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Relational structures for concurrent behaviours

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A B S T R A C T

Relational structures based on acyclic relations can successfully model fundamental aspects of concurrent systems behaviour. Examples include Elementary Net systems and Mazurkiewicz traces. There are however cases where more general relational structures are needed. In this paper, we present a general model of relational structures which can be used for a broad class of concurrent behaviours. We demonstrate how this general set-up works for combined order structures which are based on two relations, viz. an acyclic 'before' relation and a possibly cyclic 'not later than' relation.

1. Introduction

In the development of effective design and validation techniques for a wide range of concurrent computing systems, it is often necessary to capture and analyse intricate relationships between events (occurrences of actions) executed during a system run. While sequential representations of behaviours are easy to process, without additional information or structuring they cannot convey precise information about important semantical characteristics such as causality and independence between executed actions. A successful way of addressing this problem is to employ relational structures such as partial orders, where causally related events are ordered, and concurrent (or independent) events are unordered. An example of such an approach are Mazurkiewicz traces [1,2], where invariant causal dependencies between events, common to all elements of the trace (a set of related sequential executions), define an acyclic dependence graph which — through its transitive closure — determines the underlying causality structure of the trace as a (labelled) partial order [3]. As a result, each trace can be represented by a labelled partial order (see, e.g., [4–6]) identifying independence and unorderedness, and the approach uses relational structures to represent different aspects of concurrent behaviour: acyclic graphs to represent dependence graphs, partial orders to represent causality (both direct and indirect), and sets of total orders to represent sequential executions (records or individual observations of a behaviour). Mazurkiewicz traces are a particularly well fitting behavioural model for Elementary Net systems, a fundamental class of Petri Nets (see [7]).

In [8,9], a generalisation of the theory of Mazurkiewicz traces is presented for the case that actions could be observed as occurring simultaneously. Thus executions are sequences of *steps*, i.e., sets of one or more simultaneously observed actions.

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Now, to represent the invariant dependencies including 'mutually exclusive' (unordered but not simultaneous) and 'earlier or simultaneous' (not later than) occurrences, requires the introduction of new ordering relations between events. Concurrent runs can be represented by labelled relational structures which are generalisations of the labelled partial orders used in the case of Mazurkiewicz traces. They provide two intrinsic relations between elements: *mutual exclusion* and *weak causality*. Other examples where relational structures are used to model concurrent behaviours include, e.g., [10,11], which investigates relational structures with two component relationships, viz. causality and weak causality. Models of concurrent behaviours developed by these extensions are powerful enough to describe the concurrent runs of different classes of Elementary Net systems e.g., Elementary Net systems extended with inhibitor arcs (test for zero) and mutex arcs preventing simultaneous execution of actions, e.g., [10,12,13]. The initial versions of the relational structures considered in this paper were proposed in [14,15], and the version of relational structures used in this paper originates from [16].

A common feature of the semantical models mentioned above is that they deal with three kinds of *relational structures* aimed at the modelling of different aspects of concurrent behaviours. At the top level one finds dependence structures describing *direct* relationships, such as causality or mutual exclusion, between events executed in a concurrent run. Dependence structures are defined, e.g., by taking account of resource dependencies in a concurrent system during a run. There is no attempt, however, to represent *derived* relationships. This is done at the middle level, where all relationships stemming from, e.g., causal chains between events, are added. The resulting structure gives a full account of the invariant relationships between the events shared by the relational structures at the bottom level. Each relational structure at this level is treated as representing a system execution which adheres to the dependence structure at the top level and invariant structure at the middle level, providing a complete description of a single system execution. For example, a concurrent behaviour with two independent events, e and f, executed on a single processor machine generates two total order executions, 'e followed by f' and 'f followed by e'. One of the advantages of such a three-level view of concurrent behaviours is that one can, e.g., deal with a single structure at the invariant level rather than with the (exponentially) many structures at the bottom level. This property has been exploited by verification techniques such as partial order model checking [17]. The soundness of the approach we just outlined depends on providing suitable means of relating the three levels of representing concurrent histories, such as deriving an invariant structure from a dependence structure (e.g., through transitive closure of an acyclic relation in the case of causal partial orders).

In this paper, we introduce and investigate a generic model based on a class $\mathscr{R}$ of (labelled) relational structures, and all structures we consider are included in $\mathscr{R}$. Relational structures in $\mathscr{R}$ are compared w.r.t. the information they convey, and a relational structure rs is an extension of a relational structure rs' if the former is obtained from the latter by adding new relationships. Intuitively, this means that rs is more *concrete* than rs' by providing more details about the relations between events. The *maximal* relational structures $\mathscr{R}^{max}$ residing at the bottom level are those which cannot be extended (or concretised) any further, and sets of relational structures extending a given rs are then regarded as the executions conforming to a more abstract concurrent history. Finally, the middle level relational structures are the *closed* relational structures $\mathscr{R}^{clo}$ which can be seen as transitively closed dependence structures. Closed structures are in a one-to-one correspondence with sets of maximal structures generated by dependence structures. We then discuss the notion of *label-linear* relational structures $\mathscr{R}^{lin}$ such that one can attribute a strict execution order to all instances of a given action. Label-linear relational structures allow one to construct *relational spaces* where the interpretation of the execution order of events is unambiguous.

The central message conveyed by this paper is that after specifying the set $\mathscr{R}$ of relational structures which represent an application specific class of concurrent behaviours, the development of a complete framework is basically automatic.

The paper is organised as follows. The next section provides basic notions used throughout. Sections 3–7, discuss relational structures as the basis for constructing relational spaces, introducing more relevant assumptions whenever it is necessary. In particular, we will aim at characterising those relational structures which can be thought of as representations of individual system executions, and those which can be thought of as representations of sets of closely related system executions. This leads to a blueprint for developing relational spaces without making any additional assumptions, e.g., about interpretations of the relationships between events. In Section 9, we show how this blueprint can be applied in the case of relational structures comprising two relationships, viz. causality and weak causality.

2. Preliminaries

Basic notations Let $R, S \subseteq X \times X$ be two binary relations over X. Then: *(i)* the *composition* of R and S is the relation $R \circ S = \{\langle x, z\rangle \mid \exists y \in X : \langle x, y\rangle \in R \wedge \langle y, z\rangle \in S\}$; *(ii)* $R^{-1} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$ is the *inverse* of R; *(iii)* $R^{sym} = R \cup R^{-1}$ is the *symmetric closure* of R; *(iv)* $R^0 = \mathrm{id}_X = \{\langle x, x\rangle \mid x \in X\}$ and $R^n = R^{n-1} \circ R$ $(n \geq 1)$ are the *powers* of R, where $\mathrm{id}_X = \{\langle x, x\rangle \mid x \in X\}$; *(v)* $R^+ = R^1 \cup R^2 \cup \ldots$ is the *transitive closure* of R; *(vi)* $R^* = R^+ \cup R^0$ is the *reflexive transitive closure* of R; *(vii)* $(R^{sym})^*$ is the *symmetric, reflexive, and transitive closure* of R; and *(viii)* $R^{\curlywedge} = R^+ \setminus R^0$ is the *irreflexive transitive closure* of R.

Relational structures In this paper, behaviours of concurrent systems are modelled as structures consisting of a finite domain, the elements of which represent events labelled by actions, together with binary relations describing the interdependencies between the domain elements. In the literature, structures of this kind have been used (explicitly or implicitly) to represent concurrent behaviours in, e.g., [11,15,16,18]. It should be stressed that the treatment of relational structures (a general class of mathematical concepts) in this paper is motivated by their application to modelling concurrent behaviours. Although the resulting theory is not restricted to concurrency related properties, some of the concepts and results are tied to the chosen area of application.

Throughout the paper, $\mathbb{U}$ denotes a (infinite) *universe* of all possible *domain elements* that can be used in the structures we deal with, $\mathbb{A}$ is the set of *actions* which any hypothetical concurrent system might perform, and $\mathbb{E} = \{a^{(i)} \mid a \in \mathbb{A} \wedge i \geq 1\}$ is the set of *events* comprising all possible execution instances of these actions. Intuitively, an event $a^{(i)}$ represents the i-th occurrence of action a in some behaviour of a concurrent system, and it is implicitly assumed that $a^{(i)}$ occurred before $a^{(j)}$ whenever $i < j$.

A *relational structure* is a tuple $rs = \langle \Delta, Q^1, \ldots, Q^n, \ell \rangle$, where: $n \geq 1$ is the *arity* of rs; Δ is a *finite* subset of $\mathbb{U}$ called the *domain*; the Q^i's are binary relations over the domain; and $\ell : \Delta \to \mathbb{A}$ is the *labelling* of the domain elements. The components of rs can be referred to with the subscript rs. Intuitively, rs is a record of an execution of a concurrent system. The domain lists events that have been recorded, and the labelling associates a specific action with each of these events. Crucially, the relations state the nature of the relationships between the executed events, such as causality or mutual exclusion. A *concurrent history* (cf. [2,14]) consists of records that 'belong together', i.e., while they may be formally different, they are intuitively close [19] and seen as 'equivalent' observations of the same execution.

To illustrate the concepts associated with relational structures relevant to this paper, we will use three running examples. The running examples are based on three sets of relational structures, AO, DAO, and ONE.

Example 2.1. AO consists of relational structures representing concurrent behaviours where the only structural relationship between events is causality (or precedence). These relational structures have one relation that is assumed to be acyclic (to exclude 'causal cycles'). Hence, AO comprises *acyclic orders* $ao = \langle \Delta, \prec, \ell \rangle$ such that $\prec$ is an acyclic relation over Δ (i.e., $\prec^+$ is irreflexive). Moreover, ao belongs to: (i) *total orders* TO if $\prec \cup \prec^{-1} = (\Delta \times \Delta) \setminus \mathrm{id}_\Delta$; (ii) *partial orders* PO if $\prec$ is transitive; (iii) *stratified orders* SO if $\prec$ is transitive and $(\Delta \times \Delta) \setminus \prec^{sym}$ is an equivalence relation; and (iv) *interval orders* if $a \prec c$ and $b \prec d$ implies $a \prec d$ or $b \prec c$, for all $a, b, c, d \in \Delta$.

We denote by $\sqsubset_{ao}$ the relation $\{\langle x, y \rangle \in \Delta \times \Delta \mid x \neq y \not\prec x\}$, i.e., $x \sqsubset_{ao} y$ if x precedes y or if the two elements are unordered.

In many models, executions are represented by *sequences* or *step sequences* of actions, i.e., total or stratified orders (see, for example, [9,10,20,21]). When standard 'true concurrency' is assumed (i.e., if two actions a and b are deemed independent, then their simultaneous execution and the orders a followed by b and b followed by a are considered 'equivalent', i.e., they are seen as records of the same execution), then the acyclic orders AO adequately model concurrent behaviours [4,20,21]. It was argued by Wiener in 1914 [22] (and later more formally in [14]) that any execution that can be observed by a single observer must be an interval order. Generating system runs that are represented by interval orders directly is problematic for most models of concurrency [18]. Interval orders have however a natural sequence representation. In this paper, we do not exclude representing executions by interval orders of events, however we will not discuss this issue in detail.

Example 2.2. DAO consists of *distributed acyclic orders* $dao = \langle \Delta, \rightharpoonup, \rightharpoondown, \ell \rangle$, where $\rightharpoonup$ and $\rightharpoondown$ are disjoint binary relations over Δ such that their union is acyclic. One may think of distributed acyclic orders as records of dependencies between events generated at two different locations of a distributed system.

Although $\langle \Delta, \rightharpoonup \cup \rightharpoondown, \ell \rangle$ is an acyclic order, and so distributed acyclic orders represent the same dependencies between events as acyclic orders, DAO and AO are not equivalent. In particular, an acyclic order may correspond to many different distributed acyclic orders. Consider $dao = \langle \Delta, \{\langle x, y \rangle\}, \{\langle y, z \rangle\}, \ell \rangle$ modelling a behaviour where x is followed by y in one subsystem, and the same y is followed by z in the other subsystem. Intuitively, dao is equivalent to the acyclic order $ao = \langle \Delta, \{\langle x, y \rangle, \langle y, z \rangle\}, \ell \rangle$ and also $dao' = \langle \Delta, \{\langle x, y \rangle, \langle y, z \rangle\}, \varnothing, \ell \rangle$ is a distributed acyclic order equivalent to ao.

Example 2.3. ONE is the set of all relational structures of arity one such that the relations they contain are irreflexive. This example is not meant to model behaviours of a computational system.

The *actions* and *events* of a relational structure rs are respectively given by $\mathbb{A}_{rs} = \ell_{rs}(\Delta_{rs})$ and $\mathbb{E}_{rs} = \{a^{(i)} \mid a \in \mathbb{A}_{rs} \wedge 1 \leq i \leq |\ell_{rs}^{-1}(a)|\}$. Intuitively, $\mathbb{E}_{rs}$ is the set of all events involved in a concurrent behaviour represented by rs, each of these events being an instance of an underlying action in $\mathbb{A}_{rs}$.

An *extension* of a relational structure rs is any relational structure rs' with the same arity n, the same domain and labelling, and satisfying $Q^i_{rs} \subseteq Q^i_{rs'}$, for every $1 \leq i \leq n$. This is denoted by $rs \trianglelefteq rs'$, and $rs \triangleleft rs'$ means that $rs \trianglelefteq rs'$ and $rs \neq rs'$. The *set of extensions* of rs is given by $\mathrm{ext}(rs) = \{rs' \mid rs \trianglelefteq rs'\}$. We also denote $\mathrm{ext}_{\mathscr{R}}(rs) = \mathrm{ext}(rs) \cap \mathscr{R}$.

For all $a \in \mathbb{A}_{rs}$ and $1 \leq i \leq n$, $\Delta^{\lfloor a \rfloor} = \ell^{-1}(a)$, $\ell^{\lfloor a \rfloor} = \ell|_{\Delta^{\lfloor a \rfloor}}$, $(Q^i)^{\lfloor a \rfloor} = Q^i_{rs}|_{\Delta^{\lfloor a \rfloor} \times \Delta^{\lfloor a \rfloor}}$, and $rs^{\lfloor a \rfloor} = \langle \Delta^{\lfloor a \rfloor}, (Q^1)^{\lfloor a \rfloor}, \ldots, (Q^n)^{\lfloor a \rfloor}, \ell^{\lfloor a \rfloor} \rangle$.

The *intersection* of a nonempty set RS of relational structures with the same domain Δ, arity n, and labelling ℓ is $\bigcap RS = \langle \Delta, \bigcap_{rs \in RS} Q^1_{rs}, \ldots, \bigcap_{rs \in RS} Q^n_{rs}, \ell \rangle$.

When in this paper a set of relational structures with the same domain and labelling is considered, it implicitly comprises relational structures representing behaviours of some hypothetical concurrent system, such as a distributed program or a

Petri Net. Moreover, the relational structures in such set are intended to represent behaviours at *different levels of abstraction*. The approach we are about to present deals with these different levels using *the same kind of relational structures*, and the distinct nature of the abstraction levels will follow solely from their general structural (or order theoretic) properties.

From now until the end of Section 7, $\mathscr{R}$ and $\mathscr{S}$ are assumed to be nonempty sets of relational structures.

A function $f : \mathscr{R} \to \mathscr{R}$ is *non-decreasing* if $rs \trianglelefteq f(rs)$, for every $rs \in \mathscr{R}$, and $g : \mathscr{R} \to \mathscr{S}$ is *monotonic* if $g(rs) \trianglelefteq g(rs')$, for all $rs, rs' \in \mathscr{R}$ satisfying $rs \trianglelefteq rs'$.

A *renaming* of a relational structure $rs = \langle \Delta, Q^1, \ldots, Q^n, \ell\rangle$ is a bijection ψ from Δ to a subset of $\mathbb{U}$. Applying ψ to rs yields an *isomorphic* relational structure $\psi(rs) = \langle \psi(\Delta), \psi(Q^1), \ldots, \psi(Q^n), \ell \circ \psi^{-1}\rangle$, where $\psi(Q^i) = \{\langle \psi(x), \psi(y)\rangle \mid \langle x, y\rangle \in Q^i\}$, for every $1 \le i \le n$. We call ψ an *isomorphism* and denote $rs \sim_\psi rs'$. We say that $\mathscr{R}$ is *renaming-closed* if $\psi(rs) \in \mathscr{R}$, for every $rs \in \mathscr{R}$ and every renaming ψ of rs. Note that in a renaming-closed $\mathscr{R}$, we can abstract from the identities of the domain elements of a relational structure and focus on the labellings and the structure as defined through its relations.

3. Maximal relational structures

In general, a relational structure rs in $\mathscr{R}$ provides partial information about the relationships between its events, and it can thus be used to represent a group of system executions. The first kind of relational structures we distinguish are those which represent *individual behaviours (or single executions)*. By this we mean that all information about the concurrent behaviour is present in the relational structure, and any additional relationship would be inconsistent with the behaviour represented (e.g., as it would introduce a cycle to a relation intended to represent causality between executed events).

Definition 3.1 *(maximal relational structure).* A relational structure $rs \in \mathscr{R}$ is *maximal in* $\mathscr{R}$ if $\mathsf{ext}_{\mathscr{R}}(rs) = \{rs\}$. We denote this by $rs \in \mathscr{R}^{max}$. The function $\max_{\mathscr{R}} : \mathscr{R} \to 2^{\mathscr{R}^{max}}$ returns the maximal extensions of relational structures, i.e., $\max_{\mathscr{R}}(rs) = \mathsf{ext}_{\mathscr{R}^{max}}(rs)$, for every $rs \in \mathscr{R}$.

Each relational structure in $\mathscr{R}$ has at least one maximal extension in $\mathscr{R}$; extending a relational structure may only reduce the number of maximal extensions; and the intersection of all the maximal extensions of a relational structure rs is an extension of rs. Formally:

Proposition 3.2. *Let $rs, rs' \in \mathscr{R}$.*

(1) $\max_{\mathscr{R}}(rs) \neq \emptyset$.
(2) $rs \trianglelefteq rs'$ *implies* $\max_{\mathscr{R}}(rs) \supseteq \max_{\mathscr{R}}(rs')$.
(3) $rs \trianglelefteq \bigcap \max_{\mathscr{R}}(rs)$.
(4) $\max_{\mathscr{R}}(rs) = \max_{\mathscr{R}}(\bigcap \max_{\mathscr{R}}(rs))$, *provided that* $\bigcap \max_{\mathscr{R}}(rs) \in \mathscr{R}$.

Proof. *(1,2,3)* Follow directly from the definitions as well as the finiteness of relational structures (in the case of part *(1)*).

(4) Let $\overline{rs} = \bigcap \max_{\mathscr{R}}(rs)$. By parts *(2)* and *(3)*, $\max_{\mathscr{R}}(rs) \supseteq \max_{\mathscr{R}}(\overline{rs})$. Suppose that $rs'' \in \max_{\mathscr{R}}(rs)$. Then $\overline{rs} \trianglelefteq rs''$. Hence, by part *(2)*, $\max_{\mathscr{R}}(\overline{rs}) \supseteq \max_{\mathscr{R}}(rs'') = \{rs''\}$. Thus $\max_{\mathscr{R}}(rs) \subseteq \max_{\mathscr{R}}(\overline{rs})$, and so the result holds. □

Example 3.3. The total orders TO are the maximal relational structures in AO and $\max_{\mathsf{AO}}$ returns the set of all total order extensions of an acyclic order, i.e., $\mathsf{TO} = \mathsf{AO}^{max}$. For example, Proposition 3.2(1) means that each acyclic order can be extended to a total order.

DAO^{max} comprises all distributed acyclic orders dao such that $\rightharpoonup_{dao} \cup \rightharpoondown_{dao}$ is a total order relation. Moreover, $\bigcap \max_{\mathsf{DAO}}(dao) = dao$, for every $dao \in \mathsf{DAO}$. This follows from the fact that if $\langle \Delta_{dao}, \rightharpoonup_{dao} \uplus \rightharpoondown'_{dao}, \rightharpoondown_{dao} \uplus \rightharpoonup'_{dao}, \ell_{dao}\rangle$ is a maximal extension of dao, then so is $\langle \Delta_{dao}, \rightharpoonup_{dao} \uplus \rightharpoondown'_{dao}, \rightharpoondown_{dao} \uplus \rightharpoonup'_{dao}, \ell_{dao}\rangle$.

ONE^{max} comprises all relational structures of the form $\langle \Delta, \{\langle x, y\rangle \mid x \neq y \in \Delta\}, \ell\rangle$.

Intuitively, a maximal relational structure represents in full all the dependencies between events involved in a single system execution. No further relationships can be added without moving out of $\mathscr{R}$. (Note also that if $\max_{\mathscr{R}}(rs)$ consists of only one relational structure, then rs does not have to be maximal, as there may be still some implicit dependencies not included in rs.) With this interpretation of $\mathscr{R}^{max}$, the set of maximal extensions $\max_{\mathscr{R}}(rs)$ of rs represents all single system executions (in $\mathscr{R}$) that respect all relations between the events of rs.

When relating two classes of relational structures, we are interested in particular in the relationship between their maximal elements since these represent the individual behaviours defined by the class. The following result provides us with a condition that guarantees a one-to-one correspondence between the maximal relational structures in the two classes. However, this in itself would not be enough to justify behavioural equivalence between these classes, understood as reflecting the same concurrent behaviours through a relation between corresponding elements. This issue is addressed in the next result by showing that corresponding relational structures have corresponding maximal extensions.

Theorem 3.4. *Let $\mathscr{R} \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{R}$ be monotonic functions such that $g \circ f$ and $f \circ g$ are non-decreasing functions.*

(1) $\mathscr{R}^{max} \xrightarrow{f} \mathscr{S}^{max} \xrightarrow{g} \mathscr{R}^{max}$ *are inverse bijections.*
(2) $\max_{\mathscr{S}} \circ f = f \circ \max_{\mathscr{R}}$ *and* $\max_{\mathscr{R}} \circ g = g \circ \max_{\mathscr{S}}$.
(3) $\max_{\mathscr{R}} = g \circ \max_{\mathscr{S}} \circ f$ *and* $\max_{\mathscr{S}} = f \circ \max_{\mathscr{R}} \circ g$.

Proof. *(1)* Let $rs \in \mathscr{R}^{max}$. We first observe that $rs = g \circ f(rs)$, which follows from $rs \trianglelefteq g \circ f(rs)$ (as $g \circ f$ is non-decreasing) and rs being maximal. We next observe that $f(rs) \in \mathscr{S}^{max}$. Indeed, suppose that $f(rs) \trianglelefteq rs' \in \mathscr{S}^{max}$. Then, by the monotonicity of g, $g \circ f(rs) \trianglelefteq g(rs')$. Hence, as $g \circ f$ is non-decreasing, $rs \trianglelefteq g \circ f(rs) \trianglelefteq g(rs')$. Thus, since rs is maximal, $g(rs') = rs$. Hence, as $f \circ g$ is non-decreasing, $rs' \trianglelefteq f \circ g(rs') = f(rs)$. As a result, by $rs' \in \mathscr{S}^{max}$, we have $rs' = f(rs)$, and so $f(rs) \in \mathscr{S}^{max}$.

We have shown $rs = g \circ f(rs)$ and $f(rs) \in \mathscr{S}^{max}$, for every $rs \in \mathscr{R}^{max}$. By a symmetric argument, $rs' = f \circ g(rs')$ and $g(rs') \in \mathscr{R}^{max}$, for every $rs' \in \mathscr{S}^{max}$. Hence, part *(1)* holds.

(2) We only prove the first equality as the second one follows by symmetry.

Suppose that $rs \in \mathscr{R}$ and $rs' \in \max_{\mathscr{R}}(rs)$. Then, by the monotonicity of f, we have $f(rs) \trianglelefteq f(rs')$. Moreover, by part *(1)*, $f(rs') \in \mathscr{S}^{max}$. Hence $f(rs') \in \max_{\mathscr{S}} \circ f(rs)$, and so $\max_{\mathscr{S}} \circ f(rs) \supseteq f \circ \max_{\mathscr{R}}(rs)$.

Suppose now that $rs' \in \max_{\mathscr{S}} \circ f(rs)$. Then, by the monotonicity of g, we have that $g \circ f(rs) \trianglelefteq g(rs')$. Moreover, by part *(1)* and $g \circ f$ being non-decreasing, $g(rs') \in \mathscr{R}^{max}$ and $rs \trianglelefteq g \circ f(rs)$. Hence $g(rs') \in \max_{\mathscr{R}}(rs)$. Moreover, by part *(1)*, we obtain $f \circ g(rs') = rs'$. Thus $rs' = f \circ g(rs') \in f \circ \max_{\mathscr{R}}(rs)$, and so $\max_{\mathscr{S}} \circ f(rs) \subseteq f \circ \max_{\mathscr{R}}(rs)$.

(3) We only prove the first equality as the second one follows by symmetry.

By parts *(1)* and *(2)*, we have $\max_{\mathscr{R}} = g \circ f \circ \max_{\mathscr{R}} = g \circ \max_{\mathscr{S}} \circ f$. □

Theorem 3.4 employs two functions, f and g, intended to relate 'equivalent' structures. Requiring f to be monotonic, i.e., $f(rs) \trianglelefteq f(rs')$ whenever $rs \trianglelefteq rs'$, means that f (and also g) preserves structural information in extensions. And, similarly, the application of the non-decreasing functions $g \circ f$ and $f \circ g$ does not lead to loss of information.

As the definition of maximal extensions is based on structure rather than labels, the set of maximal relational structures in $\mathscr{R}$ is renaming-closed if $\mathscr{R}$ is renaming-closed.

Theorem 3.5. *If $\mathscr{R}$ is renaming-closed, then $\mathscr{R}^{max}$ is renaming-closed. Moreover, if ψ is a renaming of $rs \in \mathscr{R}$, then* $\max_{\mathscr{R}} \circ \psi(rs) = \psi \circ \max_{\mathscr{R}}(rs)$.

Proof. We first observe that, for all $rs, rs' \in \mathscr{R}$, we have: *(i)* $rs \trianglelefteq rs'$ implies $\psi(rs) \trianglelefteq \psi(rs')$; and *(ii)* $rs \triangleleft rs'$ implies $\psi(rs) \triangleleft \psi(rs')$.

Let $rs \in \mathscr{R}^{max}$. As $\mathscr{R}$ is renaming-closed, $\psi(rs) \in \mathscr{R}$. Suppose that $\psi(rs) \notin \mathscr{R}^{max}$. Then there is $rs' \in \mathscr{R}$ such that $\psi(rs) \triangleleft rs'$. Hence, by *(ii)* and $\mathscr{R}$ being renaming-closed, $\psi^{-1}(rs') \in \mathscr{R}$ and $rs = \psi^{-1} \circ \psi(rs) \triangleleft \psi^{-1}(rs')$. This, however, contradicts $rs \in \mathscr{R}^{max}$. Hence $\mathscr{R}^{max}$ is renaming-closed.

Suppose now that $rs \in \mathscr{R}$ and $\overline{rs} \in \max_{\mathscr{R}}(rs)$. Then, by $\mathscr{R}$ and $\mathscr{R}^{max}$ being renaming-closed and *(i)* and $rs \trianglelefteq \overline{rs}$, we have $\psi(rs) \in \mathscr{R}$ and $\psi(\overline{rs}) \in \mathscr{R}^{max}$ and $\psi(rs) \trianglelefteq \psi(\overline{rs})$. Hence $\psi \circ \max_{\mathscr{R}}(rs) \subseteq \max_{\mathscr{R}} \circ \psi(rs)$. By a symmetric argument for ψ^{-1} and $\psi(rs)$, we have $\psi^{-1} \circ \max_{\mathscr{R}} \circ \psi(rs) \subseteq \max_{\mathscr{R}} \circ \psi^{-1} \circ \psi(rs)$, and so $\max_{\mathscr{R}} \circ \psi(rs) = \psi \circ \psi^{-1} \circ \max_{\mathscr{R}} \circ \psi(rs) \subseteq \psi \circ \max_{\mathscr{R}} \circ \psi^{-1} \circ \psi(rs) = \psi \circ \max_{\mathscr{R}}(rs)$. Hence $\max_{\mathscr{R}} \circ \psi(rs) = \psi \circ \max_{\mathscr{R}}(rs)$. □

4. Closed relational structures

In the semantical treatment of concurrent behaviours, a central role is played by relational structures which are intersections of their maximal extensions, such as the causal partial orders in Mazurkiewicz trace theory (with partial orders being the intersection of their total order extensions), stratified orders in comtrace theory [10] and step trace theory [9], and interval orders in interval trace theory [18].

The set $\mathscr{R}$ is *intersection-closed* if $\bigcap \max_{\mathscr{R}}(rs) \in \mathscr{R}$, for every $rs \in \mathscr{R}$. Note that, by Proposition 3.2(1), $\max_{\mathscr{R}}(rs) \neq \varnothing$. Moreover, all relational structures in $\max_{\mathscr{R}}(rs)$ have the same domain and labelling. Thus $\bigcap \max_{\mathscr{R}}(rs)$ is always a relational structure.

Example 4.1. AO, DAO, and ONE are all intersection-closed sets of relational structures.

There is a simple sufficient condition for intersection-closedness. We say that $\mathscr{R}$ is *convex* if, for all $rs, rs'' \in \mathscr{R}$ and for every relational structure rs', $rs \trianglelefteq rs' \trianglelefteq rs''$ implies that $rs' \in \mathscr{R}$.

Proposition 4.2. *If $\mathscr{R}$ is convex, then it is intersection-closed.*

Proof. Let $rs \in \mathscr{R}$, $rs' = \bigcap \max_{\mathscr{R}}(rs)$, and $rs'' \in \max_{\mathscr{R}}(rs)$. By Proposition 3.2(3), we have $rs \trianglelefteq rs' \trianglelefteq rs''$. Hence, as $\mathscr{R}$ is convex, $rs' \in \mathscr{R}$. □

Convexity is not a necessary condition for intersection-closedness. Consider, for example, any three relational structures satisfying $rs \lhd rs' \lhd rs''$. Then $\mathscr{R} = \{rs, rs''\}$ is intersection-closed as $\bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs'') = \bigcap\{rs''\} = rs''$, but not convex.

Example 4.3. AO, DAO, and ONE are all convex sets of relational structures.

Definition 4.4 *(closed relational structure).* A relational structure $rs \in \mathscr{R}$ is *closed in* $\mathscr{R}$ if $rs = \bigcap \max_{\mathscr{R}}(rs)$. We denote this by $rs \in \mathscr{R}^{clo}$.

Note that each closed relational structure adheres to the 'spirit' of Szpilrajn's Theorem [23] which states that each partial order is the intersection of its total order extensions.

Example 4.5. $\mathsf{AO}^{clo} = \mathsf{PO}$, $\mathsf{DAO}^{clo} = \mathsf{DAO}$, and $\mathsf{ONE}^{clo} = \mathsf{ONE}^{max}$.

The next result captures basic relationships between maximal and closed structures. In particular, one can characterise all closed relational structures in a rather straightforward and precise way, by showing that each relational structure gives rise to a closed one through intersecting all its maximal extensions.

Proposition 4.6. *$\mathscr{R}^{max}$ and $\mathscr{R}^{clo}$ are nonempty intersection-closed sets of relational structures satisfying $(\mathscr{R}^{max})^{max} = (\mathscr{R}^{max})^{clo} = (\mathscr{R}^{clo})^{max} = \mathscr{R}^{max} \subseteq \mathscr{R}^{clo} = (\mathscr{R}^{clo})^{clo}$ and $\max_{\mathscr{R}^{clo}}(rs) = \max_{\mathscr{R}}(rs)$, for every $rs \in \mathscr{R}^{clo}$. Moreover, $\mathscr{R}^{clo} = \{\bigcap \max_{\mathscr{R}}(rs) \mid rs \in \mathscr{R}\}$ whenever $\mathscr{R}$ is intersection-closed.*

Proof. Clearly, $(\mathscr{R}^{max})^{max} = \mathscr{R}^{max}$. We then observe that $\bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}^{max}}(rs) = \bigcap\{rs\} = rs$, for every $rs \in \mathscr{R}^{max}$. Hence $\mathscr{R}^{max} \subseteq \mathscr{R}^{clo}$, $(\mathscr{R}^{max})^{clo} = \mathscr{R}^{max}$, and $\mathscr{R}^{max}$ is intersection-closed.

To show $(\mathscr{R}^{clo})^{max} = \mathscr{R}^{max}$, we observe that $\mathscr{R}^{max} \subseteq (\mathscr{R}^{clo})^{max}$ as we have $\mathscr{R}^{max} \subseteq \mathscr{R}^{clo} \subseteq \mathscr{R}$. Suppose that $rs \in (\mathscr{R}^{clo})^{max} \setminus \mathscr{R}^{max}$. Then there is $rs' \in \mathscr{R}^{max} \subseteq \mathscr{R}^{clo}$ such that $rs \lhd rs'$. Thus, $rs' \in \max_{\mathscr{R}^{clo}}(rs)$, a contradiction.

We next observe that, by $(\mathscr{R}^{clo})^{max} = \mathscr{R}^{max}$, $rs \in \mathscr{R}^{clo}$ implies $\max_{\mathscr{R}^{clo}}(rs) = \max_{\mathscr{R}}(rs)$. This, in turn, means that $\bigcap \max_{\mathscr{R}^{clo}}(rs) = \bigcap \max_{\mathscr{R}}(rs) = rs$, and so $\mathscr{R}^{clo}$ is intersection-closed.

We then observe that, for every $rs \in \mathscr{R}$, $rs \in (\mathscr{R}^{clo})^{clo}$ iff $rs \in \mathscr{R}^{clo} \wedge rs = \bigcap \max_{\mathscr{R}^{clo}}(rs)$ iff $rs \in \mathscr{R}^{clo} \wedge rs = \bigcap \max_{\mathscr{R}}(rs)$ iff $rs \in \mathscr{R}^{clo}$. Hence $(\mathscr{R}^{clo})^{clo} = \mathscr{R}^{clo}$.

We note that $\mathscr{R}^{max}$ is nonempty by Proposition 3.2(1) and $\mathscr{R} \neq \varnothing$. This and $\mathscr{R}^{max} \subseteq \mathscr{R}^{clo}$ implies that $\mathscr{R}^{clo}$ is nonempty.

Finally, we observe that $\mathscr{R}^{clo} = \{rs \mid rs \in \mathscr{R}^{clo}\} = \{\bigcap \max_{\mathscr{R}}(rs) \mid rs \in \mathscr{R}^{clo}\} \subseteq \{\bigcap \max_{\mathscr{R}}(rs) \mid rs \in \mathscr{R}\}$. To show the reverse inclusion, suppose that $rs \in \mathscr{R}$ and $rs' = \bigcap \max_{\mathscr{R}}(rs)$. Then, by Proposition 3.2(4), $\bigcap \max_{\mathscr{R}}(rs') = \bigcap \max_{\mathscr{R}}(rs) = rs'$, and so $rs' \in \mathscr{R}^{clo}$. □

5. Closing relational structures

The discussion in the previous two sections, in particular Proposition 4.6, suggests a way of 'closing' the relational structures belonging to $\mathscr{R}$ by using the *structure closure* function $\mathrm{clo}_{\mathscr{R}} : \mathscr{R} \to \mathscr{R}^{clo}$ defined by $\mathrm{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs)$ for all $rs \in \mathscr{R}$. As an example, recall that $\mathsf{AO}^{clo} = \mathsf{PO}$ and, indeed, all partial orders can be obtained by intersecting the linearisations of an acyclic order.

Among the properties listed in the following proposition, we find that closing a closed structure has no effect and that the maximal extensions of a relational structure are precisely the maximal extensions of the closure of that structure. The latter property is particularly interesting as it shows that $\mathrm{clo}_{\mathscr{R}}(rs)$ is the maximal relational structure among all those which have the same maximal extensions as rs. One may interpret this as saying that $\mathrm{clo}_{\mathscr{R}}(rs)$ contains all the explicit and *implicit* dependencies between events in the system executions consistent with rs.

Proposition 5.1. *Let $\mathscr{R}$ be intersection-closed, $rs, rs' \in \mathscr{R}$ and $rs'' \in \mathscr{R}^{clo}$.*

(1) $rs \unlhd \mathrm{clo}_{\mathscr{R}}(rs)$.
(2) $rs \unlhd rs''$ *implies* $\mathrm{clo}_{\mathscr{R}}(rs) \unlhd rs''$.
(3) $\mathrm{clo}_{\mathscr{R}}(rs'') = rs''$.
(4) $\mathrm{ext}_{\mathscr{R}^{clo}}(rs) = \mathrm{ext}_{\mathscr{R}^{clo}}(\mathrm{clo}_{\mathscr{R}}(rs))$.
(5) $\max_{\mathscr{R}}(rs) = \max_{\mathscr{R}} \circ \mathrm{clo}_{\mathscr{R}}(rs)$.
(6) $rs \unlhd rs'$ *implies* $\mathrm{clo}_{\mathscr{R}}(rs) \unlhd \mathrm{clo}_{\mathscr{R}}(rs')$.
(7) $rs'' \lhd rs$ *implies* $\max_{\mathscr{R}}(rs) \neq \max_{\mathscr{R}}(rs'')$.
(8) $\max_{\mathscr{R}}(rs) = \max_{\mathscr{R}}(rs')$ *implies* $\mathrm{clo}_{\mathscr{R}}(rs) = \mathrm{clo}_{\mathscr{R}}(rs')$ *and* $rs' \unlhd \mathrm{clo}_{\mathscr{R}}(rs)$.

Proof. *(1)* Follows from Proposition 3.2(3).

(2) By Proposition 3.2(2), we have $\max_{\mathscr{R}}(rs'') \subseteq \max_{\mathscr{R}}(rs)$. Hence, since $rs'' \in \mathscr{R}^{clo}$, we obtain $\mathrm{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs) \trianglelefteq \bigcap \max_{\mathscr{R}}(rs'') = rs''$.

(3) By part *(1)*, $rs'' \trianglelefteq \mathrm{clo}_{\mathscr{R}}(rs'')$. Moreover, by $rs'' \trianglelefteq rs''$ and part *(2)*, we obtain $\mathrm{clo}_{\mathscr{R}}(rs'') \trianglelefteq rs''$. Hence $\mathrm{clo}_{\mathscr{R}}(rs'') = rs''$.

(4) Let $rs''' \in \mathscr{R}^{clo}$. We then observe that $rs \trianglelefteq rs'''$ iff $\mathrm{clo}_{\mathscr{R}}(rs) \trianglelefteq rs'''$. Indeed, the right-to-left implication follows from part *(1)*, and the left-to-right implication from part *(2)*.

(5) Follows from Proposition 3.2(4).

(6) By $rs \trianglelefteq rs'$ and part *(1)*, $rs \trianglelefteq \mathrm{clo}_{\mathscr{R}}(rs')$. Hence, by part *(2)*, $\mathrm{clo}_{\mathscr{R}}(rs) \trianglelefteq \mathrm{clo}_{\mathscr{R}}(rs')$.

(7) Suppose that $\max_{\mathscr{R}}(rs) = \max_{\mathscr{R}}(rs'')$. Then, by part *(5)*, we obtain that the following holds: $\mathrm{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}} \circ \mathrm{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs'') = rs''$. On the other hand, by part *(1)*, $rs \trianglelefteq \mathrm{clo}_{\mathscr{R}}(rs) = rs'' \triangleleft rs$. Hence $rs \triangleleft rs$, yielding a contradiction.

(8) We have $\mathrm{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs') = \mathrm{clo}_{\mathscr{R}}(rs')$. Hence, by part *(1)*, $rs' \trianglelefteq \mathrm{clo}_{\mathscr{R}}(rs') = \mathrm{clo}_{\mathscr{R}}(rs)$. □

Example 5.2. The structure closure for the acyclic orders in AO is the transitive closure. For example, Proposition 5.1(5) means that the total order extensions of an acyclic order are the same as the total order extensions of its transitive closure.

The structure closure for the distributed acyclic orders is the identity function.

$\mathrm{clo}_{\mathsf{ONE}}(rs) = \langle \Delta_{rs}, \{\langle x, y\rangle \mid x \neq y \in \Delta_{rs}\}, \ell_{rs}\rangle$, for every $rs \in \mathsf{ONE}$ (see Example 3.3).

The next result employs a pair of functions, f and g, as in Theorem 3.4. It can be used to demonstrate a one-to-one correspondence between the closed structures in different sets of relational structures, and also shows that the closure of a relational structure can be derived from a corresponding relational structure in another class of relational structures.

Theorem 5.3. *Let $\mathscr{R}$ and $\mathscr{S}$ be intersection-closed and $\mathscr{R} \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{R}$ be monotonic functions such that $g \circ f$ and $f \circ g$ are non-decreasing functions.*

(1) $\mathscr{R}^{clo} \xrightarrow{f} \mathscr{S}^{clo} \xrightarrow{g} \mathscr{R}^{clo}$ *are inverse bijections.*
(2) $\mathrm{clo}_{\mathscr{R}} = g \circ \mathrm{clo}_{\mathscr{S}} \circ f$ *and* $\mathrm{clo}_{\mathscr{S}} = f \circ \mathrm{clo}_{\mathscr{R}} \circ g$.

Proof. *(1)* Let $rs \in \mathscr{R}^{clo}$. By Theorem 3.4(3,2), we have $\max_{\mathscr{R}}(rs) = g \circ \max_{\mathscr{S}} \circ f(rs) = \max_{\mathscr{R}} \circ g \circ f(rs)$. Hence $g \circ f(rs) \trianglelefteq \bigcap \max_{\mathscr{R}} \circ g \circ f(rs) = \bigcap \max_{\mathscr{R}}(rs) = rs$, by Proposition 3.2(3) and $rs \in \mathscr{R}^{clo}$. Moreover, $rs \trianglelefteq g \circ f(rs)$, as $g \circ f$ is non-decreasing. Hence $g \circ f(rs) = rs$ *(*)*.

By a symmetric argument, for every $rs' \in \mathscr{S}^{clo}$, $f \circ g(rs') = rs'$ *(**)*.

We next show that $f(rs) \in \mathscr{S}^{clo}$. Let $\overline{rs} = \bigcap \max_{\mathscr{S}} \circ f(rs)$. By Propositions 3.2(3) and 4.6, we have $f(rs) \trianglelefteq \overline{rs}$ and $\overline{rs} \in \mathscr{S}^{clo}$. Hence, by the monotonicity of g and *(*)*, $rs = g \circ f(rs) \trianglelefteq g(\overline{rs})$. Moreover, by Theorem 3.4(2,3) and Proposition 3.2(4), we have $\max_{\mathscr{R}}(rs) = g \circ \max_{\mathscr{S}} \circ f(rs) = g \circ \max_{\mathscr{S}}(\bigcap \max_{\mathscr{S}} \circ f(rs)) = g \circ \max_{\mathscr{S}}(\overline{rs}) = \max_{\mathscr{R}} \circ g(\overline{rs})$. Hence, by Proposition 3.2(3) and $rs \in \mathscr{R}^{clo}$, $g(\overline{rs}) \trianglelefteq \bigcap \max_{\mathscr{R}} \circ g(\overline{rs}) = \bigcap \max_{\mathscr{R}}(rs) = rs$. As a result, we obtained $g(\overline{rs}) = rs$. Thus, by *(**)*, $f(rs) = f \circ g(\overline{rs}) = \overline{rs} \in \mathscr{S}^{clo}$.

We have shown $rs = g \circ f(rs)$ and $f(rs) \in \mathscr{S}^{clo}$, for every $rs \in \mathscr{R}^{clo}$. By a symmetric argument, $rs' = f \circ g(rs')$ and $g(rs') \in \mathscr{R}^{clo}$, for every $rs' \in \mathscr{S}^{clo}$. Hence the result holds.

(2) We only prove the first equality as the second one follows by symmetry.

Let $rs \in \mathscr{R}$ and $rs' = g \circ \mathrm{clo}_{\mathscr{S}} \circ f(rs)$. By Proposition 4.6 and part *(1)*, we obtain $rs' \in \mathscr{R}^{clo}$. Moreover, by Theorem 3.4(2) and Proposition 5.1(5), and Theorem 3.4(3), we have $\max_{\mathscr{R}}(rs') = \max_{\mathscr{R}} \circ g \circ \mathrm{clo}_{\mathscr{S}} \circ f(rs) = g \circ \max_{\mathscr{S}} \circ \mathrm{clo}_{\mathscr{S}} \circ f(rs) = g \circ \max_{\mathscr{S}} \circ f(rs) = \max_{\mathscr{R}}(rs)$. Hence, $\mathrm{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs') = rs' = g \circ \mathrm{clo}_{\mathscr{S}} \circ f(rs)$. □

As with maximal extensions, closure is defined on basis of structure rather than labelling. And so, similar to Theorem 3.5, the class of closed relational structures is renaming-closed, if $\mathscr{R}$ is renaming-closed. Moreover, and also similar to Theorem 3.5, computing the closure of the renamings of $rs \in \mathscr{R}$, can be done by computing the closure of a single representative of the isomorphism class and then applying the renaming.

Theorem 5.4. *If $\mathscr{R}$ is intersection-closed and renaming-closed, then $\mathscr{R}^{clo}$ is renaming-closed. Moreover, if ψ is a renaming of $rs \in \mathscr{R}$, then $\mathrm{clo}_{\mathscr{R}} \circ \psi(rs) = \psi \circ \mathrm{clo}_{\mathscr{R}}(rs)$.*

Proof. Suppose that $rs \in \mathscr{R}$. Then, by Theorem 3.5, the second part holds as we have $\mathrm{clo}_{\mathscr{R}} \circ \psi(rs) = \bigcap \max_{\mathscr{R}} \circ \psi(rs) = \bigcap \psi \circ \max_{\mathscr{R}}(rs) = \psi(\bigcap \max_{\mathscr{R}}(rs)) = \psi \circ \mathrm{clo}_{\mathscr{R}}(rs)$.

To show the first part, suppose that $rs \in \mathscr{R}^{clo}$. Then $rs = \bigcap \max_{\mathscr{R}}(rs)$ and $\psi(rs) \in \mathscr{R}$ (as $\mathscr{R}$ is renaming-closed). Hence, by the second part, $\mathrm{clo}_{\mathscr{R}} \circ \psi(rs) = \psi(\bigcap \max_{\mathscr{R}}(rs)) = \psi(rs)$. Thus $\psi(rs) \in \mathscr{R}^{clo}$, and so $\mathscr{R}^{clo}$ is renaming-closed. □

Intuitively, the next two results demonstrate that structure closure and closed structure rely on maximal relational structures. In what follows, the set $\mathscr{R}$ is *upward-closed in* $\mathscr{S}$ if $\mathscr{R} \subseteq \mathscr{S}$ and $rs \trianglelefteq rs'$ implies $rs' \in \mathscr{R}$, for all $rs \in \mathscr{R}$ and $rs' \in \mathscr{S}$.

Proposition 5.5. *Let $\mathscr{R}$ and $\mathscr{S}$ be intersection-closed such that $\mathscr{R} \subseteq \mathscr{S}$ and one of the following holds: (i) $\max_{\mathscr{R}}(rs) = \max_{\mathscr{S}}(rs)$, for every $rs \in \mathscr{R}$; or (ii) $\mathscr{R}^{max} \subseteq \mathscr{S}^{max}$. Then $\mathscr{R}^{clo} = \mathscr{S}^{clo} \cap \mathscr{R}$ and $\text{clo}_{\mathscr{R}}(rs) = \text{clo}_{\mathscr{S}}(rs)$, for every $rs \in \mathscr{R}$.*

Proof. Suppose that *(i)* holds. We first observe that $\text{clo}_{\mathscr{R}}(rs) = \text{clo}_{\mathscr{S}}(rs)$, for every $rs \in \mathscr{R}$. Indeed, we have $\text{clo}_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{S}}(rs) = \text{clo}_{\mathscr{S}}(rs)$. We then observe that $rs \in \mathscr{R}^{clo} \iff rs \in \mathscr{R} \wedge rs = \bigcap \max_{\mathscr{R}}(rs) \iff rs \in \mathscr{R} \wedge rs = \bigcap \max_{\mathscr{S}}(rs) \iff rs \in \mathscr{R} \wedge rs \in \mathscr{S}^{clo}$, and so $\mathscr{R}^{clo} = \mathscr{S}^{clo} \cap \mathscr{R}$. We finally observe that *(ii)* implies *(i)*. □

Proposition 5.6. *Let $\mathscr{S}$ be intersection-closed and $\mathscr{R} \subseteq \mathscr{S}$ be upward-closed in $\mathscr{S}$.*

(1) $\mathscr{R}$ is intersection-closed.
(2) $\mathscr{R}^{max} = \mathscr{S}^{max} \cap \mathscr{R}$ and $\max_{\mathscr{R}}(rs) = \max_{\mathscr{S}}(rs)$, for every $rs \in \mathscr{R}$.
(3) $\mathscr{R}^{clo} = \mathscr{S}^{clo} \cap \mathscr{R}$ and $\text{clo}_{\mathscr{R}}(rs) = \text{clo}_{\mathscr{S}}(rs)$, for every $rs \in \mathscr{R}$.

Proof. *(2)* Follows from $\mathscr{R}$ being upward-closed in $\mathscr{S}$.

(1) Let $rs \in \mathscr{R}$ and $rs' = \bigcap \max_{\mathscr{R}}(rs)$. Then, by the already shown part *(2)* and $\mathscr{S}$ being intersection-closed, $rs' = \bigcap \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{S}}(rs) \in \mathscr{S}$. Hence, since $rs \trianglelefteq rs'$ by Proposition 3.2(3), we have $rs' \in \mathscr{R}$.

(3) By parts *(1)* and *(2)*, we can apply Proposition 5.5. □

As in the case of closed relational structures, the intersection-based definition of structure closure is both inconvenient for algorithmic purposes and hardly useful from the point of view of gaining insight into causal dependencies between events involved in concurrent histories. The next result addresses both points by providing sufficient conditions for showing that a given function and a given set of relational structures are, respectively, the structure closure and the set of closed relational structures. Recall that Q^i_{rs} denotes the i-th relation in the relational structure rs.

Proposition 5.7. *Let $\mathscr{S} \subseteq \mathscr{R}$ and let $f : \mathscr{R} \to \mathscr{S}$ be a monotonic and non-decreasing function. Moreover, (i) for all $rs \in \mathscr{S}$, $f(rs) \trianglelefteq rs$; and (ii) for all $rs \in \mathscr{S}$ and all $x \neq y \in \Delta_{rs}$: if $\langle x, y\rangle \notin Q^i_{rs}$, then there is $rs' \in \text{ext}_{\mathscr{R}}(rs)$ satisfying $\langle x, y\rangle \notin Q^i_{\bigcap \max_{\mathscr{R}}(rs')}$. Then f is the structure closure of $\mathscr{R}$, i.e., $\mathscr{S} = \mathscr{R}^{clo}$ and $f(rs) = \text{clo}_{\mathscr{R}}(rs)$, for every $rs \in \mathscr{R}$.*

Proof. We first observe that, for every $rs \in \mathscr{R}^{max} \cup \mathscr{S}$, $f(rs) = rs$ *(*)*. Indeed, if $rs \in \mathscr{R}^{max}$ then, by f being non-decreasing, we have $rs \trianglelefteq f(rs)$, and so $rs = f(rs)$. If $rs \in \mathscr{S}$ then, by *(i)* and f being non-decreasing, we have $rs \trianglelefteq f(rs) \trianglelefteq rs$, and so $rs = f(rs)$.

Let $rs \in \mathscr{R}$. Since f is non-decreasing, $rs \trianglelefteq f(rs)$, and so the domain of rs and $f(rs)$ is the same, and the arity n of rs and $f(rs)$ is the same. Below we show that $f(rs) = \text{clo}_{\mathscr{R}}(rs)$.

By the monotonicity of f, $f(rs) \trianglelefteq f(rs')$, for every $rs' \in \max_{\mathscr{R}}(rs)$. Hence, by *(*)*, $f(rs) \trianglelefteq \bigcap f \circ \max_{\mathscr{R}}(rs) = \bigcap \max_{\mathscr{R}}(rs) = \text{clo}_{\mathscr{R}}(rs)$, and so $f(rs) \trianglelefteq \text{clo}_{\mathscr{R}}(rs)$.

To show $\text{clo}_{\mathscr{R}}(rs) \trianglelefteq f(rs)$, suppose that $x \neq y \in \Delta_{f(rs)}$ and $1 \leq i \leq n$ satisfy $\langle x, y\rangle \notin Q^i_{f(rs)}$. (If such $x \neq y$ and i do not exist, $\text{clo}_{\mathscr{R}}(rs) \trianglelefteq f(rs)$ holds trivially.) Then, by $f(rs) \in \mathscr{S}$ and *(ii)*, there is $rs' \in \text{ext}_{\mathscr{R}}(f(rs))$ satisfying $\langle x, y\rangle \notin Q^i_{\bigcap \max_{\mathscr{R}}(rs')}$. By $f(rs) \trianglelefteq rs'$, we have $\langle x, y\rangle \notin Q^i_{\bigcap \max_{\mathscr{R}} \circ f(rs)}$. Therefore, from f being non-decreasing, it follows that $\langle x, y\rangle \notin Q^i_{\bigcap \max_{\mathscr{R}}(rs)} = Q^i_{\text{clo}_{\mathscr{R}}(rs)}$. Consequently, $\text{clo}_{\mathscr{R}}(rs) \trianglelefteq f(rs)$, and so $f(rs) = \text{clo}_{\mathscr{R}}(rs)$.

We demonstrated that $f(rs) = \text{clo}_{\mathscr{R}}(rs)$, for every $rs \in \mathscr{R}$. All we need to show now is that $\mathscr{S} = \mathscr{R}^{clo}$. By Proposition 4.6, $\mathscr{R}^{clo} = \text{clo}_{\mathscr{R}}(\mathscr{R})$. This means that $\mathscr{R}^{clo} = f(\mathscr{R}) \subseteq \mathscr{S}$. Moreover, by *(*)* and $\mathscr{S} \subseteq \mathscr{R}$, $\mathscr{S} = f(\mathscr{S}) \subseteq f(\mathscr{R}) = \mathscr{R}^{clo}$. Hence $\mathscr{S} = \mathscr{R}^{clo}$. □

The assumptions of Proposition 5.7 are satisfied in the case of transitive closure of acyclic orders. In particular, if po is a partial order and $x \neq y$ are two domain elements such that $x \not\prec_{po} y$, then either $y \prec_{po} x$ and in *(ii)* we take $rs' = rs$, or $y \not\prec_{po} x$ and po extended with the relationship $\langle y, x\rangle$ can be taken as the acyclic order rs'. In any case, no total order extending such an rs' contains the relationship $\langle x, y\rangle$ as this would violate its acyclicity.

6. Relational spaces

When dealing with sequences or step sequences only, one does not need any explicit enumeration of the actions, i.e., the set $\mathbb{E}$. When a sequence of actions *ababac* represents some execution, it implicitly indicates that a occurred three times (as the first, the third and the fifth event), b occurred twice (as the second and the fourth event), and c occurred only once (as the sixth event). However, when we want to represent this sequence as a total order, we need to make all action occurrences explicitly different, and the most natural (or canonical) total order representation seems to be $a^{(1)} \prec b^{(1)} \prec a^{(2)} \prec b^{(2)} \prec a^{(3)} \prec c^{(1)}$, where the elements of $\mathbb{E}$ are used to denote events (i.e., action occurrences). In general, one aims at representing concurrent behaviours using relational structures which are more expressive than total orders, and our next definition introduces a necessary condition for such a representation to be sound. In the approach adopted in this paper

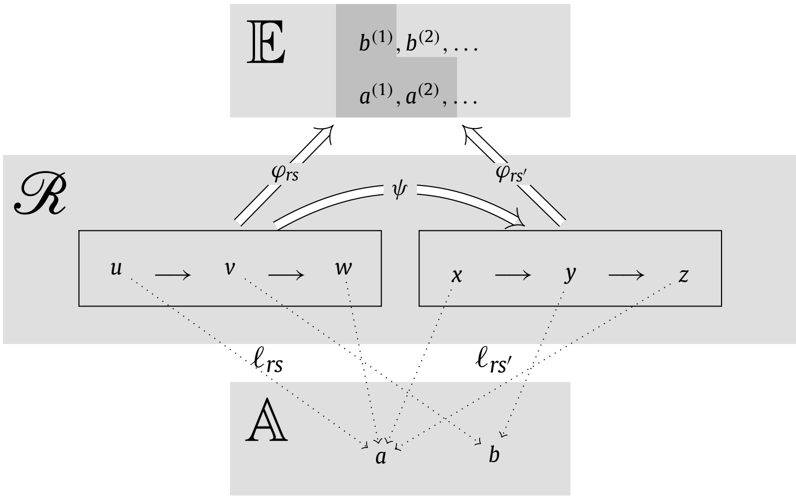

Fig. 1. Relations between the relational space $\mathscr{R}$, and the sets of events $\mathbb{E}$ and actions $\mathbb{A}$. The factors are defined by: $\varphi_{rs}(u) = \varphi_{rs'}(x) = a^{(1)}$, $\varphi_{rs}(v) = \varphi_{rs'}(y) = b^{(1)}$, and $\varphi_{rs}(w) = \varphi_{rs'}(z) = a^{(2)}$.

it is implicitly assumed that in any concurrent run occurrences of the same action are always ordered (and so cannot be simultaneous), excluding what is usually referred to as *auto-concurrency*.

A *relational space* is a pair $\langle \mathscr{R}, \Phi \rangle$ comprising a nonempty renaming-closed set $\mathscr{R}$ of relational structures, and a set of bijections $\Phi = \{\varphi_{rs} : \Delta_{rs} \to \mathbb{E}_{rs} \mid rs \in \mathscr{R}\}$ called *factors* and indexed by the relational structures in $\mathscr{R}$ such that, for all $rs, rs' \in \mathscr{R}$ and $x \in \Delta_{rs}$ with $\varphi_{rs}(x) = a^{(i)}$:

$$\ell_{rs}(x) = a \quad \text{and} \quad (rs \sim_{\psi} rs' \implies \psi = \varphi_{rs'}^{-1} \circ \varphi_{rs}) \ . \tag{1}$$

Intuitively, and as illustrated in Fig. 1, $\mathscr{R}$ is a set of relational structures modelling behaviours of concurrent systems, with each relational structure $rs \in \mathscr{R}$ specifying causal or temporal relationships between the observed events (recorded as domain elements Δ_{rs}). It is intended that, for any $x \in \Delta_{rs}$, one can state — by only looking at the relationships between domain elements — whether it is the record of the first, or the second, or the third, etc, occurrence of the corresponding action, recorded as $\ell_{rs}(x)$. Such information is conveyed by the function φ_{rs}, e.g., $\varphi_{rs}(x) = a^{(12)}$ indicates that x was the twelfth occurrence of action a. Given all that, the first part of Eq. (1) simply requires consistency between the two ways of identifying the action corresponding to an event: one using the labelling ℓ_{rs}, and the other using the function φ_{rs}. Furthermore, as it is desirable (and indeed expected in a definition based on structural properties) to equate isomorphic relational structures, it is natural to require that action occurrences associated with two corresponding recorded events in isomorphic relational structures are identical. That is, if $rs' = \psi(rs)$, for some renaming ψ, then one would expect that $\varphi_{rs'}(y) = \varphi_{rs}(x)$ whenever $y = \psi(x)$. In other words, that $\varphi_{rs'} \circ \psi = \varphi_{rs}$ should hold. Hence, as all three functions are bijections, this implies that ψ can be factored into $\varphi_{rs'}^{-1}$ and φ_{rs}, $\psi = \varphi_{rs'}^{-1} \circ \varphi_{rs}$. We have therefore provided a 'proof' justifying the inclusion of the second part of Eq. (1) in the definition of a relational space, and also calling each $\varphi_{rs} \in \Phi$ the *factor* of $rs \in \mathscr{R}$. The choice of factors for a set of relational structures is not necessarily unique. However, in the areas of application with which this paper is concerned, it will be determined by a well-motivated notion of event precedence (that is, if $\varphi_{rs}(x) = a^{(i)}$ and $\varphi_{rs}(y) = a^{(j)}$, then $i < j$ whenever x 'precedes' y in rs).

As the next example shows, not every relational structure can belong to a relational space, and the factors of relational structures need to be chosen with care.

Example 6.1. Note: Fig. 2 depicts the relational structures considered below.

(1) $rs_0 = \langle \{x, y\}, \varnothing, \{x \mapsto a, y \mapsto a\} \rangle$ cannot belong to any relational space. Indeed, suppose that this is not the case and, without loss of generality, $\varphi_{rs_0} = \{x \mapsto a^{(1)}, y \mapsto a^{(2)}\}$. Then $rs_0 \sim_{\psi_0} rs_0$, where $\psi_0 = \{x \mapsto y, y \mapsto x\}$, but the second part of Eq. (1) is not satisfied as we have $\varphi_{rs_0}^{-1} \circ \varphi_{rs_0} = \{x \mapsto x, y \mapsto y\} \neq \psi_0$.

(2) $\varphi_{rs_1} = \{x \mapsto a^{(1)}, y \mapsto a^{(2)}\}$ and $\varphi_{rs_2} = \{z \mapsto a^{(1)}, w \mapsto a^{(2)}\}$ are not suitable factors for the relational structures $rs_1 = \langle \{x, y\}, \{\langle x, y \rangle\}, \{x \mapsto a, y \mapsto a\} \rangle$ where, intuitively, x precedes y, and $rs_2 = \langle \{w, z\}, \{\langle w, z \rangle\}, \{w \mapsto a, z \mapsto a\} \rangle$ with w preceding z. Indeed, $rs_1 \sim_{\psi_1} rs_2$, where $\psi_1 = \{x \mapsto w, y \mapsto z\}$, but the second part of Eq. (1) is not satisfied as we have $\varphi_{rs_2}^{-1} \circ \varphi_{rs_1} = \{x \mapsto z, y \mapsto w\} \neq \psi_1$. However, $\varphi_{rs_1} = \{x \mapsto a^{(1)}, y \mapsto a^{(2)}\}$ and $\varphi_{rs_2} = \{w \mapsto a^{(1)}, z \mapsto a^{(2)}\}$ are suitable factors as they are both consistent with the structure of rs_1 and rs_2.

(3) The choice of factors for relational structures in $\mathscr{R} = \{rs_{vu} \mid v \neq u \in \mathbb{U}\}$, where, for all $v \neq u \in \mathbb{U}$, $rs_{vu} = \langle \{v, u\}, \{\langle v, u \rangle\}, \{v \mapsto a, u \mapsto a\} \rangle$, can be made in exactly two ways:

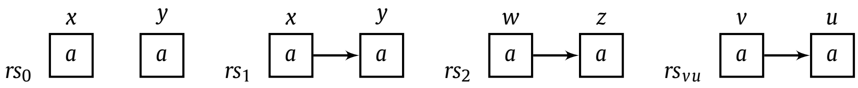

Fig. 2. Relational structures in Example 6.1.

$$\Phi \;=\; \{\varphi_{rs_{vu}} = \{v \mapsto a^{(1)}, u \mapsto a^{(2)}\} \mid v \neq u \in \mathbb{U}\}$$
$$\Phi' \;=\; \{\varphi_{rs_{vu}} = \{v \mapsto a^{(2)}, u \mapsto a^{(1)}\} \mid v \neq u \in \mathbb{U}\}\,.$$

However, if one interprets the relationship $\langle v, u\rangle$ as stating that v precedes u in the observation recorded by rs_{vu}, then only the first option would be semantically justifiable.

We then consider the running examples.

Example 6.2. Example 6.1(1) means that AO cannot possibly be turned into a relational space. However, its subset AO′ comprising all acyclic orders ao such that $x \prec_{ao}^{+} y$, for all domain elements x and y with the same label, can be equipped with suitable factors. This can be seen by defining factor φ_{ao} as follows. For each $a \in \mathbb{A}_{ao}$, we can line up in a unique way all the elements labelled by a in a chain $x_1 \prec_{ao}^{+} x_2 \prec_{ao}^{+} \cdots \prec_{ao}^{+} x_m$, and then define $\varphi_{ao}(x_i) = a^{(i)}$, for every $1 \le i \le m$. Interestingly, one could also define factors by setting $\varphi_{ao}(x_i) = a^{(m+1-i)}$, for every $1 \le i \le m$. However, the latter version would not be suitable for concurrent system applications.

Like AO, neither DAO nor ONE can be turned into a relational space. However, all distributed acyclic orders dao such that $x(\rightharpoondown_{dao} \cup \rightharpoonup_{dao})^{+} y$, for all domain elements x and y with the same label, can be equipped with suitable factors. And, similarly, $\mathsf{AO}' \subseteq \mathsf{ONE}$.

It is important to point out that each isomorphism class RS of relational structures belonging to a relational space has a unique (canonical) representative can_{RS} whose factor is an identity function (clearly, $\Delta_{can_{RS}} = \mathbb{E}_{rs}$ and $can_{RS} = \varphi_{rs}(rs)$, for every $rs \in RS$). This allows, for instance, to establish a one-to-one correspondence between sequences of actions and canonical total orders. A detailed discussion of such a correspondence and its generalisations is outside the scope of this paper.

We end this section with an example showing that relational spaces can be formed on the basis of structural properties other than precedence.

Example 6.3. Consider again the first running example AO and, for all $ao \in \mathsf{AO}$ and $x \in \Delta_{rs}$, denote by $out_{rs}(x)$ the outdegree of x in rs. Let AO″ be the set of all acyclic orders such that, for all $a \in \mathbb{A}_{rs}$, $out_{rs}(\Delta_{rs}^{\lfloor a\rfloor}) = \{1, \dots, |\Delta_{rs}^{\lfloor a\rfloor}|\}$. We can then define $\varphi_{rs}(x) = a^{(i)}$, where $a = \ell_{rs}(x)$ and $i = out_{rs}(x)$. In other words, the structural property of the outdegree, rather than the structural property of the number of predecessors labelled by the same action, can be used to 'calculate' the sequence numbers.

7. Label-linear relational structures

In this section, we introduce a property of relational structures reflecting an ordering of domain elements with the same label. We demonstrate how with relational structures that satisfy this property, a factor can be associated, which thus leads to a relational space. In the case of acyclic orders, the notion which allows us to attribute a strict execution order to all instances of a given action is 'label-linearity' defined directly on the basis of the transitive closure of acyclic orders. In the general case, however, such a direct definition is difficult to formalise as it is not clear what might be a satisfactory notion of transitive closure. We will therefore introduce label-linearity indirectly, using the maximal extensions of relational structures to 'discover' orderings of domain elements labelled with the same label. The rationale behind such an approach comes from the view that any ordering of this kind should be preserved by extensions, i.e., if $rs \trianglelefteq rs'$ and x occurred before y in rs, then x should also occur before y in rs', as all information about relative ordering between x and y present in rs is also present in rs'. Therefore, all the maximal extensions of a label-linear relational structure rs should share with rs the orderings of domain elements labelled with the same label.

Recall that a relational structure is a tuple $\langle \Delta, Q^1, \dots, Q^n, \ell\rangle$, where $n \ge 1$ is the arity of the relational structure, i.e., the number of binary relations associated with rs. Although in the examples in this paper, the arity of the reaction systems involved is not more than two, our results are valid for reaction systems which specify any number of relevant relations between pairs of actions. At this point, we assume without loss of generality (see also below) that the last relation of a relational structure can be used to identify the ordering of events.

In what follows, Q_{rs}^{last} denotes *the last* component relation Q_{rs}^{n} of a relational structure rs of arity n, and trunc_{rs} denotes the relational structure $\langle \Delta_{rs}, Q_{rs}^{last}, \ell_{rs}\rangle$ obtained from rs by *truncating* all but the last component relation. Then the *fingerprint* of rs is the set of relational structures $\mathrm{fngpr}_{rs} = \{\mathrm{trunc}_{rs}^{\lfloor a\rfloor} \mid a \in \mathbb{A}\}$, comprising projections of the last component relation of rs onto sets of domain elements labelled by the same action.

Definition 7.1 *(label-linear relational structure).* A relational structure $rs \in \mathscr{R}$ is *label-linear in* $\mathscr{R}$ if there is a set of total orders $\mathrm{totalfngpr}_{rs}$ such that $\mathrm{totalfngpr}_{rs} = \mathrm{fngpr}_{\overline{rs}}$, for every $\overline{rs} \in \max_{\mathscr{R}}(rs)$. We denote this by $rs \in \mathscr{R}^{lin}$.

The definition is sound as, by Proposition 3.2(1), $\max_{\mathscr{R}}(rs)$ is nonempty. Note that $\mathscr{R}^{lin}$ includes all relational structures in $\mathscr{R}$ with injective labellings.

For every action a of a label-linear relational structure rs, total^a_{rs} denotes the unique total order in $\mathrm{totalfngpr}_{rs}$ with the domain $\Delta^{\lfloor a \rfloor}_{rs}$. Note that total^a_{rs} is nonempty iff $a \in \mathbb{A}_{rs}$. Intuitively, total^a_{rs} represents a linear ordering according to which all occurrences of action a were executed in the concurrent behaviour represented by rs. In other words, total^a_{rs} describes the behaviour represented by rs restricted to the action a and it can be obtained from *any* maximal extension of rs by restricting it to occurrences of a only. Label-linearity is therefore very much linked to the idea of representing sets of 'equivalent' executions by a single relational structure, and so inevitably less general than other notions concerning relational structures discussed thus far.

Example 7.2. AO^{lin} is the set of all acyclic orders ao such that, for all $x \neq y \in \Delta_{ao}$ labelled by the same action, either $x \prec^+_{ao} y$ or $y \prec^+_{ao} x$.

DAO^{lin} is the set of all distributed acyclic orders dao such that, for all $x \neq y \in \Delta_{dao}$ labelled by the same action, either $x \rightharpoonup^+_{dao} y$ or $y \rightharpoonup^+_{dao} x$.

ONE^{lin} is the set of all relational structures in ONE with injective labellings.

Choosing Q^{last}_{rs} as the component relation from which the ordering of occurrences of the same action is derived, does not in practical terms limit the applicability of label-linearity. Suppose, e.g., that the information about such an ordering can only be derived from all the relations Q^i_{rs} through a binary relation $f(Q^1_{rs}, \dots, Q^n_{rs})$ Then, one can extend rs by another relation $Q^{n+1}_{rs} = f(Q^1_{rs}, \dots, Q^n_{rs})$, and use the above notion of label-linearity.

Example 7.3. According to Example 7.2, $\langle \{x, y\}, \{\langle x, y\rangle\}, \varnothing, \{x \mapsto a, y \mapsto a\}\rangle \in \mathsf{DAO}$ is not a label-linear distributed acyclic order even though it corresponds to the label-linear total order $\langle \{x, y\}, \{\langle x, y\rangle\}, \{x \mapsto a, y \mapsto a\}\rangle$. The reason is that only the second ordering relation is taken into account in Definition 7.1. However, the introduction of a third component relation defined as the union of the two original orderings solves the problem. Indeed, we then obtain $\langle \{x, y\}, \{\langle x, y\rangle\}, \varnothing, \{\langle x, y\rangle\}, \{x \mapsto a, y \mapsto a\}\rangle$ which is a label-linear relational structure.

As stated in the next result, the total order relations used in the definition of label-linearity are unique, and label-linearity is preserved by extensions.

Proposition 7.4. *Let $\mathscr{R}$ be renaming- and intersection-closed, $rs \in \mathscr{R}^{lin}$, and $rs' \in \mathscr{R}$.*

(1) $\max_{\mathscr{R}}(rs') \subseteq \max_{\mathscr{R}}(rs)$ *implies* $rs' \in \mathscr{R}^{lin}$ *and* $\mathrm{totalfngpr}_{rs'} = \mathrm{totalfngpr}_{rs}$.
(2) $\mathrm{fngpr}_{rs} \subseteq \mathsf{AO}$.
(3) $\mathrm{fngpr}_{rs} \subseteq \mathsf{TO}$ *implies* $\mathrm{fngpr}_{rs} = \mathrm{totalfngpr}_{rs}$.
(4) $\mathscr{R}^{lin}$ *and* $(\mathscr{R}^{lin})^{max}$ *are renaming-closed.*
(5) $\mathscr{R}^{lin}$ *is intersection-closed.*
(6) $(\mathscr{R}^{lin})^{max} = \mathscr{R}^{lin} \cap \mathscr{R}^{max} = (\mathscr{R}^{max})^{lin}$ *and* $\max_{\mathscr{R}^{lin}}(rs) = \max_{\mathscr{R}}(rs)$.

Proof. *(1)* Follows directly from the definitions.

(2) Follows from the fact that only acyclic orders can have total order extensions.

(3) Follows from the fact that total orders do not have proper total order extensions.

(4) By Theorem 3.5, it suffices to show that $\mathscr{R}^{lin}$ is renaming-closed. Let ψ be a renaming of rs. By $\mathscr{R}$ being renaming-closed, we have $\psi(rs) \in \mathscr{R}$. Below, for every $a \in \mathbb{A}$, ψ_a denotes the bijection obtained by restricting ψ to the domain $\Delta^{\lfloor a \rfloor}_{rs}$ and codomain $\Delta^{\lfloor a \rfloor}_{\psi(rs)}$.

To show $\psi(rs) \in \mathscr{R}^{lin}$, it suffices to observe that if $Z = \{\psi_a(\mathrm{total}^a_{rs}) \mid \mathrm{total}^a_{rs} \in \mathrm{totalfngpr}_{rs}\}$ and $\overline{rs} \in \max_{\mathscr{R}} \circ \psi(rs)$, then $Z = \mathrm{fngpr}_{\overline{rs}}$. Indeed, by Theorem 3.5, $\psi^{-1}(\overline{rs}) \in \max_{\mathscr{R}}(rs)$. Hence, by $rs \in \mathscr{R}^{lin}$, we have $\psi_a(\mathrm{total}^a_{rs}) = \psi_a(\mathrm{trunc}^{\lfloor a \rfloor}_{\psi^{-1}(\overline{rs})}) = \psi_a \circ \psi_a^{-1}(\mathrm{trunc}^{\lfloor a \rfloor}_{\overline{rs}}) = \mathrm{trunc}^{\lfloor a \rfloor}_{\overline{rs}}$.

(5,6) By Proposition 3.2(2) and part *(1)*, $\mathscr{R}^{lin}$ is upward-closed in $\mathscr{R}$ which is intersection-closed. Hence, Proposition 5.6(1,2) implies parts *(5)* and *(6)*, except for the equality $\mathscr{R}^{lin} \cap \mathscr{R}^{max} = (\mathscr{R}^{max})^{lin}$. We show the latter as follows. Since $(\mathscr{R}^{max})^{lin} \subseteq \mathscr{R}^{max}$ it suffices to consider $\overline{rs} \in \mathscr{R}^{max}$. Then we observe that from $\max_{\mathscr{R}}(\overline{rs}) = \max_{\mathscr{R}^{max}}(\overline{rs}) = \{rs\}$ it follows that $\overline{rs}$ is label-linear in $\mathscr{R}$ iff $\overline{rs}$ is label-linear in $\mathscr{R}^{max}$. □

The next result gathers together properties involving closed relational structures.

Proposition 7.5. *Let $\mathcal{R}$ be renaming- and intersection-closed, $rs \in \mathcal{R}^{lin}$, and $rs' \in \mathcal{R}$.*

(1) $rs \in \mathcal{R}^{clo}$ implies $\mathrm{totalfngpr}_{rs} = \mathrm{fngpr}_{rs}$.
(2) $rs' \in \mathcal{R}^{lin}$ if and only if $\mathrm{clo}_{\mathcal{R}}(rs') \in \mathcal{R}^{lin}$.
(3) $(\mathcal{R}^{lin})^{clo}$ is renaming-closed.
(4) $(\mathcal{R}^{lin})^{clo} = \mathcal{R}^{lin} \cap \mathcal{R}^{clo} = (\mathcal{R}^{clo})^{lin}$ and $\mathrm{clo}_{\mathcal{R}^{lin}}(rs) = \mathrm{clo}_{\mathcal{R}}(rs)$.

Proof. *(1)* Follows directly from the definitions.

(2) Follows from Proposition 5.1(5).

(3) Follows from Proposition 7.4(4) and Theorem 5.4.

(4) Propositions 3.2(2), 7.4(1), and 5.6(3) imply part *(4)*, except for the equality $\mathcal{R}^{lin} \cap \mathcal{R}^{clo} = (\mathcal{R}^{clo})^{lin}$. We show the latter as follows. Since $(\mathcal{R}^{clo})^{lin} \subseteq \mathcal{R}^{clo}$ it suffices to consider $\overline{rs} \in \mathcal{R}^{clo}$. We observe that $\max_{\mathcal{R}}(\overline{rs}) = \max_{\mathcal{R}^{clo}}(\overline{rs})$. Indeed, by Proposition 4.6, $\max_{\mathcal{R}}(\overline{rs}) \subseteq \max_{\mathcal{R}^{clo}}(\overline{rs})$. Suppose that $rs'' \in \max_{\mathcal{R}^{clo}}(\overline{rs})$. Then there is $rs''' \in \max_{\mathcal{R}^{max}}(rs'') \in \mathcal{R}^{max} \subseteq \mathcal{R}^{clo}$. Hence $rs'' = rs'''$. We therefore have $\max_{\mathcal{R}}(\overline{rs}) = \max_{\mathcal{R}^{clo}}(\overline{rs})$, and so $\overline{rs}$ is label-linear in $\mathcal{R}$ iff $\overline{rs}$ is label-linear in $\mathcal{R}^{clo}$. □

At the beginning of this section we mentioned that it is not clear what might be a general notion of transitivity leading to a satisfactory notion of label-linearity. For example, being inspired by Proposition 7.5(2), one might define a *closure-based label-linearity* of a relational structure $rs \in \mathcal{R}$ by requiring $\mathrm{fngpr}_{\mathrm{clo}_{\mathcal{R}}(rs)} \subseteq \mathsf{TO}$. Such a definition coincides, e.g., with label-linearity for acyclic orders. However, as the next example shows, it would not provide a fully satisfactory notion, as the set of closure-based label-linear relational structures is not always upward closed in $\mathcal{R}$ (see also the discussion at the beginning of this section).

Example 7.6. Let TWO be the set of all relational structures $two \in \mathsf{ONE}$ such that $|Q^{last}_{two}| \le 2$. Then $\mathsf{TWO}^{clo} = \mathsf{TWO}$ and $\mathsf{TWO}^{max} = \{two \in \mathsf{TWO} \mid |Q^{last}_{two}| = 2\}$. Consider a pair of closed relational structures in TWO^{clo}, $two = \langle\{x, y, z\}, \{\langle x, y\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\}\rangle$ and $two' = \langle\{x, y, z\}, \{\langle x, y\rangle, \langle y, x\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\}\rangle$. Clearly, two is closure-based label-linear and $two \lhd two'$, but two' is not closure-based label-linear.

Since each total^a_{rs} is a total order, we can define a bijection $\varphi_{rs} : \Delta_{rs} \to \mathbb{E}_{rs}$ associating the events of a label-linear relational structure $rs \in \mathcal{R}^{lin}$ with the domain elements:

$$\varphi_{rs}(x) = a^{(i)}, \quad \text{where } a = \ell_{rs}(x) \text{ and } i = 1 + |\{y \in \Delta^{\lfloor a \rfloor}_{rs} \mid y \prec_{\mathrm{total}^a_{rs}} x\}| \,. \tag{2}$$

The bijection φ_{rs} will be called a *factor* of rs in view of results proved later on. Moreover, in the rest of this section it will be *fixed*.

It follows from Proposition 7.4(3) that if $\mathrm{fngpr}_{rs} \subseteq \mathsf{TO}$, then Eq. (2) can be simplified, in the following way:

$$\varphi_{rs}(x) = a^{(i)}, \quad \text{where } a = \ell_{rs}(x) \text{ and } i = 1 + |\{y \in \Delta^{\lfloor a \rfloor}_{rs} \mid \langle y, x\rangle \in Q^{last}_{rs}\}| \,. \tag{3}$$

Also, by Proposition 7.5(1), we can derive the same factor of a label-linear relational structure $rs \in \mathcal{R}^{lin}$ using the last component relation in its closure, as follows:

$$\varphi_{rs}(x) = a^{(i)}, \quad \text{where } a = \ell_{rs}(x) \text{ and } i = 1 + |\{y \in \Delta^{\lfloor a \rfloor}_{rs} \mid \langle y, x\rangle \in Q^{last}_{\mathrm{clo}_{\mathcal{R}}(rs)}\}| \,. \tag{4}$$

The function φ_{rs} introduced in Eq. (2) is not the only way to define factors of label-linear relational structures. For example, one could use the formula $i = 1 + |\{y \in \Delta^{\lfloor a \rfloor}_{rs} \mid x \prec_{\mathrm{total}^a_{rs}} y\}|$ (see also Example 6.2). However, the function φ_{rs} in Eq. (2) reflects an intuitive meaning of label-linearity in the modelling of system behaviours, assuming that each total order total^a_{rs} represents the order of executions of all the instances of action a. Such a view is fully consistent with the fact that $x \prec_{\mathrm{total}^a_{rs}} y$ if and only if $i < j$, for all $x, y \in \Delta_{rs}$ with $\varphi_{rs}(x) = a^{(i)}$ and $\varphi_{rs}(y) = a^{(j)}$.

Each isomorphism between two label-linear relational structures can be factored onto the bijections defined in Eq. (2), justifying the usefulness of the notion of label-linearity proposed in this section.

Proposition 7.7. *Let $\mathcal{R}$ be intersection-closed and renaming-closed, and $rs, rs' \in \mathcal{R}^{lin}$ be such that $rs \sim_\psi rs'$. Then $\psi = (\varphi_{rs'})^{-1} \circ \varphi_{rs}$, where the factors are defined as in Eq. (2) or Eq. (4).*

Proof. For every $a \in \mathbb{A}$, let $\Delta = \Delta^{\lfloor a \rfloor}_{rs}$, $\Delta' = \Delta^{\lfloor a \rfloor}_{rs'}$, and $\psi_a = \psi|_{\Delta \to \Delta'}$. By Proposition 3.2(1), we can take $\overline{rs} \in \max_{\mathcal{R}}(rs)$. Then, by Theorem 3.5, $\psi(\overline{rs}) \in \max_{\mathcal{R}}(rs')$. Consider $a \in \mathbb{A}$. We have $\psi_a(\mathrm{total}^a_{rs}) = \psi_a(\mathrm{trunc}^{\lfloor a \rfloor}_{\overline{rs}}) = \mathrm{trunc}^{\lfloor a \rfloor}_{\psi(\overline{rs})} = \mathrm{total}^a_{\psi(rs)} = \mathrm{total}^a_{rs'}$, and so $\mathrm{total}^a_{rs} \sim_{\psi_a} \mathrm{total}^a_{rs'}$. Hence, since $\mathrm{total}^a_{rs}, \mathrm{total}^a_{rs'} \in \mathsf{TO}$, we obtain $\psi_a = (\varphi_{rs'}|_{\Delta'})^{-1} \circ \varphi_{rs}|_{\Delta}$. As a result, $\psi = (\varphi_{rs'})^{-1} \circ \varphi_{rs}$. □

We end this section with an example showing that, in general, there is no real hope to define a unique 'reasonable' notion of label-linearity.

Example 7.8. Consider all the finite sets of points in $\mathbb{R}^2$ with two relations, 'being to the left' and 'being under'. A relational space would then contain tuples $rs = \langle \Delta, \prec_h, \prec_v, \ell \rangle$, where $\Delta \subseteq \mathbb{R}^2$, ℓ is such that two points with a common coordinate never get the same label, $\langle x, y \rangle \prec_h \langle x', y' \rangle$ if $x < x'$, and $\langle x, y \rangle \prec_v \langle x', y' \rangle$ if $y < y'$. One could then use either $\prec_h$ or $\prec_v$ to define label-linearity. Clearly, the resulting total order relations and factors would be completely different.

8. Constructing relational spaces

We will now take advantage of the results presented so far in this paper, by providing what may be seen as a blueprint for constructing relational spaces. We start by showing that the approach based on label-linearity leads to three different kinds of relational spaces.

Theorem 8.1. *Let $\mathscr{R}$ be a nonempty renaming- and intersection-closed set of relational structures.*

(1) $\mathscr{R}^{lin}$ *forms a relational space with the factors defined as in Eq. (2) or Eq. (4).*
(2) $(\mathscr{R}^{lin})^{max} = \mathscr{R}^{lin} \cap \mathscr{R}^{max} = (\mathscr{R}^{max})^{lin}$ *forms a relational space with the factors defined as in Eq. (3).*
(3) $(\mathscr{R}^{lin})^{clo} = \mathscr{R}^{lin} \cap \mathscr{R}^{clo} = (\mathscr{R}^{clo})^{lin}$ *forms a relational space with the factors defined as in Eq. (3).*

Proof. *(1)* $\mathscr{R}^{lin}$ is renaming-closed by Proposition 7.4(4). Conditions involving the factors follow from Proposition 7.7, Eq. (2), and Eq. (4).

(2) The equalities follow from Proposition 7.4(6). $(\mathscr{R}^{lin})^{max}$ is renaming-closed by Proposition 7.4(4). Conditions involving the factors follow from Propositions 7.4(3) and 7.7, and Eq. (3).

(3) The equalities follow from Proposition 7.5(4). $(\mathscr{R}^{lin})^{clo}$ is renaming-closed by Proposition 7.5(3). Conditions involving the factors follow from $(\mathscr{R}^{lin})^{clo} = \mathscr{R}^{lin} \cap \mathscr{R}^{clo}$, Propositions 7.5(1) and 7.7, and Eq. (3). □

There is a subtle difference between the two ways of deriving the factors in Theorem 8.1(1). Basically, using Eq. (2) would normally require one to maximally extend a relational structure, whereas using Eq. (4) would normally require to close it. And the size of the closure is never greater than that of a maximal extension. On the other hand, deriving the closure may require more computational effort than deriving a maximal extension.

In addition to being relational spaces, the calculation of maximal extensions and closures in $\mathscr{R}^{lin}$ can be carried out in the same way as in $\mathscr{R}$.

Theorem 8.2. *Let $\mathscr{R}$ be a nonempty renaming- and intersection-closed set of relational structures.*

(1) $\mathscr{R}^{lin} \xrightarrow{\max_{\mathscr{R}}} 2^{\mathscr{R}^{lin} \cap \mathscr{R}^{max}}$ *is the maximal extension function for* $\mathscr{R}^{lin}$.
(2) $\mathscr{R}^{lin} \xrightarrow{\mathrm{clo}_{\mathscr{R}}} \mathscr{R}^{lin} \cap \mathscr{R}^{clo}$ *is the closure function for* $\mathscr{R}^{lin}$.

Proof. *(1)* $\mathscr{R}^{lin} \xrightarrow{\max_{\mathscr{R}^{lin}}} 2^{\mathscr{R}^{lin} \cap \mathscr{R}^{max}}$ is the maximal extension function for $\mathscr{R}^{lin}$. Hence the result holds by Proposition 7.4(6).

(2) We first observe that $\mathscr{R}^{lin}$ is intersection-closed by Propositions 7.4(5). Hence, $\mathscr{R}^{lin} \xrightarrow{\mathrm{clo}_{\mathscr{R}^{lin}}} \mathscr{R}^{lin} \cap \mathscr{R}^{clo}$ is well-defined as the closure function for $\mathscr{R}^{lin}$. Thus, the result holds by Proposition 7.5(4). □

It is worth emphasising the generic nature of the results obtained so far. First, if one is given a nonempty set of relational structures which fit a specific area of application, then not only the notions of maximal and closed relational structures are determined, but also the notion of closure. Thus, for each nonempty set of relational structures $\mathscr{R}$ there is a unique subset $\mathscr{R}^{clo}$ of precisely those structures which can be represented by their maximal extensions. Second, in the specific area of application to concurrent systems' executions, after defining a suitable notion of label-linearity, one automatically obtains three different types of relational spaces with maximal extension and closure functions as in $\mathscr{R}$.

9. Combined order structures

Combined order structures employ two relationships, viz. causality ($\prec$) and weak causality ($\sqsubset$), admitting a limited kind of cycles. A relational structure $\langle \Delta, \prec, \sqsubset, \ell \rangle$ is *weakly cyclic* if there is no sequence $x_1, \ldots, x_n$ ($n \geq 2$) of domain elements such that $x_n \prec x_1$ and $\langle x_i, x_{i+1} \rangle \in (\prec \cup \sqsubset)$, for every $1 \leq i < n$. In other words, all cycles in the graph of the relation $\prec \cup \sqsubset$ are cycles of $\sqsubset$.

Weak cyclicity has a straightforward interpretation in operational terms. More precisely, it means that in a given system run there are no events $x_1, x_2, \ldots, x_n$ such that each x_i 'happened before or simultaneously' with x_{i+1}, while x_n 'happened (strictly) before' x_1. Thus, only weak causality cycles are possible. Intuitively, each such cycle involves a set of simultaneous events.

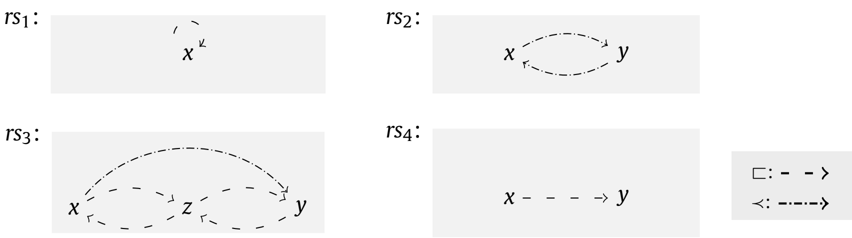

Fig. 3. Relational structures demonstrating the minimality of axioms LC:1–4.

Definition 9.1 *(combined order structure). A combined order structure (or* co*-structure)* is a weakly cyclic relational structure $cos = \langle \Delta, \prec, \sqsubset, \ell \rangle$ such that $\sqsubset$ is irreflexive. We denote this by $cos \in \mathsf{COS}$.

In what follows, we denote $\mathrm{pre}_{\mathrm{cos}}(Q, R) = (Q \cup R)^* \circ Q \circ (Q \cup R)^*$, for any relations Q and R. It is straightforward to check that:

$$Q \subseteq Q' \wedge R \subseteq R' \implies \mathrm{pre}_{\mathrm{cos}}(Q, R) \subseteq \mathrm{pre}_{\mathrm{cos}}(Q', R') \tag{5}$$

$$\langle \Delta, \prec, \sqsubset, \ell \rangle \text{ is weakly cyclic} \iff \mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset) \text{ is irreflexive .} \tag{6}$$

Proposition 9.2. COS *is intersection-closed, renaming-closed, and convex.*

Proof. The first part follows from Eq. (5) and Eq. (6), and the fact that a subset of an irreflexive relation is also an irreflexive relation. The second part follows from the fact that removing relationships cannot create new cycles, nor make an irreflexive relation reflexive. The third part follows directly from the definitions, and the last from the second. □

We will now provide a direct characterisation of both maximal and closed co-structures, as well as a direct definition of the closure of co-structures. Crucially, we will not intersect sets of maximal extensions of co-structures in the characterisations of the latter two notions. We start by introducing an axiomatisation of maximal co-structures.

Definition 9.3 *(*LC*-structure). A layered concurrent structure (or* LC*-structure)* is a relational structure $lcs = \langle \Delta, \prec, \sqsubset, \ell \rangle$ such that, for all $x, y, z \in \Delta$:

(LC:1) $x \not\sqsubset x$	(LC:2) $x \prec y \implies (x \sqsubset y \wedge y \not\sqsubset x)$
(LC:3) $x \prec y \implies (x \prec z \vee z \prec y)$	(LC:4) $x \neq y \implies (x \sqsubset y \sqsubset x \vee x \prec^{sym} y)$.

We denote this by $lcs \in \mathsf{LCS}$.

Examples of relational structures $rs_1, \ldots, rs_4 \notin \mathsf{LCS}$ in Fig. 3 demonstrate that the set of axioms is minimal, as each rs_i satisfies all the axioms in Definition 9.3 except for LC:i.

LC-structures are not only maximal, but also are closely related to stratified orders, which provides a justification for some of the terminology used in this section.

Proposition 9.4. $\mathsf{LCS} = \mathsf{COS}^{max} = \{\langle \Delta_{so}, \prec_{so}, \sqsubset_{so}, \ell_{so} \rangle \mid \langle \Delta_{so}, \prec_{so}, \ell_{so} \rangle = so \in \mathsf{SO}\}$, *where* $\sqsubset_{so} = (\Delta_{so}^2 \setminus \mathrm{id}_{\Delta_{so}}) \setminus \prec_{so}^{-1}$.

Proof. One can show that a partial order ao is a stratified order if and only if, for all $x, y, z \in \Delta$, $x \prec y \implies x \prec z \vee z \prec y$ *(*)*.

Below we denote $RS = \{\langle \Delta_{so}, \prec_{so}, \sqsubset_{so}, \ell_{so} \rangle \mid so \in \mathsf{SO}\}$. Recall that $x \sqsubset_{so} y$ iff $y \not\prec_{so} x$, for all $x \neq y \in \Delta_{so}$ *(**)*.

($\mathsf{LCS} \subseteq \mathsf{COS}^{max}$) Let $lcs = \langle \Delta, \prec, \sqsubset, \ell \rangle \in \mathsf{LCS}$.

We first observe that $\mathsf{LCS} \subseteq \mathsf{COS}$. Indeed, suppose that $lcs = \langle \Delta, \prec, \sqsubset, \ell \rangle \in \mathsf{LCS} \setminus \mathsf{COS}$. Then, since $\sqsubset$ is irreflexive by LC:1, it must be the case that lcs is not weakly cyclic. This and the first part of LC:2 means that there is a shortest sequence $x_1, \ldots, x_n$ of distinct domain elements such that $x_1 \sqsubset \ldots \sqsubset x_n \prec x_1$. Since $\prec$ is irreflexive (by LC:1/2), $n \geq 2$. In fact, $n \geq 3$ as for $n = 2$ we would have a contradiction with the second part of LC:2. We then observe that, by LC:3 and $x_n \prec x_1$, we have $x_n \prec x_2$ (contradicting the minimality of n) or $x_2 \prec x_1$ (contradicting the second part of LC:2).

Hence $\mathsf{LCS} \subseteq \mathsf{COS}$, and so, in view of Proposition 9.2, to show $lcs \in \mathsf{COS}^{max}$, it suffices to observe that adding any new relationship between $x \neq y \in \Delta$ would result in a relational structure outside COS (due to violating weak cyclicity). This follows since, by LC:4 and the first part of LC:2, we have: *(i)* $x \sqsubset y \sqsubset x$; or *(ii)* $x \prec y \wedge x \sqsubset y$; or *(iii)* $y \prec x \wedge y \sqsubset x$ (this case

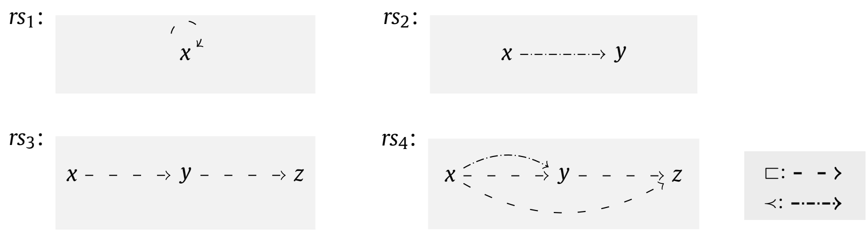

Fig. 4. Relational structures demonstrating the minimality of axioms SO:1–4.

is symmetric to *(ii)*). If *(i)* holds, then adding $\langle x, y\rangle$ or $\langle y, x\rangle$ to $\prec$ would contradict the second part of LC:2. If *(ii)* holds, then adding $\langle y, x\rangle$ to $\prec$ or $\sqsubset$ would contradict the second part of LC:2.

($\mathsf{COS}^{max} \subseteq RS$) Let $rs = \langle \Delta, \prec, \sqsubset, \ell\rangle \in \mathsf{COS}^{max}$ and $so = \langle \Delta, \prec, \ell\rangle$. We begin by showing that $so \in \mathsf{SO}$.

We first observe that $so \in \mathsf{PO}$. Indeed, as $rs \in \mathsf{COS}$, $\prec$ is irreflexive, and so suppose that $x \prec y \prec z$. By LC:1/2, we have $x \neq y \neq z$. Moreover, $x = z$ would imply $x \prec y \prec x$ and so $x \sqsubset y \prec x$ (by the first part of LC:2), contradicting the second part of LC:2. Thus $x \neq y \neq z \neq x$. If $x \not\prec z$ then, since rs is maximal, adding $\langle x, z\rangle$ to $\prec$ invalidates weak cyclicity. Hence, as $x \neq z$, we must have $\langle z, x\rangle \in (\prec \cup \sqsubset)^+$. But this and $x \prec y \prec z$ contradicts the weak cyclicity of rs. Thus $so \in \mathsf{PO}$.

Suppose that $so \notin \mathsf{SO}$. Then, by *(*)*, there are distinct $x, y, z \in \Delta$ such that $x \prec y$ and $x \not\prec z \not\prec y$. Hence, as $x \neq z$ and rs is maximal, $\langle z, x\rangle \in (\prec \cup \sqsubset)^+$ and $\langle y, z\rangle \in (\prec \cup \sqsubset)^+$. Therefore $\langle y, x\rangle \in (\prec \cup \sqsubset)^+$ which, together with $x \prec y$, contradicts the weak cyclicity of rs. Thus $so \in \mathsf{SO}$.

To conclude this part of the proof, we show that $x \sqsubset y$ iff $x \sqsubset_{so} y$, for all $x \neq y \in \Delta$. This, by *(**)* is equivalent to $x \sqsubset y$ iff $y \not\prec x$, for all $x \neq y \in \Delta$. Suppose that $x \sqsubset y$ and $y \prec x$. Then we obtain a contradiction with the second part of LC:2. Suppose next that $y \not\prec x$ and $x \not\sqsubset y$. Then, since rs is maximal, we obtain $\langle x, y\rangle \in (\prec \cup \sqsubset)^+$ and $\langle y, x\rangle \in \mathrm{pre}_{\mathsf{cos}}(\prec, \sqsubset)$. Hence $\langle x, x\rangle \in \mathrm{pre}_{\mathsf{cos}}(\prec, \sqsubset)$ and so, by Eq. (6), rs is not weakly cyclic, yielding a contradiction. As a result, $rs \in RS$.

($RS \subseteq \mathsf{LCS}$) Let $so = \langle \Delta, \prec, \ell\rangle \in \mathsf{SO}$ and $rs = \langle \Delta, \prec, \sqsubset, \ell\rangle$, where $\sqsubset = \sqsubset_{so}$. We observe that LC:1/2 follow from $so \in \mathsf{PO}$ and the definition of $\sqsubset_{so}$. LC:3 follows from *(*)* and $so \in \mathsf{SO}$. To show LC:4, suppose $x \neq y \in \Delta$ and $y \not\prec x \not\prec y$. Then $y \sqsubset_{so} x \sqsubset_{so} y$, and so $y \sqsubset x \sqsubset y$. □

Therefore, LC-structures can be seen as faithful representations of stratified orders.

For each co-structure, the LC-structures extending it are given by the function $\mathrm{cos2LCS} : \mathsf{COS} \to 2^{\mathsf{LCS}}$ such that $\mathrm{cos2LCS}(cos) = \mathrm{ext}_{\mathsf{LCS}}(cos)$, for every $cos \in \mathsf{COS}$. By Propositions 3.2(1) and 9.4, cos2LCS returns nonempty sets of LC-structures, and it is the $\max_{\mathsf{COS}}$ function in the terminology of Section 3. We then introduce an axiomatisation of the closed co-structures.

Definition 9.5 *(SO-structure). A stratified order structure*[1] *(or SO-structure)* is a relational structure $sos = \langle \Delta, \prec, \sqsubset, \ell\rangle$ such that, for all $x, y, z \in \Delta$:

(SO:1) $x \not\sqsubset x$	(SO:2) $x \prec y \implies x \sqsubset y$
(SO:3) $(x \neq z \wedge x \sqsubset y \sqsubset z) \implies x \sqsubset z$	(SO:4) $(x \sqsubset y \prec z \vee x \prec y \sqsubset z) \implies x \prec z.$

We denote this by $sos \in \mathsf{SOS}$.

Examples of relational structures $rs_1, \ldots, rs_4 \notin \mathsf{SOS}$ in Fig. 4 show that the set of axioms in the last result is minimal, as each rs_i satisfies all the axioms in Proposition 9.5 except for SO:i.

The next result shows that the so-structures are precisely all the closed co-structures. Moreover, the closure of co-structures can be expressed directly by adding all the implied causalities and weak causalities.

Theorem 9.6. *The structure closure of* COS *is* $\mathrm{cos2sos} : \mathsf{COS} \to \mathsf{SOS}$ *such that we have* $\mathrm{cos2sos}(cos) = \langle \Delta, (\prec \cup \sqsubset)^* \circ \prec \circ (\prec \cup \sqsubset)^*, (\prec \cup \sqsubset)^{\curlywedge}, \ell\rangle$, *for every* $cos = \langle \Delta, \prec, \sqsubset, \ell\rangle \in \mathsf{COS}$.

Proof. Let $cos = \langle \Delta, \prec, \sqsubset, \ell\rangle \in \mathsf{COS}$. We first observe that cos2sos is well-defined since $\mathrm{cos2sos}(cos)$ satisfies SO:1–4. Indeed, SO:1 clearly holds, and SO:2 follows from $\mathrm{pre}_{\mathsf{cos}}(\prec, \sqsubset) \subseteq (\prec \cup \sqsubset)^{\curlywedge}$ and the weak cyclicity of cos. Moreover, SO:3/4 respectively follow from:

[1] Stratified order structures were first independently proposed in [24] and [25]. In [10] they were successfully used to model concurrent behaviours with 'not later than' relationship between events. They were extended to deal with the mutex (mutual exclusion) relationship between events in [9] and [16].

$((\prec \cup \sqsubset)^{\curlywedge} \circ (\prec \cup \sqsubset)^{\curlywedge}) \setminus \mathrm{id}_\Delta \subseteq (\prec \cup \sqsubset)^{\curlywedge}$ and
$((\prec \cup \sqsubset)^{\curlywedge} \circ \mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset)) \cup (\mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset) \circ (\prec \cup \sqsubset)^{\curlywedge}) \subseteq \mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset)$.

To show that $\mathsf{COS} \xrightarrow{\mathrm{cos2sos}} \mathsf{SOS}$ is the structure closure of COS, it suffices to check the assumptions of Proposition 5.7. To start with, we observe that $\mathsf{SOS} \subseteq \mathsf{COS}$. Indeed, suppose that $sos = \langle \Delta, \prec, \sqsubset, \ell \rangle \in \mathsf{SOS}$. The irreflexivity of $\sqsubset$ follows from SO:1. Suppose that sos is not weakly cyclic. Hence, by SO:1/2, there is a sequence $x_1, \ldots, x_n$ $(n \geq 2)$ of distinct domain elements such that $x_1 \sqsubset \ldots \sqsubset x_n \prec x_1$. Thus, by applying SO:4 $n-1$ times, we can obtain $x_1 \prec x_1$. Hence, by SO:2, we have $x_1 \sqsubset x_1$, contradicting SO:1.

Thus $\mathsf{SOS} \subseteq \mathsf{COS}$ and cos2sos is clearly monotonic (see, in particular, Eq. (5)). To show that cos2sos is non-decreasing, we observe that $\prec \subseteq \mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset)$. Moreover, since $\sqsubset$ is irreflexive, $\sqsubset \subseteq \sqsubset^+ \setminus \mathrm{id}_\Delta \subseteq (\prec \cup \sqsubset)^{\curlywedge}$.

Let $sos = \langle \Delta, \prec, \sqsubset, \ell \rangle \in \mathsf{SOS}$. To show Proposition 5.7(i), suppose that $\langle x, y \rangle \in \mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset)$. Then there are distinct $x = x_1, \ldots, x_m, x_{m+1}, \ldots, x_n = y$ $(n \geq 2)$ such that $x_1 \sqsubset \ldots \sqsubset x_m \prec x_{m+1} \sqsubset \ldots \sqsubset x_n$. Hence, by repetitive application of SO:2–4, we can obtain $x_1 \prec x_n$, and so $x \prec y$. Suppose next that $\langle x, y \rangle \in (\prec \cup \sqsubset)^{\curlywedge}$. Then there are distinct $x = x_1, \ldots, x_n = y$ $(n \geq 2)$ such that $\langle x_i, x_{i+1} \rangle \in (\prec \cup \sqsubset)$, for every $1 \leq i \leq n$. Hence, by SO:2, we obtain $x_1 \sqsubset \ldots \sqsubset x_n$. Thus, by repetitive application of SO:3, we obtain $x_1 \sqsubset x_n$, and so $x \sqsubset y$. As a result, $\mathrm{cos2sos}(sos) \trianglelefteq sos$.

To show Proposition 5.7(ii), we take $x \neq y \in \Delta$ and consider two cases.

Case 1: $x \not\prec y$. Then $rs = \langle \Delta, \prec, \sqsubset \cup \{\langle y, x \rangle\}, \ell \rangle \in \mathsf{COS}$. Indeed, otherwise is not weakly cyclic, and so $\langle x, y \rangle \in \mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset)$. Hence, by SO:2–4, we have $x \prec y$, yielding a contradiction. Moreover, $x \not\prec_{lcs} y$, for every $lcs \in \mathrm{cos2LCS}(rs)$, as $y \sqsubset_{lcs} x$ and $lcs \in \mathsf{COS}$. Hence, we have $\langle x, y \rangle \notin \prec_{\bigcap \mathrm{cos2LCS}(rs)}$.

Case 2: $x \not\sqsubset y$. Then $rs = \langle \Delta, \prec \cup \{\langle y, x \rangle\}, \sqsubset, \ell \rangle \in \mathsf{COS}$. Indeed, otherwise rs is not weakly cyclic, and so $\langle x, y \rangle \in (\sqsubset \cup \prec)^+$. Hence, by SO:2, $x \sqsubset^+ y$. Thus, by SO:3, we have $x \sqsubset y$, yielding a contradiction. Moreover, $x \not\sqsubset_{lcs} y$, for every $lcs \in \mathrm{cos2LCS}(rs)$, as $y \prec_{lcs} x$ and $lcs \in \mathsf{COS}$. Hence $\langle x, y \rangle \notin \sqsubset_{\bigcap \mathrm{cos2LCS}(rs)}$. □

From the point of view of applying co-structures to the modelling of concurrent behaviours, not all co-structures are relevant. What we need is to identify those which can be used to form relational spaces. The first part of this paper prepared us for this task through the introduction of the notions of label-linearity in the general setting. We will now take advantage of the results obtained there in the specific setting of co-structures.

We start by providing a full characterisation of label-linear co-structures.

Proposition 9.7. *The following are equivalent, for every* co-structure $cos = \langle \Delta, \prec, \sqsubset, \ell \rangle$ *and its closure* $sos = \mathrm{cos2sos}(cos)$.

(1) cos *belongs to* COS^{lin}.
(2) sos *belongs to* COS^{lin}.
(3) $\langle x, y \rangle$ *belongs to* $\prec_{sos}^{sym}$, *for all* $x \neq y \in \Delta$ *with the same label.*
(4) $\langle x, y \rangle$ *belongs to* $\mathrm{pre}_{\mathrm{cos}}(\prec, \sqsubset)^{sym}$, *for all* $x \neq y \in \Delta$ *with the same label.*

Proof. (1) $\Longleftrightarrow$ (2) Follows from Proposition 7.5(2).

(3) $\Longleftrightarrow$ (4) Follows from Theorem 9.6.

(2) $\Longrightarrow$ (3) Suppose that $x \neq y \in \Delta$ satisfy $\ell(x) = \ell(y) = a$. We observe that if $x \not\prec_{sos} y$ then $rs = \langle \Delta, \prec_{sos}, \sqsubset_{sos} \cup \{\langle y, x \rangle\}, \ell \rangle \in \mathsf{COS}$. Indeed, otherwise rs would not be weakly cyclic, and so $\langle x, y \rangle \in \mathrm{pre}_{\mathrm{cos}}(\prec_{sos}, \sqsubset_{sos})$. Hence, by SO:2–4, we would have $x \prec_{sos} y$, yielding a contradiction. Hence $y \prec_{\mathrm{total}^a_{sos}} x$ as we have $y \sqsubset_{lcs} x$, for every $lcs \in \mathrm{cos2LCS}(rs) \subseteq \mathrm{cos2LCS}(sos)$. By a symmetric argument, $y \not\prec_{sos} x$ implies $x \prec_{\mathrm{total}^a_{sos}} y$. Therefore, if $x \prec_{sos}^{sym} y$ does not hold, then total^a_{sos} is not a total order, yielding a contradiction.

(3) $\Longrightarrow$ (2) Let $a \in \mathbb{A}$ and $\mathrm{total}^a = \langle \Delta_{sos}, \prec_{sos}, \ell_{sos} \rangle^{\lfloor a \rfloor}$. It suffices to observe that $\mathrm{total}^a \in \mathsf{TO}$ and $\mathrm{total}^a = \mathrm{trunc}^{\lfloor a \rfloor}_{lcs}$, for every $lcs \in \mathrm{cos2LCS}(sos)$. Indeed, by SO:1/2/4, we have $\mathrm{total}^a \in \mathsf{PO}$ and so, by the assumption made, $\mathrm{total}^a \in \mathsf{TO}$. We also have $\mathrm{total}^a \trianglelefteq \mathrm{trunc}^{\lfloor a \rfloor}_{lcs}$ as, by SO:2, we have $\prec_{sos} \subseteq \sqsubset_{sos}$. Hence, by $\mathrm{total}^a \in \mathsf{TO}$ and LC:1/2, we have $\mathrm{total}^a = \mathrm{trunc}^{\lfloor a \rfloor}_{lcs}$. □

Note that one cannot replace $x \prec^{sym} y$ by $x \sqsubset^{sym} y$ in Proposition 9.7(3). A counterexample is provided by the following so-structure and one of its maximal extensions:

$$\begin{aligned} sos &= \langle \{x, y\}, \varnothing, \{\langle x, y \rangle\}, \{x \mapsto a, y \mapsto a\} \rangle \in \mathsf{SOS} \\ lcs &= \langle \{x, y\}, \varnothing, \{\langle x, y \rangle, \langle y, x \rangle\}, \{x \mapsto a, y \mapsto a\} \rangle \in \mathrm{cos2LCS}(sos)\ . \end{aligned}$$

We conclude with a construction of relational spaces based on different sets of co-structures, and the corresponding maximal extension and closure functions.

Theorem 9.8.

(1) COS^{lin} *forms a relational space with the factor of each* co*-structure cos being defined so that, for every* $x \in \Delta_{cos}$ *with* $\ell_{cos}(x) = a$*:*

$$\varphi_{cos}(x) = a^{(i)} \quad \textit{where} \quad i = 1 + |\{y \in \Delta_{cos}^{\lfloor a \rfloor} \mid y \sqsubset_{cos}^{+} x\}| \,.$$

(2) LCS^{lin} *and* SOS^{lin} *form relational spaces with the factor of each* co*-structure cos being defined so that, for every* $x \in \Delta_{cos}$ *with* $\ell_{cos}(x) = a$*:*

$$\varphi_{cos}(x) = a^{(i)} \quad \textit{where} \quad i = 1 + |\{y \in \Delta_{cos}^{\lfloor a \rfloor} \mid y \sqsubset_{cos} x\}| \,.$$

(3) $\mathsf{COS}^{lin} \xrightarrow{\text{cos2LCS}} 2^{\mathsf{COS}^{lin} \cap \mathsf{LCS}}$ *is the maximal extension function for* COS^{lin}*.*

(4) $\mathsf{COS}^{lin} \xrightarrow{\text{cos2sos}} \mathsf{COS}^{lin} \cap \mathsf{SOS}$ *is the closure function for* COS^{lin}*.*

Proof. Follows from Theorems 8.1, 8.2, and 9.6. □

Combined order structures provide a model for dealing with the behaviours of safe Petri Nets with *activator arcs* operating under the step sequence semantics. An activator arc (sometimes called a read arc) from a place p to a transition t means that t can only be executed if p contains a token, but the execution itself does not affect that token. This models testing for the presence of tokens and, in particular, several transitions can do this simultaneously. The following example illustrates how co-structures are applied to reflect this.

Example 9.9. Fig. 5(a) depicts a Petri Net N modelling a concurrent system consisting of a producer, a buffer of capacity one, and a consumer, together with the (initial) marking $M = \{p_2, p_3, p_6\}$. The producer can execute two actions: m (making an item), and a (adding a new item to the buffer). The consumer can execute three actions: g (getting an item from the buffer), u (using the acquired item), and f (finishing the work). Positioned in-between the producer and consumer, the buffer can cyclically execute the a and g actions. Intuitively, the three components progress independently but any action shared by two components can be executed only if both of them do so. This behaviour is further constrained by an activator arc (indicated by an arrow ending with a black arrowhead) from place p_1 to transition u which indicates that u can only be executed if p_1 contains a token. It is also important that the testing for the presence of this token does not prevent transition a from simultaneously consuming it. The semantical model of N we assume is that of step sequence semantics, where a set of (simultaneous) transitions can be executed at any computational move.

A possible execution of N is the step sequence $\sigma = \{m\}\{u\}\{g\}\{u\}\{a\}$ comprising singleton steps. Using the standard net unfolding technique, σ generates an occurrence net with activator arcs ON shown in Fig. 5(b). From this occurrence net ON, one can derive in a purely structural way the co-structure cos depicted in Fig. 5(c). It shows both direct causality and direct weak causality relationships between the events involved in ON (one can show that such a co-structure is always label-linear). The co-structure cos can be closed and the resulting so-structure $sos = \text{cos2sos}(cos)$ underlying ON is shown in Fig. 5(d). In particular, the indirect causality relationship $e_2 \prec_{sos} e_4$ can be derived from one of two chains of direct relationships: $e_2 \prec_{cos} e_3 \prec_{cos} e_4$ and $e_2 \prec_{cos} e_3 \prec_{cos} e_5 \sqsubset_{cos} e_4$. Finally, there are exactly two LC-structures extending sos, viz. lcs in Fig. 5(e) and lcs' in Fig. 5(f). The former corresponds to σ and the latter to the step sequence $\sigma' = \{m\}\{u\}\{g\}\{u, a\}$, where u and a are executed simultaneously.

10. Conclusions

In this paper, we introduced a generic framework for constructing relational spaces that can be used for the formal modelling of the behaviours of concurrent systems. In the process of developing this framework, we highlighted the fundamental role played by maximal and closed relational structures. We introduced a general notion of label-linearity which is instrumental in defining relational spaces. We also included results showing that monotonic and non-decreasing functions can provide a useful tool when comparing expressiveness of different classes of relational structures.

Relational structures as understood in this paper have been introduced and studied in the last 15 years of the last century [14,15,24–26], and recently they have been substantially revised and generalized [9,16]. Much of the earlier research has been focused on the modelling of concurrent behaviours emphasising assumptions about executions (are they total, stratified or interval orders), concrete relationships between events (e.g., mutex), and the potential to model phenomena like 'not later than'. The approach used in this paper is different. While still being motivated by the modelling of concurrent system behaviours, we started out from a fairly general mathematical definition of relational structures from which we then directly derived relevant properties. As a consequence, the results of this paper are more general and could be applied to new, more sophisticated models of concurrency. For example, the maximal relational structures, that correspond to observations of system executions, are not necessarily derived from partial orders (in disguise).

The combined order structures of Section 9 are based on two possible relationships between domain elements (causality and weak causality). The order structures from [16,9] on the other hand are based on weak causality and a mutex relation. Whereas causality and weak causality can be seen as ordering relations, the mutex relation expresses only that two events

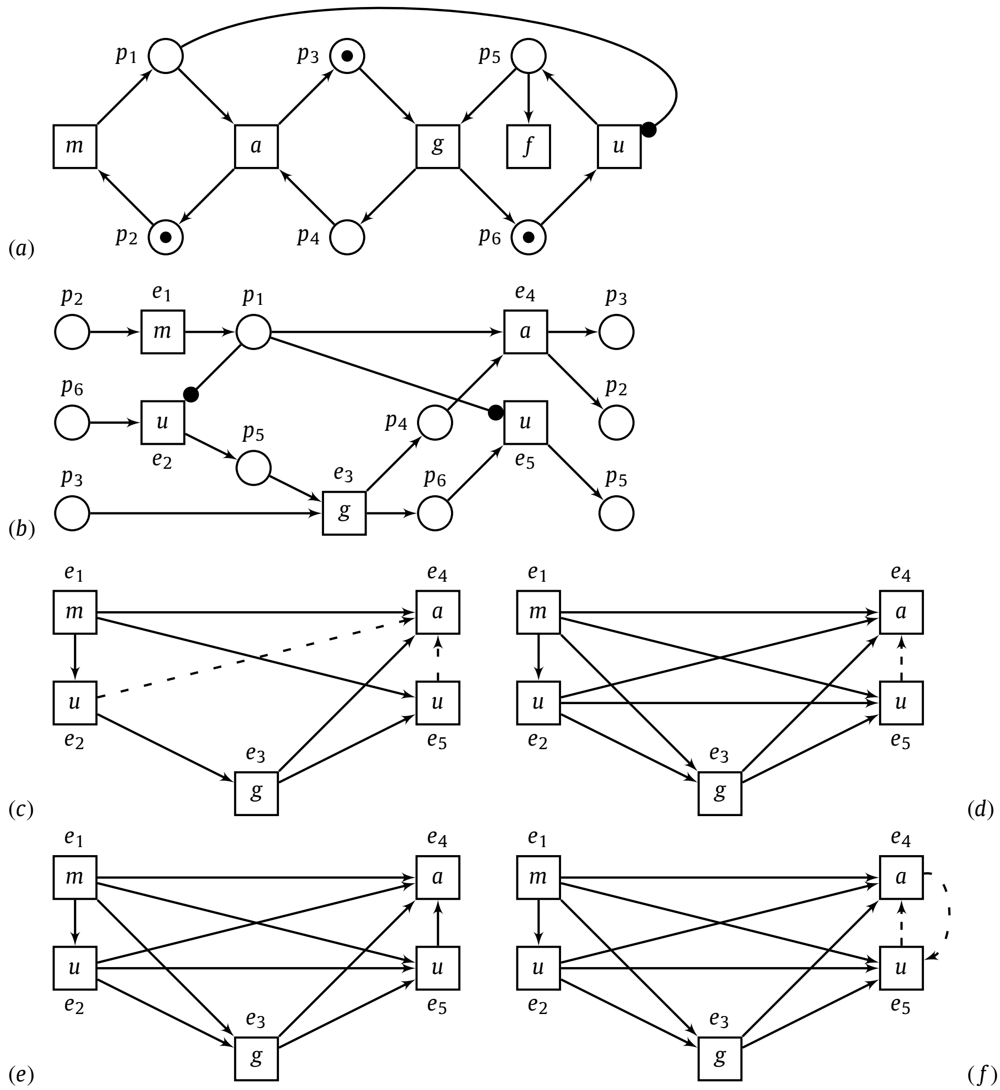

Fig. 5. A Petri Net with activator arcs N modelling a producer/consumer system (a); an occurrence net with activator arcs ON modelling the execution of step sequence $\sigma = \{m\}\{u\}\{g\}\{u\}\{a\}$ by N (b); structurally derived co-structure cos involving the events which occurred in the behaviour captured by ON (c); the so-structure $sos = \text{cos2sos}(cos)$ underlying ON (d); and the only two LC-structures extending sos: lcs in (e) corresponding to the step sequence σ of N, and lcs' in (f) corresponding to the step sequence $\sigma' = \{m\}\{u\}\{g\}\{u, a\}$ of N (i.e., $\text{cos2LCS}(sos) = \text{cos2LCS}(cos) = \{lcs, lcs'\}$). Note that in ($d, e, f$) a solid arc implies a dashed arc.

did not occur simultaneously. This allows one to express precedence (causality) through weak causality ('not later than') and mutex ('not together'). Moreover, the mutex relation makes it possible to express 'interleaving' (two events can be observed in any order) without the implication of simultaneity. So, order structures are strictly more expressive than combined order structures. However, combined order structures turn out to be equivalent to order structures that have the property that their mutex relation is included in the transitive closure of their weak causality relation. To establish this equivalence one can take advantage of two results of this paper, viz. Theorems 3.4 and 5.3.

The assumption that relational structures have only binary relations between events stems from the intuition that in a system's execution, the occurrence of an action relates to other individual occurrences of actions and that dependencies between two or more events can be fully expressed through binary relations — as is the case e.g., in Elementary Net systems (see [7]) and Mazurkiewicz traces ([1,2]) where these relations are derived from a fixed, global, independence relation between actions. Relaxing this assumption would lead to a new line of research comparable to the effort of lifting the concept of Mazurkiewicz traces with their global binary independence relation to more general traces that could describe the behaviour of non-safe Petri Nets using a local non-binary independence relation (see e.g., [27]). Another extension would be to allow relational structures with infinite domains. Also this would require a significant effort, e.g., it would be necessary to extend the work on 'initial finiteness' of relational structures investigated in [15]. We leave these topics for future research.

Currently, we are interested in adding to the existing framework different notions of conflict between executed events as well as in experimenting with relationships between events captured by relations involving sets or tuples of events rather than only pairs of events. We also plan to use our set-up to further investigate the case of interval orders representing system executions with mutex.

Declaration of competing interest

There is no conflict of interest to declare.

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