{y}_k^2.\end{aligned} \tag{64}$$

Furthermore, for the covariance matrix P_k in (11) it is obtained that

$$\boldsymbol{\zeta}_{k-1}^T P_k \boldsymbol{\zeta}_{k-1} = \frac{\xi_{k-1}}{1 + \alpha_{k-1}\rho_{k-1}\mu_{k-1}}. \tag{65}$$

Substitution of (65) in (64) and following a procedure similar to that shown in [21], it is obtained that

$$\begin{aligned}\Delta V_k &\leq \alpha_{k-1}\rho_{k-1}\tilde{y}_k^2\left[-1 + \frac{\alpha_{k-1}\rho_{k-1}\xi_{k-1}}{1 + \alpha_{k-1}\rho_{k-1}\xi_{k-1}}\right] \\ &\leq -\frac{\alpha_{k-1}\rho_{k-1}}{1 + \alpha_{k-1}\rho_{k-1}\xi_{k-1}}\tilde{y}_k^2\end{aligned} \tag{66}$$

which is true when $|\tilde{y}_k| \geq \nu_{\max}$. From the result (66) it is evident that ΔV_k is always non-positive and, hence V_k, is non-increasing. Therefore, the parameter estimation error $\tilde{\boldsymbol{\theta}}_k$ and the parameter estimate $\hat{\boldsymbol{\theta}}_k$ are bounded. Furthermore, the value of V_k is the accumulation of changes ΔV_k to its initial value V_{k_0}

$$V_k = V_{k_0} + \sum_{i=1}^{k-k_0} \Delta V_i. \qquad (67)$$

Substituting (66) into (67) gives

$$V_k \leq V_{k_0} - \sum_{i=1}^{k-k_0} \frac{\alpha_{i-1}\rho_{i-1}}{1+\alpha_{i-1}\rho_{i-1}\xi_{i-1}} \tilde{y}_i^2 \qquad (68)$$

and using the result in [21] it is concluded that

$$\lim_{k\to\infty} \Delta V_k \leq \lim_{k\to\infty} \frac{\alpha_{k-1}\rho_{k-1}}{1+\alpha_{k-1}\rho_{k-1}\xi_{k-1}} \tilde{y}_k^2 = 0. \qquad (69)$$

□

6.2 Proof of Lemma 3

Proof: Consider the adaptive law given by (10)-(12), it is shown in [25], that for the type of adaptive laws as (10)-(12) the difference between two consecutive values of the adaptive parameters vanish asymptotically, i.e.,

$$\lim_{k\to\infty} \left\| \Delta\hat{\boldsymbol{\theta}}_k \right\| = \lim_{k\to\infty} \left\| \hat{\boldsymbol{\theta}}_k - \hat{\boldsymbol{\theta}}_{k-1} \right\| = 0. \qquad (70)$$

Consider now the matrix $\hat{\Phi}_k$ given as

$$\hat{\Phi}_k = \begin{bmatrix} \hat{\phi}_{1,k} & \hat{\phi}_{2,k} & \cdots & \cdots & \hat{\phi}_{n,k} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad (71)$$

then the difference $\Delta\hat{\Phi}_k$ is given as

$$\Delta\hat{\Phi}_k = \begin{bmatrix} \Delta\hat{\phi}_{1,k} & \Delta\hat{\phi}_{2,k} & \cdots & \cdots & \Delta\hat{\phi}_{n,k} \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \qquad (72)$$

and from (70), $\lim_{k\to\infty} \left| \Delta\hat{\phi}_{i,k} \right| = 0 \ \forall i \in [1, n]$, then

$$\lim_{k\to\infty} \left\| \Delta\hat{\Phi}_k \right\| = 0. \qquad (73)$$

Next, to show that $\|\Delta\hat{\Phi}_k^{-1}\| \to 0$, first consider that the inverse of $\hat{\Phi}_k$ is given as

$$\hat{\Phi}_k^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ \frac{1}{\hat{\phi}_{n,k}} & -\frac{\hat{\phi}_{1,k}}{\hat{\phi}_{n,k}} & \cdots & \cdots & -\frac{\hat{\phi}_{n-1,k}}{\hat{\phi}_{n,k}} \end{bmatrix}. \qquad (74)$$

Thus the difference $\Delta\hat{\Phi}_k^{-1} = \hat{\Phi}_k^{-1} - \hat{\Phi}_{k-1}^{-1}$ is

$$\Delta\hat{\Phi}_k^{-1} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ \frac{1}{\hat{\phi}_{n,k}} - \frac{1}{\hat{\phi}_{n,k-1}} & \frac{\hat{\phi}_{1,k-1}}{\hat{\phi}_{n,k-1}} - \frac{\hat{\phi}_{1,k}}{\hat{\phi}_{n,k}} & \cdots & \frac{\hat{\phi}_{n-1,k-1}}{\hat{\phi}_{n,k-1}} - \frac{\hat{\phi}_{n-1,k}}{\hat{\phi}_{n,k}} \end{bmatrix} \qquad (75)$$

The adaptive parameter $\hat{\phi}_{n,k}$ is just the sum of its value in the previous time step and the change up to the current one, i.e.

$$\hat{\phi}_{n,k} = \hat{\phi}_{n,k-1} + \Delta\hat{\phi}_{n,k} \qquad (76)$$

Thus, the first entry in the bottom row of $\Delta\hat{\Phi}_k^{-1}$ are rearranged as

$$\frac{1}{\hat{\phi}_{n,k}} - \frac{1}{\hat{\phi}_{n,k-1}} = \frac{1}{\hat{\phi}_{n,k-1} + \Delta\hat{\phi}_{n,k}} - \frac{1}{\hat{\phi}_{n,k-1}} = -\frac{\Delta\hat{\phi}_{n,k}}{\hat{\phi}_{n,k-1}(\hat{\phi}_{n,k-1} + \Delta\hat{\phi}_{n,k})}. \qquad (77)$$

The rest of the entries in the bottom row of $\Delta\hat{\Phi}_k^{-1}$ are rearranged as

$$\frac{\hat{\phi}_{i,k-1}}{\hat{\phi}_{n,k-1}} - \frac{\hat{\phi}_{i,k}}{\hat{\phi}_{n,k}} = \frac{\hat{\phi}_{i,k-1}}{\hat{\phi}_{n,k-1}} - \frac{\hat{\phi}_{i,k}}{\hat{\phi}_{n,k-1} + \Delta\hat{\phi}_{n,k}} = \frac{-\Delta\hat{\phi}_{i,k}\hat{\phi}_{n,k-1} + \hat{\phi}_{i,k-1}\Delta\hat{\phi}_{n,k}}{\hat{\phi}_{n,k}(\hat{\phi}_{n,k-1} + \Delta\hat{\phi}_{n,k})} \qquad (78)$$

where $i \in [1, n-1]$. In this form it is evident that all terms in the bottom row of $\Delta\hat{\Phi}_k^{-1}$ vanish, because 1) there exists a lower bound $\phi_{n,\min}$ such that $|\hat{\phi}_{n,k}| \geq |\phi_{n,\mathrm{m}}|$, 2) by *Lemma 2* all parameters are bounded, and 3) from (70) it is known that all terms $\Delta\hat{\phi}_{i,k}$ $\forall i \in [1, n]$ vanish. Therefore,

$$\lim_{k\to\infty} \left\| \Delta\hat{\Phi}_k^{-1} \right\| = 0. \qquad (79)$$

Finally, from (20) it is obtained that $\hat{\boldsymbol{\beta}}_{p_{\mathrm{m}},k} = \hat{\Phi}_k^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k}$ and the difference between two consecutive values is given as

$$\begin{aligned} \Delta\hat{\boldsymbol{\beta}}_{p_{\mathrm{m}},k} &= \hat{\boldsymbol{\beta}}_{p_{\mathrm{m}},k} - \hat{\boldsymbol{\beta}}_{p_{\mathrm{m}},k-1} \\ &= \hat{\Phi}_k^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} - \hat{\Phi}_{k-1}^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k-1} \\ &= \hat{\Phi}_k^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} - \hat{\Phi}_{k-1}^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} + \hat{\Phi}_{k-1}^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} \\ &\quad - \hat{\Phi}_{k-1}^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k-1} \\ &= \left(\hat{\Phi}_k^{-1} - \hat{\Phi}_{k-1}^{-1}\right)\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} + \hat{\Phi}_{k-1}^{-1}\left(\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} \right. \\ &\quad \left. - \hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k-1}\right) \\ &= \Delta\hat{\Phi}_k^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} + \hat{\Phi}_{k-1}^{-1}\Delta\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} \end{aligned} \qquad (80)$$

and from (70) and (79), it is obtained that

$$\lim_{k\to\infty} \left\| \Delta\hat{\boldsymbol{\beta}}_{p_{\mathrm{m}},k} \right\| = \lim_{k\to\infty} \left\| \Delta\hat{\Phi}_k^{-1}\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} + \hat{\Phi}_{k-1}^{-1}\Delta\hat{\boldsymbol{\psi}}_{p_{\mathrm{m}},k} \right\| = 0. \qquad (81)$$

Following similar steps then it is also true that $\lim_{k\to\infty} \left\| \Delta\hat{\boldsymbol{\beta}}_{i,k} \right\| = 0 \ \forall i \in [0, p_{\mathrm{m}} - 1]$. □

6.3 Proof of Lemma 4

Proof: Consider (15) and (16), the difference of the two vectors results in the expression

$$\begin{aligned} \boldsymbol{\eta}_k &= \hat{\boldsymbol{\eta}}_k + \left[\tilde{y}_k \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^\top + \sum_{i=1}^{p_{\mathrm{m}}} \left(\hat{\boldsymbol{\beta}}_{i,k} - \hat{\boldsymbol{\beta}}_{i,k-1}\right) u_{k-i} \\ &= \hat{\boldsymbol{\eta}}_k + \left[\tilde{y}_k \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right]^\top + \sum_{i=1}^{p_{\mathrm{m}}} \Delta\hat{\boldsymbol{\beta}}_{i,k} u_{k-i} \end{aligned} \qquad (82)$$

where $\Delta\hat{\boldsymbol{\beta}}_{i,k} \triangleq \hat{\boldsymbol{\beta}}_{i,k} - \hat{\boldsymbol{\beta}}_{i,k-1}$. Substitution of (21) and (32) in (82)

$$\begin{aligned}\boldsymbol{\eta}_k &= \hat{\Phi}_{\mathrm{m},k-1}\boldsymbol{\eta}_{k-1} + \sum_{i=1}^{p_\mathrm{m}} \Delta\hat{\boldsymbol{\beta}}_{i,k} u_{k-i} + \hat{\boldsymbol{\beta}}_{0,k-1}\varphi_{\mathrm{r},k-1} r_{k-1} \\ &\quad + \left[\tilde{y}_k \;\vdots\; 0 \;\vdots\; \cdots \;\vdots\; 0\right]^\top \\ &= \hat{\Phi}_{\mathrm{m},k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}\varphi_{\mathrm{r},k-1} r_{k-1} + \left[\tilde{y}_k \;\vdots\; 0 \;\vdots\; \cdots \;\vdots\; 0\right]^\top \\ &\quad - \sum_{i=1}^{p_\mathrm{m}} \Delta\hat{\boldsymbol{\beta}}_{i,k} \left(\boldsymbol{\varphi}_{\mathrm{x},k-i}^\top \boldsymbol{\eta}_{k-i} - \varphi_{\mathrm{r},k-i} r_{k-i}\right). \end{aligned} \quad (83)$$

Expressing (83) in augmented form and defining $\hat{\Gamma}_{i,k-1} \triangleq \Delta\hat{\boldsymbol{\beta}}_{i,k}\boldsymbol{\varphi}_{\mathrm{x},k-i}^\top \in \mathbb{R}^{n\times n}$ such that,

$$\begin{aligned}\bar{\boldsymbol{\eta}}_k &= \begin{bmatrix} \hat{\Phi}_{\mathrm{m},k-1} - \hat{\Gamma}_{1,k-1} & -\hat{\Gamma}_{2,k-1} & \cdots & -\hat{\Gamma}_{p_\mathrm{m},k-1} \\ I & [0] & \cdots & [0] \\ [0] & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & [0] \end{bmatrix} \bar{\boldsymbol{\eta}}_{k-1} \\ &\quad + \left[\hat{\boldsymbol{\beta}}_{0,k-1} \;\vdots\; [0] \;\vdots\; \cdots \;\vdots\; [0]\right]^\top \varphi_{\mathrm{r},k-1} r_{k-1} + \sum_{i=1}^{p_\mathrm{m}} \left[\Delta\hat{\boldsymbol{\beta}}_{i,k}^\top \;\vdots\right. \\ &\quad \left.[0] \;\vdots\; \cdots \;\vdots\; [0]\right]^\top \varphi_{\mathrm{r},k-i} r_{k-i} + \left[\tilde{y}_k \;\vdots\; 0 \;\vdots\; \cdots \;\vdots\; 0\right]^\top \end{aligned} \quad (84)$$

where $\bar{\boldsymbol{\eta}}_{k-1}^\top \triangleq \left[\boldsymbol{\eta}_{k-1}^\top \;\vdots\; \boldsymbol{\eta}_{k-2}^\top \;\vdots\; \cdots \;\vdots\; \boldsymbol{\eta}_{k-p_\mathrm{m}}^\top\right] \in \mathbb{R}^{n\cdot p_\mathrm{m}}$. Using the results in *Lemma 2* and [21], $\lim_{k\to\infty}\left\|\hat{\Gamma}_{i,k-1}\right\| = 0$ and that implies that the augmented system, (84), can be reduced to the form

$$\begin{aligned}\bar{\boldsymbol{\eta}}_k &= \begin{bmatrix} \hat{\Phi}_{\mathrm{m},k-1} & [0] & \cdots & [0] \\ I & [0] & \cdots & [0] \\ [0] & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & [0] \end{bmatrix} \bar{\boldsymbol{\eta}}_{k-1} + \left[\hat{\boldsymbol{\beta}}_{0,k-1}^\top \;\vdots\; [0] \;\vdots\right. \\ &\quad \left.\cdots \;\vdots\; [0]\right]^\top \varphi_{\mathrm{r},k-1} r_{k-1} + \left[\tilde{y}_k \;\vdots\; 0 \;\vdots\; \cdots \;\vdots\; 0\right]^\top. \end{aligned} \quad (85)$$

The system (85) will the have n eigenvalues of the matrix $\hat{\Phi}_{\mathrm{m},k}$ which are stable and the remaining $n \cdot p_\mathrm{m} - n$ eigenvalues are 0 and, as a result, can be further reduced to an n^{th}-order dynamics of the form

$$\boldsymbol{\eta}_k = \hat{\Phi}_{\mathrm{m},k-1}\boldsymbol{\eta}_{k-1} + \hat{\boldsymbol{\beta}}_{0,k-1}\varphi_{\mathrm{r},k-1} r_{k-1} + \left[\tilde{y}_k \;\vdots\; 0 \;\vdots\; \cdots \;\vdots\; 0\right]^\top. \quad (86)$$

Futhermore, $\hat{\boldsymbol{\beta}}_{0,k}$, $\varphi_{\mathrm{r},k}$ and r_k are bounded, therefore, the system (86) is stable and a bound on $\boldsymbol{\eta}_k$ exists such that

$$\|\boldsymbol{\eta}_k\| \leq c_0 + c_1 \max_{i\in[0,k]} |\tilde{y}_{k-i}| \quad (87)$$

for some positive constants c_0 and c_1. This establishes the bound on $\boldsymbol{\eta}_k$.

Consider now the control law (32). From (87) and the fact that $\boldsymbol{\varphi}_{\mathrm{x},k}$, $\varphi_{\mathrm{r},k}$ and r_k are bounded then the control input is bounded as

$$|u_k| \leq c_2 + c_3 \max_{i\in[0,k]} |\tilde{y}_{k-i}| \quad (88)$$

for some positive constants c_2 and c_3. Using (82) and the fact that $y_k = \hat{y}_k + \tilde{y}_k$ a bound on y_k is obtained as

$$|y_k| \leq c_4 + c_5 \max_{i\in[0,k]} |\tilde{y}_{k-i}| \quad (89)$$

for some positive constants c_4 and c_5. From the definition of $\boldsymbol{\zeta}_k$ and using (88), (89) there exists positive constants c_0^0 and c_1^0 such that

$$\|\boldsymbol{\zeta}_k\| \leq c_0^0 + c_1^0 \max_{i\in[0,k]} |\tilde{y}_{k-i}|. \quad (90)$$

Consequently, from (35), (90) and the *Key Technical Lemma*, [25], it is obtained that

$$\lim_{k\to\infty} |\tilde{y}_k| \leq \nu_{\max}. \quad (91)$$

□

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