Fundamenta Informaticae XX (2020) 1–30
DOI 10.3233/FI-2016-0000
IOS Press

Investigating Reversibility of Steps in Petri Nets

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Abstract. In reversible computations one is interested in the development of mechanisms allowing to undo the effects of executed actions. The past research has been concerned mainly with reversing single actions. In this paper, we consider the problem of reversing the effect of the execution of groups of actions (steps).

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Using Petri nets as a system model, we introduce concepts related to this new scenario, generalising notions used in the single action case. We then present properties arising when reverse actions are allowed in place/transition nets (PT-nets). We obtain both positive and negative results, showing that allowing steps makes reversibility more problematic than in the interleaving/sequential case. In particular, we demonstrate that there is a crucial difference between reversing steps which are sets and those which are true multisets. Moreover, in contrast to sequential semantics, splitting reverses does not lead to a general method for reversing bounded PT-nets. We then show that a suitable solution can be obtained by combining split reverses with weighted read arcs.

Keywords: Petri net, reversible computation, step semantics, action splitting, net synthesis, direct reversibility, mixed reversibility, weighted activator arcs

1. Introduction

Reversibility of (partial) computations has been extensively studied during the past years, looking for mechanisms that allow to (partially) undo some actions executed during a computational process, that for some reason one needs to cancel. As a result, the execution can then continue from a consistent state as if that suppressed action had not been executed at all. In particular, these mechanisms allow for the correct implementation of transactions [9, 10], that are partial computations which either are totally executed or not executed at all. This includes updating in databases, so that one never commits an 'incomplete' set of related updates that might produce an inconsistent state (in which one could infer contradictory facts). Another example would be money transfers between banks, or modern e-commerce platforms, where the payments received should match the goods distributed [7].

Within Formal Methods, reversibility has been investigated, for instance, in the framework of process calculi [24, 19], event structures [25], DNA-computing [6], category theory [11], and quantum computing [27]. In the latter case, it plays a central role due to the inherent reversibility of the mechanisms on which quantum computing is based. This paper is concerned with reversibility in place/transition nets (PT-*nets*), which are a fundamental class of *Petri nets*, operating according to the step semantics in which multisets of actions (*steps*) are executed simultaneously.

In Petri nets, reversibility is usually understood as a global property resembling cyclicity. It was also considered in a manner closer to its process calculi meaning using symmetric nets [14] (symmetric nets have later been used to study structural symmetries of state spaces [8]). Locally defined reversibility has not yet been extensively studied within the Petri net framework. This is rather surprising as the formalisation of an action by means of a pair of *pre-places* and *post-places* provides an immediate way of defining the *reverse* of the actions simply by interchanging these two sets of places. There are, however, some more recent works in which reversibility is understood as cyclicity (i.e., an ability to return to the initial state from any reachable state). They are usually based on the structure theory of Petri nets [17], or an algebraic study by means of invariants [22].

From the operational point of view, one can distinguish three essential ways of reversing computational processes: backtracking, causal reversibility, and out of causal reversibility. For concurrent systems, the backtracking mode was considered, for example, in [9], where the RCCS process algebra is introduced. An investigation of causal reversibility in the Petri net context can be found, for example, in [20], where it was implemented using occurrence nets. All three ways of reversing computations were studied in [23], where biologically motivated reversing Petri nets were introduced. In all these works, one needs to enrich the original model by additional annotations or constructs. It is the memory of monitored processes for RCCS, the computation stack encoded through colours for folded occurrence nets, and atoms and bonds together with the history function for reversing Petri nets. In our approach, we are interested in studying the possibility of reversing computations in step semantics emphasizing reversing the effects, and avoiding the reachability of new states. The latter ensures that one can reach only states that are reachable by forward computations, which differentiates our approach from the out of causal reversibility discussed in [23]. We also do not equip our nets with additional external monitors which help to ensure causality. As a result, it may happen that reverses of actions that were not yet executed become enabled. This inconvenience can, however, be easily removed by suitably augmenting a PT-net being reversed to yield another net, as described in [4].

The approach presented in this paper is closer to inverse nets presented in [5], and so more *operational*. It extends the study of reversing (sequential) transition systems initiated in [4], where it was shown that the apparent simplicity of this approach is far from trivial, mainly due to the difficulty of avoiding situations where an added reverse action is executed in an inconsistent manner, e.g., before the action being reversed has been executed. Further investigation of this problem can be found in [21], while [3] considers *bounded* PT-nets, distinguishing between the *strict* reverses and *effect* reverses of actions. The latter deliver the effect of reversing the original actions, but possibly with a change in the way action enabling is carried out. It was shown that some transition systems which can be *solved* by bounded nets allow the reversal of their actions by means of single reverse actions, while in other cases the reversal is only possible if *splitting* of reverses is allowed (i.e., each action has a set of reverses which collectively provide means of reversing the original action).

In [3] only the sequential (*interleaving*) semantics of nets was considered and, in fact, several of the presented examples were just (finite) *linear transition systems*, taking advantage of the results presented in [2, 13], where binary words representable by Petri net were characterised. The latter problem and its consequences for reversibility has been further investigated in [15].

About this paper We consolidate and extend the results of [16], where the study of *step reversing* in PT-nets and (step) transition systems was initiated. We assume that the transition systems to be synthesized include information about the multisets of actions (steps) that should be executed in parallel. Reversing of the actions should preserve this step information so that the simultaneous firing of several reverse actions should correspond to the original steps at the system represented by a PT-net.

We introduce several concepts related to this new scenario, generalising notions used in the single action case. A number of straightforward definition which worked in the sequential case are no longer adequate. When looking for their adequate generalisations, we identify two ‘natural’ notions of step reversibility. The former (*direct reversibility*) only allows steps which comprise either the original actions, or the reverse actions. The latter (*mixed reversibility*) allows also mixing of the original and reverse actions. It turns out that these two ways of interpreting step reversibility are fundamentally different. Crucially, the direct reversibility cannot be implemented for steps which are true multisets, and so in such cases one has to look for mixed reversibility solutions. In this way, we identified a striking difference between reversing steps which are sets and those which are true multisets (when autoconcurrency of actions in system executions is allowed). However, there is still a general positive result which basically applies whenever sequential reversing is possible and the original steps can be be satisfactorily represented.

We also adapt split reverses introduced in [3]. Unfortunately, splitting is not enough to deal with all bounded PT-nets (also adding inhibitor arcs to the PT-net model does not always help). A general solution we propose uses *weighted read arcs* [18] (the further development of this model is out of the scope of this paper, and is left as a topic for the future work).

The paper is organised as follows. Section 2 recalls notions and notations used throughout the paper. Moreover, some basic results concerning the step transition model are given. Section 3 introduces four different ways of defining reversibility in step transition systems, including direct step reversibility and mixed step reversibility, as well as set reversibility (where a true multiset of actions is reversed in stages) and split reversibility. Section 4 demonstrates that the direct reversibility cannot be achieved in the presence of autoconcurrency. Moreover, it characterises cases where mixed reversibility can be replaced by (more desirable) direct reversibility or set reversibility. Section 5 provides result allowing one to deal with mixed reversibility and step reversibility in an effective way, by reducing the reversibility problem to the net synthesis problem. This approach is further continued Section 6, where lifting of sequential reversibility to step reversibility is discussed. Section 7 proposes a general solution to the step reversibility of bounded PT-nets which relies on the weighted read arcs. Finally, Section 8 contains concluding remarks.

2. Preliminaries

Vectors, multisets and actions An X-*vector* over a set X is a mapping $\alpha : X \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. For two X-vectors, α and β, the *sum* $(\alpha + \beta)$, *difference* $(\alpha - \beta)$, and *less-than-or-equal* relationship $(\alpha \leq \beta)$ are defined component-wise. The *support* of an X-vector α is the set $\mathrm{supp}(\alpha) = \{x \in X \mid \alpha(x) \neq 0\}$. The *empty* X-vector has the empty support and is denoted by $\varnothing_X$ or simply by $\varnothing$, and $-\alpha$ denotes $\varnothing_X - \alpha$. The *union* of an X-vector α and a Y-vector β, where $X \cap Y = \varnothing$, is the $(X \cup Y)$-vector $\alpha \sqcup \beta$ such that $\alpha \sqcup \beta|_X = \alpha$ and $\alpha \sqcup \beta|_Y = \beta$.

Multisets over X are X-vectors returning non-negative integers in $\mathbb{N}$, the subsets of X can be identified with multisets returning 0 or 1, and the elements of X with singleton sets. The set of all multisets over X is denoted by $\mathrm{mult}(X)$. The *size* of $\alpha \in \mathrm{mult}(X)$ is given by $|\alpha| = \sum_{x \in X} \alpha(x)$. For $x \in X$, we denote $x \in \alpha$ whenever $\alpha(x) \geq 1$.

In what follows, e.g., (xxz) denotes a multiset α with the support $\{x, z\}$ satisfying $\alpha(x) = 2$ and $\alpha(z) = 1$. Moreover, x^k denotes a multiset α with the support $\{x\}$ satisfying $\alpha(x) = k$.

Throughout the paper, $\mathcal{A}$ denotes an infinite set *actions*, including the *reverse actions* and *indexed reverse actions* introduced in Section 3, used in step transition systems and PT-nets to model events occurring in concurrent behaviours. To simplify the presentation, we will treat a vector or multiset α over $T \subseteq \mathcal{A}$ as a vector or multiset over $\mathcal{A}$, assuming that $\alpha|_{\mathcal{A} \setminus T} = \varnothing_{\mathcal{A} \setminus T}$.

Step transition systems A *step transition system* is a tuple $STS = (S, T, \to, s_0)$ such that S is a nonempty set of *states*, T is a finite set of *actions*, $\to \subseteq S \times \mathrm{mult}(T) \times S$ is the set of *transitions*, and $s_0 \in S$ is the *initial state*. The transition labels in $\mathrm{mult}(T)$ represent simultaneous executions of groups of actions, called *steps*. Rather than $(s, \alpha, r) \in \rightarrow$, we can denote $s \xrightarrow{\alpha}_{STS} r$. Moreover, $s \xrightarrow{\alpha}_{STS}$ means that there is some r such that $s \xrightarrow{\alpha}_{STS} r$. STS is:

- a *set transition system* if α is a set, for every transition (s, α, r); and
- *state-finite* if S is finite, *step-finite* if $\{\alpha \mid s \xrightarrow{\alpha}_{STS}\}$ is finite, and *finite* if it is both state- and step-finite (and so $\rightarrow$ is finite).

In the diagrams, step transition systems are depicted as labelled directed graphs. Arcs labelled by the empty multiset are omitted.

A state r is *reachable* from state s if there are steps $\alpha_1, \ldots, \alpha_k$ ($k \geq 0$) and states $s_1, \ldots, s_{k+1}$ such that $(s =)s_1 \xrightarrow{\alpha_1}_{STS} s_2 \ldots s_k \xrightarrow{\alpha_k}_{STS} s_{k+1}(= r)$. We denote this by $s \xrightarrow{\alpha_1 \cdots \alpha_k}_{STS} r$.

The set of all states from which a state s is reachable is denoted by $\mathrm{pred}_{STS}(s)$, s is a *home state* if $\mathrm{pred}_{STS}(s) = S$, and $R \subseteq S$ is a *home cover* of STS if $S = \bigcup_{s \in R} \mathrm{pred}_{STS}(s)$.

An *(undirected) path* from a *source* state s to *target* state r is a sequence $\pi = \tau_1 \ldots \tau_k$ ($k \geq 0$), where each τ_i is a pair $((s_i, \alpha_i, r_i), \zeta_i) \in (\rightarrow \times \{+, -\})$ such that either $k = 0$ and $s = r$, or $k \geq 1$ and $s = \widehat{s}_1, \widehat{r}_1 = \widehat{s}_2, \ldots, \widehat{r}_{k-1} = \widehat{s}_k, \widehat{r}_k = r$, assuming that $\widehat{s}_i = s_i$ and $\widehat{r}_i = r_i$ if $\zeta_i = +$, and otherwise $\widehat{s}_i = r_i$ and $\widehat{r}_i = s_i$, for every $1 \leq i \leq k$. We denote this by $\pi \in \mathrm{paths}_{STS}(s, r)$. The *signature* of π is the $\mathcal{A}$-vector $\mathrm{sign}(\pi) = \varnothing_{\mathcal{A}} \zeta_1 \alpha_1 \ldots \zeta_k \alpha_k$, where the ζ_i's are being treated as addition and subtraction operations. For example, if $\pi = ((s', \alpha, s), -)((s', \beta, s''), +) \in \mathrm{paths}_{STS}(s, s'')$, then $\mathrm{sign}(\pi) = \varnothing_{\mathcal{A}} - \alpha + \beta = \beta - \alpha$.

Intuitively, $\mathrm{sign}(\pi)$ records the 'net contribution (or effect)' made by each action along the path π, with $a \in \alpha_i$ making a 'positive' contribution if the transition (s_i, α_i, r_i) agrees with the direction of the path, and otherwise making a 'negative' contribution. Note that r is reachable from s iff there is $\pi \in \mathrm{paths}_{STS}(s, r)$ with all the ζ_i's being equal to $+$.

In this paper, step transition systems are intended to capture (step) reachability graphs of PT-nets. We will now introduce a property of step transition systems which is motivated by the *state equation* which holds, in particular, for PT-nets. The basic idea is that the effect of executing an action is fixed, and so does not depend on the global state in which this happens (we will make this more precise later). Capturing such a constant effect is straightforward for PT-nets, but not for step transition systems. One can, however, approximate the concept of having 'the same effect' by considering as equivalent all undirected paths with the same source and target states.

Let $\bowtie_{STS}$ be the least equivalence relation on the set of all $\mathcal{A}$-vectors such that: *(i)* $\mathrm{sign}(\pi) \bowtie_{STS} \mathrm{sign}(\pi')$, for all $s, r \in S$ and $\pi, \pi' \in \mathrm{paths}_{STS}(s, r)$; and *(ii)* $\alpha \bowtie_{STS} \beta$ and $\alpha' \bowtie_{STS} \beta'$ imply $\alpha + \alpha' \bowtie_{STS} \beta + \beta'$, for all $\mathcal{A}$-vectors α, α', β, and β'. Intuitively, $\alpha \bowtie_{STS} \beta$ means that executing α has the same effect as executing β. This leads to the following property of a step transition STS:

CE $\mathrm{sign}(\pi) \bowtie_{STS} \mathrm{sign}(\pi')$ implies $r = r'$, for all $s, r, r' \in S$, $\pi \in \mathrm{paths}_{STS}(s, r)$, and $\pi' \in \mathrm{paths}_{STS}(s, r')$. *(constant effect)*

It is the case that $\alpha \bowtie_{STS} \beta$ implies $-\alpha \bowtie_{STS} -\beta$ since $\pi \in \mathrm{paths}_{STS}(s, r)$ means that there is $\pi' \in \mathrm{paths}_{STS}(r, s)$ such that $\mathrm{sign}(\pi') = -\mathrm{sign}(\pi)$. Hence we also have the following 'backward' version of the 'forward' constant effect property *CE*: $\mathrm{sign}(\pi) \bowtie_{STS} \mathrm{sign}(\pi')$ implies $s = s'$, for all $s, s', r \in S$, $\pi \in \mathrm{paths}_{STS}(s, r)$, and $\pi' \in \mathrm{paths}_{STS}(s', r)$.

We are now in a position to introduce a class of step transition systems used throughout the rest of this paper. A step transition system $STS = (S, T, \to, s_0)$ is a *constant effect step transition system* (or CEST-system) if it satisfies *CE* as well as the following three properties, for every $s \in S$:

REA $s_0 \in \mathrm{pred}_{STS}(s)$. *(reachability)*

EL $s \xrightarrow{\varnothing}_{STS} s$. *(empty loops)*

SEQ $s \xrightarrow{\alpha+\beta}_{STS}$ implies $s \xrightarrow{\alpha\beta}_{STS}$. *(sequentialisability)*

We then obtain two immediate properties of CEST-systems.

Proposition 2.1. Let STS be a CEST-system.

1. $r = r'$ whenever $s \xrightarrow{\alpha}_{STS} r$ and $s \xrightarrow{\alpha}_{STS} r'$.
2. $s = r$ whenever $s \xrightarrow{\varnothing}_{STS} r$.

Proof:
Part (1) follows from *CE*, and part (2) follows from part (1) and *EL*. □

Proposition 2.1(1) captures the property of *forward determinism* (*FD*) which allows one to unambiguously denote $s \oplus_{STS} \alpha$, or $s \oplus \alpha$ if STS is clear from the context, as the state r satisfying $s \xrightarrow{\alpha}_{STS} r$ whenever $s \xrightarrow{\alpha}_{STS}$.

Being a CEST-system still does not mean that it can be generated by a PT-net. A complete characterisation can be obtained using, e.g., theory of regions [1, 12].

Proposition 2.2. Let s be a state of a CEST-system STS. If $s \oplus \alpha$ is defined and $\beta + \gamma \leq \alpha$, then $s \oplus \beta$, $s \oplus (\beta + \gamma)$ and $(s \oplus \beta) \oplus \gamma$ are also defined, and $(s \oplus \beta) \oplus \gamma = s \oplus (\beta + \gamma)$.

Proof:
By $s \xrightarrow{\alpha}_{STS}$ as well as *SEQ* and *CE*, we have $s \xrightarrow{\beta}_{STS} s \oplus \beta \xrightarrow{\gamma}_{STS} (s \oplus \beta) \oplus \gamma$ as well as $s \xrightarrow{\beta+\gamma}_{STS} s \oplus (\beta + \gamma)$. We therefore have $\pi = ((s, \beta, s \oplus \beta), +)((s \oplus \beta, \gamma, (s \oplus \beta) \oplus \gamma), +) \in \mathrm{paths}_{STS}(s, (s \oplus \beta) \oplus \gamma)$ and $\pi' = ((s, \beta+\gamma, s \oplus (\beta+\gamma)), +) \in \mathrm{paths}_{STS}(s, s \oplus (\beta+\gamma))$. Moreover, $\mathrm{sign}(\pi) = \beta + \gamma = \mathrm{sign}(\pi')$. Hence, by *CE*, $(s \oplus \beta) \oplus \gamma = s \oplus (\beta + \gamma)$. □

We use different ways of removing transitions from a step transition system $STS = (S, T, \to, s_0)$:

$$
\begin{array}{lcll}
STS^{seq} & = & (S, T,\ \{(s, \alpha, r) \in \to \ \mid\ |\alpha| \leq 1\}, s_0) & \\
STS^{set} & = & (S, T,\ \{(s, \alpha, r) \in \to \ \mid\ \mathrm{supp}(\alpha) = \alpha\}, s_0) & \\
STS^{spike} & = & (S, T,\ \{(s, \alpha, r) \in \to \ \mid\ |\mathrm{supp}(\alpha)| \leq 1\}, s_0) & \\
STS|_{T'} & = & (S, T', \{(s, \alpha, r) \in \to \ \mid\ \alpha \in \mathrm{mult}(T')\}, s_0) & (\textit{for } T' \subseteq T)\ .
\end{array}
$$

That is, STS^{seq} is obtained by only retaining singleton steps and $\varnothing$-labelled steps, STS^{set} by only retaining steps which are sets, and STS^{spike} by removing all steps which use more than one action.

Moreover, STS is a *sequential* / *set* / *spiking* step transition system if respectively $STS = STS^{seq}$ / $STS = STS^{set}$ / $STS = STS^{spike}$.[1]

For step transition systems satisfying *SEQ*, checking the satisfaction of the constant effect property can be done by restricting oneself to the sequential steps.

Proposition 2.3. Let STS be a step transition system satisfying *SEQ*. Then STS satisfies *CE* if and only if STS^{seq} satisfies *CE*.

Proof:
We first observe that from *SEQ* for STS it follows that, for every $\pi \in \mathrm{paths}_{STS}(s,r)$, there is $\pi' \in \mathrm{paths}_{STS^{seq}}(s,r)$ such that $\mathrm{sign}(\pi') = \mathrm{sign}(\pi)$ *(*)*. Hence, we also have $\bowtie_{STS} = \bowtie_{STS^{seq}}$ *(**)*.

($\Longrightarrow$) Follows from *(**)* and $\pi \in \mathrm{paths}_{STS^{seq}}(s,r) \subseteq \pi \in \mathrm{paths}_{STS}(s,r)$.

($\Longleftarrow$) Follows from *(*)* and *(**)*. □

The essence of the next result is that adding reverses of some transitions labelled by the same action in a sequential step transition system preserves the constant effect property.

Proposition 2.4. Let $STS = (S, T, \rightarrow, s_0)$ be a sequential step transition system satisfying *CE* and $STS' = (S, T \cup \{\widetilde{a}\}, \rightarrow \cup \rightarrow', s_0)$, where $\rightarrow' \subseteq \{(r, \widetilde{a}, s) \mid (s, a, r) \in \rightarrow\}$ for some $a \in T$ and $\widetilde{a} \notin T$. Then STS' satisfies *CE*.

Proof:
The result clearly holds when $\rightarrow'$ is empty. Otherwise, we have $a \bowtie_{STS'} -\widetilde{a}$. For every $\mathcal{A}$-vector α, let $\widehat{\alpha}$ be the $\mathcal{A}$-vector such that $\widehat{\alpha}|_{\mathcal{A}\setminus\{a,\widetilde{a}\}} = \alpha|_{\mathcal{A}\setminus\{a,\widetilde{a}\}}$, $\widehat{\alpha}(a) = \alpha(a) - \alpha(\widetilde{a})$, and $\widehat{\alpha}(\widetilde{a}) = 0$.

We observe that, for all $s, r \in S$ and $\pi \in \mathrm{paths}_{STS'}(s,r)$, there is $\pi' \in \mathrm{paths}_{STS}(s,r)$ such that $\mathrm{sign}(\pi') = \widehat{\mathrm{sign}(\pi)}$ *(*)*. Hence, we also have that $\alpha \bowtie_{STS'} \beta$ iff $\widehat{\alpha} \bowtie_{STS} \widehat{\beta}$, for all $\mathcal{A}$-vectors α and β *(**)*. The result then follows from *CE* for STS together with *(*)* and *(**)*. □

Let $STS = (S, T, \rightarrow, s_0)$ and $STS' = (S', T', \rightarrow', s'_0)$ be two step transition systems such that $T \subseteq T'$. Then STS is *included* in STS' if there is a bijection $\psi\colon S \rightarrow S'$ such that $\psi(s_0) = s'_0$ and $\{(\psi(s), \alpha, \psi(s')) \mid s \xrightarrow{\alpha}_{STS} s'\} \subseteq \rightarrow'$.[2] This is denoted by $STS \lhd_\psi STS'$ or $STS \lhd STS'$, and if ψ is the identity on S, we denote $STS \blacktriangleleft STS'$. Also, STS is *isomorphic* with STS' if there is ψ such that $STS \lhd_\psi STS'$ and $STS' \lhd_{\psi^{-1}} STS$. This is denoted by $STS \simeq_\psi STS'$ or $STS \simeq STS'$.

PT-nets A PT-*net* (short for place/transition net [26]) is a tuple $N = (P, T, F, M_0)$, where P is a finite set of *places*, $T \subseteq \mathcal{A}$ is a disjoint finite set of *actions*,[3] F is the *flow function* $F\colon (P \times T) \cup (T \times P) \rightarrow \mathbb{N}$ specifying the arc weights between places and actions, and M_0 is the *initial marking* (*markings* are multisets over P representing global states). It is assumed that, for every $a \in T$, there is $p \in P$ such that $F(p, a) > 0$.

[1] If STS is a CEST-system, then STS^{seq}, STS^{set}, and STS^{spike} satisfy *REA* since STS satisfies *REA* and *SEQ*.

[2] If STS and STS' are CEST-systems, then ψ is unique due to *REA* and *FD*.

[3] We use the term 'actions' rather than 'transitions' when referring to the elements of T, in order to avoid confusion with the triples (s, α, r) used in the definition of step transition systems.

The triple (P, T, F) is an *unmarked* PT-net, and $N|_{T'} = (P, T', F|_{(P\times T')\cup(T'\times P)}, M_0)$ is the *subnet* of N induced by $T' \subseteq T$.

In the diagrams, PT-nets are depicted as labelled directed graphs, with circles representing places and boxes to representing actions. Markings are represented by black tokens or numbers drawn inside the circles, the arc weight of 1 is omitted, and the 0-weight arcs are not drawn.

Multisets over T, again called *steps*, represent executions of groups of actions. The *effect* of a step $\alpha \in \mathrm{mult}(T)$ (and, in general, a T-vector α) is the P-vector $\mathrm{eff}_N(\alpha) = \mathrm{post}_N(\alpha) - \mathrm{pre}_N(\alpha)$, where $\mathrm{pre}_N(\alpha)$ and $\mathrm{post}_N(\alpha)$ are multisets of places such that, for every $p \in P$:

$$\mathrm{pre}_N(\alpha)(p) = \sum_{a\in T} \alpha(a) \cdot F(p, a) \quad \text{and} \quad \mathrm{post}_N(\alpha)(p) = \sum_{a\in T} \alpha(a) \cdot F(a, p)\ .$$

A step α is *enabled* at a marking M if $\mathrm{pre}_N(\alpha) \leq M$, and the *firing* of such a step leads to the marking $M' = M + \mathrm{eff}_N(\alpha)$.[4] This is respectively denoted by $M[\alpha\rangle_N$ and $M[\alpha\rangle_N\, M'$. Note that it is always the case that $M[\varnothing\rangle_N\, M$, and that $M[\alpha + \beta\rangle_N$ implies $M[\alpha\rangle_N\, M'[\beta\rangle_N$, where $M' = M + \mathrm{eff}_N(\alpha)$. These two facts motivated the inclusion of *EL* and *SEQ* in the definition of CEST-systems.

The *reachable* markings of N are the smallest set of markings reach_N such that $M_0 \in \mathrm{reach}_N$ and if $M \in \mathrm{reach}_N$ and $M[\alpha\rangle_N$, then $M + \mathrm{eff}_N(\alpha) \in \mathrm{reach}_N$. N is *bounded* if the set reach_N of all the reachable markings is finite.

The overall behaviour of N can be captured by its *concurrent reachability graph* which is the step transition system $CRG_N = (\mathrm{reach}_N, T, \{(M, \alpha, M') \mid M \in \mathrm{reach}_N \wedge M[\alpha\rangle_N\, M'\}, M_0)$. In what follows, $M \xrightarrow{\alpha}_N M'$ denotes $M \xrightarrow{\alpha}_{CRG_N} M'$. Note that the concurrent reachability graphs of bounded PT-nets are finite.

The concept of *marking equation* can be explained in the following way. Suppose that a marking M' can be reached from marking M by firing a sequence of steps, e.g., $M \xrightarrow{\alpha_1\cdots\alpha_n}_{CRG_N} M'$. Then

$$M' = M + \mathrm{eff}_N(\alpha) \qquad M = M' - \mathrm{eff}_N(\alpha) \qquad \mathrm{eff}_N(\alpha) = M' - M\ , \tag{1}$$

where $\alpha = \alpha_1 + \cdots + \alpha_n$. This means that the *effect* of executing a multiset of actions α is constant, as it does not depend on the starting marking nor the ending marking nor any particular way in which the actions making up α were fired. Moreover, the effect of actions fired along any path from M to M' is constant. This motivated the inclusion of *CE* in the definition of CEST-systems.

It is straightforward to see that CRG_N is a CEST-system. In particular, by Eq.(1), we have $\mathrm{eff}_N(\mathrm{sign}(\pi)) = M' - M$, for every $\pi \in \mathrm{paths}_{CRG_N}(M, M')$. Hence, in particular, $\alpha \bowtie_{CRG_N} \beta$ implies $\mathrm{eff}_N(\alpha) = \mathrm{eff}_N(\beta)$. As a result, *CE* holds.

Solving step transition systems A step transition system STS is *solvable* if there is a PT-net N such that $STS \simeq CRG_N$. This is the standard definition used in several works concerned with the synthesis of Petri nets from transition systems. In this paper, we will also use a more general notion of solvability, defined for step transition systems with multiple initial states.

[4] M' is a multiset due to $\mathrm{pre}_N(\alpha) \leq M$.

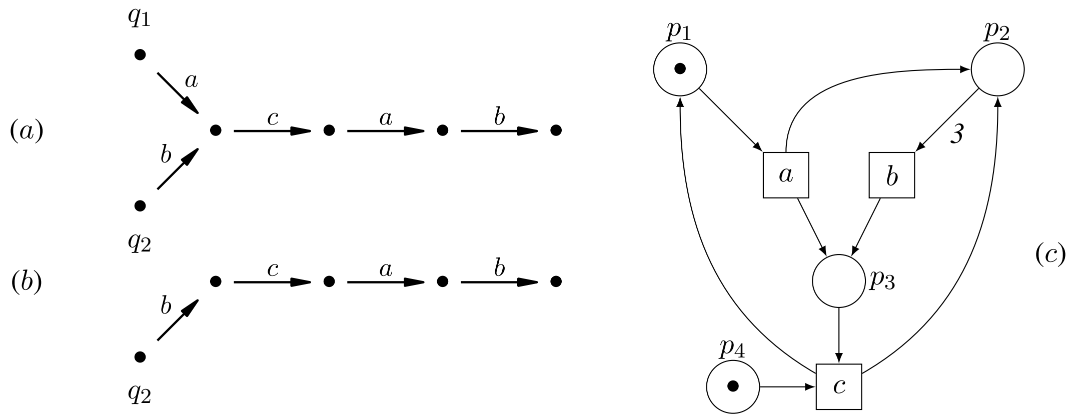


Figure 1. A step transition system with multiple initial states STS (a); step transition system STS_{q_2} (b); and Petri net solving STS_{q_1} (c).

A *step transition system with multiple initial states* is a tuple $STS = (S, T, \to, S_0)$ such that the first three components are as in the definition of a step transition system, and $S_0 \subseteq S$ is a nonempty set of initial states. Moreover, for every $r \in S_0$, $STS_r = (S_r, T, \to_r, r)$ is the step transition system such that $S_r = \{s \in S \mid r \in \mathrm{pred}_{STS}(s)\}$ and $\to_r = \to \cap (S_r \times \mathrm{mult}(T) \times S_r)$. That is, STS_r is STS restricted to those states which are reachable from r.

A step transition system with multiple initial states STS is *solvable* if there is an unmarked PT-net (P, T, F) and a mapping $\psi : S \to \mathrm{mult}(P)$ such that $STS_r \simeq_{\psi|_{S_r}} CRG_{(P,T,F,\psi(r))}$, for every $r \in S_0$. That is, a solution in this case is an unmarked PT-net which can be 'started' in different initial markings, each such initial marking solving one of the step transition systems which make up STS.

Example 2.5. Let us consider $STS = (\{q_1, \ldots, q_6\}, \{a, b, c\}, \to, \{q_1, q_2\})$, a step transition system with multiple initial states depicted in Figure 1(a) (for simplicity, all nonempty steps are singletons).

The step transition system STS_{q_2}, depicted on Figure 1(b), is obtained from STS by removing all the states which are not reachable from q_2. STS_{q_1} is constructed in similar way. The PT-net $N = (P, T, F, (p_1 p_4))$ solving STS_{q_1} is depicted on Figure 1(c). As $N = (P, T, F, p_2^4 + p_4)$ is a solution for STS_{q_2}, it follows that STS is solvable. $\diamondsuit$

3. Reversing steps

The reverse action of an action a in a step transition system STS or a PT-net N will be denoted by $\overline{a}$. Intuitively, $\overline{a}$ cancels the effect of a which corresponds to $a + \overline{a} \bowtie_{STS} \varnothing$ and $\mathrm{eff}_N(a) + \mathrm{eff}_N(\overline{a}) = 0$, respectively.

We consider four ways of modifying step transition systems to capture the effect of reversing actions. In the first three, each action a has a unique *reverse action* $\overline{a}$. Moreover, the reverse $\overline{\alpha}$ of a multiset α of actions is obtained by replacing each action occurrence in α by its reverse. In

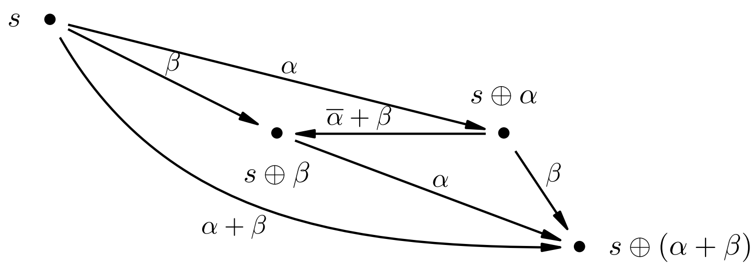


Figure 2. A mixed reverse transition $s \oplus \alpha \xrightarrow{\overline{\alpha}+\beta}_{mrev} s \oplus \beta$ derived from $s \xrightarrow{\alpha+\beta}_{STS}$.

the fourth one, an action a has possibly multiple unique *indexed reverse actions* $\overline{a}_{\langle idx \rangle}$. The *index-free* version $\mathrm{noidx}(\alpha)$ of a multiset α is obtained by replacing each $\overline{a}_{\langle idx \rangle}$ in α by $\overline{a}$. For example, $\mathrm{noidx}((\overline{a}_{\langle 7 \rangle}\, \overline{b}_{\langle s,w \rangle}\, \overline{b}\, \overline{a}_{\langle f \rangle})) = (\overline{a}\,\overline{b}\,\overline{b}\,\overline{a}) = \overline{(abba)}$.

In the domain of step transition systems, reversing is introduced at the behavioural level. The *direct/set/mixed reverse* of a CEST-system $STS = (S, T, \rightarrow, s_0)$ is respectively given by:

$$
\begin{aligned}
STS^{rev} &= (S, T \uplus \overline{T}, \rightarrow \cup \rightarrow_{rev}, s_0) \text{ with } \rightarrow_{rev} = \{(s \oplus \alpha, \overline{\alpha}, s) \mid s \xrightarrow{\alpha}_{STS}\}\\
STS^{srev} &= (S, T \uplus \overline{T}, \rightarrow \cup \rightarrow_{srev}, s_0) \text{ with } \rightarrow_{srev} = \{(s \oplus \alpha, \overline{\alpha}, s) \mid s \xrightarrow{\alpha}_{STS} \wedge \mathrm{supp}(\alpha) = \alpha\}\\
STS^{mrev} &= (S, T \uplus \overline{T}, \rightarrow_{mrev}, s_0) \quad \text{ with } \rightarrow_{mrev} = \{(s \oplus \alpha, \overline{\alpha} + \beta, s \oplus \beta) \mid s \xrightarrow{\alpha+\beta}_{STS}\}\,.
\end{aligned}
$$

That is, $\rightarrow_{rev}$ reverses *all* the (original) *steps*, $\rightarrow_{srev}$ *only* reverses the steps that are *sets*, and $\rightarrow_{mrev}$ introduces *partial* reverses with *mixed* steps, including both the original and reverse actions. Figure 2 illustrates mixed reversing. Note that $s \oplus \alpha$ and $s \oplus \beta$ are states in STS due to *SEQ* and *CE*.

A *split reverse* of STS is a step transition system $STS^{split} = (S, T \uplus T', \rightarrow', s_0)$ satisfying *SEQ* and such that $T \cap \mathrm{noidx}(T') = \varnothing$ and $\mathrm{noidx}(STS^{split}) = STS^{rev}$, where $\mathrm{noidx}(STS^{split}) = (S, T \cup \mathrm{noidx}(T'), \{(s, \mathrm{noidx}(\alpha), s') \mid (s, \alpha, s') \in \rightarrow'\}, s_0)$ is the step transition system obtained from STS by replacing each occurrence of an indexed reverse action $\overline{a}_{\langle idx \rangle}$ by $\overline{a}$. That is, $\rightarrow'$ introduces split reverses allowing one or more reverses of a step, possibly using different reverses of the same action when reversing a step that contains its multiple copies.

In the domain of PT-nets, reversing is introduced structurally rather than behaviourally, by adding reverses at the level of actions:

- A PT-net N with *reverses* is such that, for each original action a, there is a reverse action $\overline{a}$ such that $\mathrm{eff}_N(\overline{a}) = -\,\mathrm{eff}_N(a)$.

- A PT-net N with *strict reverses* is such that, for each original action a, there is a reverse action $\overline{a}$ such that $\mathrm{pre}_N(\overline{a}) = \mathrm{post}_N(a)$ and $\mathrm{post}_N(\overline{a}) = \mathrm{pre}_N(a)$.

- A PT-net N with *split reverses* is such that, for each original action a, there is at least one indexed reverse action $\overline{a}_{\langle idx \rangle}$ such that $\mathrm{eff}_N(\overline{a}_{\langle idx \rangle}) = -\,\mathrm{eff}_N(a)$.

A key problem which then arises is that of characterising relationships between statically defined reversing of PT-nets and the behavioural reversing of their concurrent reachability graphs. In the rest of this paper, we will address this problem by providing both negative and positive results. First, however, we show basic properties of the reversed step transition systems. In particular, that all such step transition systems are CEST-systems, and that the solvability of a reversed step transition system implies the solvability of the original step transition system.

Theorem 3.1. Let STS be a CEST-system, and STS^{split} be any of its split reverses.

1. $STS \blacktriangleleft STS^{srev} \blacktriangleleft STS^{rev} \blacktriangleleft STS^{mrev}$ and $STS \blacktriangleleft STS^{split}$.
2. STS^{mrev}, STS^{srev}, STS^{rev}, and STS^{split} are CEST-systems.
3. If any step transition system among STS^{mrev}, STS^{srev}, STS^{rev}, and STS^{split} is solvable, then STS is also solvable.

Proof:
Let $STS = (S, T, \to, s_0)$ and STS' be any step transition system among STS^{mrev}, STS^{srev}, STS^{rev}, and STS^{split}. We start with an auxiliary result.

Lemma 3.2. Let $\alpha, \beta, \gamma, \delta \in \mathrm{mult}(T)$.

1. $s \xrightarrow{\alpha}_{STS^{mrev}} s'$ iff $s \xrightarrow{\alpha}_{STS^{rev}} s'$ iff $s \xrightarrow{\alpha}_{STS^{split}} s'$ iff $s \xrightarrow{\alpha}_{STS} s'$.
2. $s \xrightarrow{\overline{\alpha}}_{STS^{mrev}} s'$ iff $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} s'$.

Proof:
[Lemma 3.2] (1) The second and third equivalences are obvious, so we only show the first one.

($\Longrightarrow$) Suppose that $s \xrightarrow{\alpha}_{STS^{mrev}} s'$. Then, by the definition of STS^{mrev}, there is $r \in S$ such that $r \xrightarrow{\varnothing+\alpha}_{STS}$ and $(s =) r \oplus \varnothing \xrightarrow{\overline{\varnothing}+\alpha}_{STS^{mrev}} r \oplus \alpha (= s')$. By Proposition 2.1(2), $s = r$. Hence, by Proposition 2.1(2), $r \oplus \alpha = s \oplus \alpha = s'$. As a result, $s \xrightarrow{\alpha}_{STS} s'$.

($\Longleftarrow$) Suppose that $s \xrightarrow{\alpha}_{STS} s'$. Then $s \xrightarrow{\varnothing+\alpha}_{STS}$ and so, by the definition of STS^{mrev}, $s \oplus \varnothing \xrightarrow{\overline{\varnothing}+\alpha}_{STS^{mrev}} s \oplus \alpha$. By Proposition 2.1(1), $s' = s \oplus \alpha$, and, by Proposition 2.1(2), $s = s \oplus \varnothing$. Hence $s \xrightarrow{\alpha}_{STS^{mrev}} s'$.

(2) ($\Longrightarrow$) Suppose that $s \xrightarrow{\overline{\alpha}}_{STS^{mrev}} s'$. Then, by the definition of STS^{mrev}, there is $r \in S$ such that $(s =) r \oplus \alpha \xrightarrow{\overline{\alpha}+\varnothing}_{STS^{mrev}} r \oplus \varnothing (= s')$ and $r \xrightarrow{\alpha+\varnothing}_{STS}$. By Proposition 2.1(2), $s' = r$. Hence $s' \xrightarrow{\alpha}_{STS} s$. Thus, by the definition of STS^{rev}, $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} s'$.

($\Longleftarrow$) Suppose that $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} s'$. Then, by the definition of STS^{rev}, $s' \xrightarrow{\alpha+\varnothing}_{STS} s$. Hence, by definition of STS^{mrev}, $s' \oplus \alpha \xrightarrow{\overline{\alpha}+\varnothing}_{STS^{mrev}} s' \oplus \varnothing$. By Proposition 2.1(1), $s = s' \oplus \alpha$, and, by Proposition 2.1(2), $s' = s' \oplus \varnothing$. Hence $s \xrightarrow{\alpha}_{STS^{mrev}} s'$. [Lemma 3.2] □

(1) Follows directly from the definitions and Lemma 3.2(1,2).

(2) We discuss in turn the four properties defining CEST-systems.

(*EL* and *REA*) Follow directly from part (1) and the fact that STS satisfies *EL* and *REA*.

(*SEQ*) For STS^{srev}, STS^{rev}, and STS^{split}, *SEQ* holds directly from the definitions. To show *SEQ* for STS^{mrev}, suppose that:

$$s \xrightarrow{\alpha_1+\alpha_2+\beta_1+\beta_2}_{STS} \text{ and } s \oplus (\alpha_1+\alpha_2) \xrightarrow{\overline{\alpha}_1+\overline{\alpha}_2+\beta_1+\beta_2}_{STS^{mrev}} s \oplus (\beta_1+\beta_2)\ .$$

Then, by *SEQ* for STS, we have $s \oplus \alpha_2 \xrightarrow{\alpha_1+\beta_1}_{STS}$ and $s \oplus \beta_1 \xrightarrow{\alpha_2+\beta_2}_{STS}$. Hence, by the definition of STS^{mrev},

$$\begin{array}{lcl} (s \oplus \alpha_2) \oplus \alpha_1 & \xrightarrow{\overline{\alpha}_1+\beta_1}_{STS^{mrev}} & (s \oplus \alpha_2) \oplus \beta_1 \\ (s \oplus \beta_1) \oplus \alpha_2 & \xrightarrow{\overline{\alpha}_2+\beta_2}_{STS^{mrev}} & (s \oplus \beta_1) \oplus \beta_2\ . \end{array}$$

Moreover, by Proposition 2.2, we have:

$$\begin{array}{l} s \oplus (\alpha_2+\alpha_1) = (s \oplus \alpha_2) \oplus \alpha_1 \\ (s \oplus \beta_1) \oplus \beta_2 = s \oplus (\beta_1+\beta_2) \\ (s \oplus \alpha_2) \oplus \beta_1 = s \oplus (\alpha_2+\beta_1) = (s \oplus \beta_1) \oplus \alpha_2\ . \end{array}$$

Hence, $s \oplus (\alpha_1+\alpha_2) \xrightarrow{\overline{\alpha}_1+\beta_1}_{STS^{mrev}} s \oplus (\alpha_2+\beta_1) \xrightarrow{\overline{\alpha}_2+\beta_2}_{STS^{mrev}} s \oplus (\beta_1+\beta_2)$.

(*CE*) We first observe that $s \xrightarrow{\overline{a}}_{STS^{mrev}} s'$ implies $s' \xrightarrow{\overline{a}}_{STS^{mrev}} s$, by Lemma 3.2 and the definition of STS^{rev} *(*)*.

We have already demonstrated that *SEQ* holds for STS'. Hence, by Propositions 2.3, it suffices to show that *CE* holds for $(STS')^{seq}$.

By Propositions 2.3, we have that STS^{seq} satisfies *CE*. Moreover, by Lemma 3.2(1) as well as the definition of STS' and *(*)*, $(STS')^{seq}$ can be derived by a successive application of the construction from the formulation of Proposition 2.4 (once for each reverse action and indexed reverse action). Hence, by Propositions 2.4, $(STS')^{seq}$ satisfies *CE*.

(3) Let $N' = (P, T', F, M_0)$ be a PT-net such that $STS' \simeq_\psi CRG_{N'}$. We will show that $STS \simeq_\psi CRG_N$, where $N = N'|_T$. Note that the enabling and firing of steps over T is exactly the same in both N and N' *(*)*.

We first observe that $\psi(s_0) = M_0$. Suppose then that $s \in S$ and $\psi(s) \in \mathrm{reach}_N$. To show that the executions of steps are preserved by ψ in both directions, we consider two cases for $\alpha \in \mathrm{mult}(T)$.

Case 1: $s \xrightarrow{\alpha}_{STS} s'$. Then, by part (1), $s \xrightarrow{\alpha}_{STS'} s'$. Hence, by $STS' \simeq_\psi CRG_{N'}$, we have $\psi(s) \xrightarrow{\alpha}_{N'} \psi(s')$. Thus, by *(*)*, $\psi(s) \xrightarrow{\alpha}_N \psi(s')$.

Case 2: $\psi(s) \xrightarrow{\alpha}_N M$. Then, by *(*)*, $\psi(s) \xrightarrow{\alpha}_{N'} M$. Hence, by $STS' \simeq_\psi CRG_{N'}$, we have $M \in \psi(S)$ and $s \xrightarrow{\alpha}_{STS'} \psi^{-1}(M)$. Thus, by Lemma 3.2(1), $s \xrightarrow{\alpha}_{STS} \psi^{-1}(M)$. □

4. Multiset and set reversibility

The investigation of different notions of step reversibility starts with a straightforward but important negative result stating that, in the domain of PT-nets, the concept of direct reversibility — which directly generalises sequential reversibility and should be considered as the preferred way of reversing step transition systems — cannot handle steps which are true multisets.

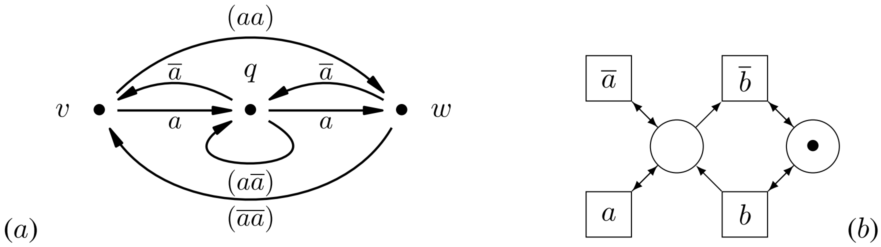


Figure 3. An illustration of the proof of Proposition 4.1 (a), and PT-net generating concurrent reachability graph which is not step-finite (b).

Proposition 4.1. Let STS be a CEST-system which is not a set transition system. Then STS^{rev} is not solvable.

Proof:
[Figure 3(a) illustrates the idea of the proof.] Let $STS = (S, T, \to, s_0)$. Suppose that STS^{rev} is solvable. Then there is a PT-net N such that $STS^{rev} \simeq_\psi CRG_N$ *(*)*. As STS is not a set transition system, there are $v \in S$ and $\alpha \in \mathrm{mult}(T)$ such that $v \xrightarrow{\alpha}_{STS}$ and $(aa) \leq \alpha$, for some $a \in T$.

By *SEQ* for STS and Theorem 3.1(1), there are $w, q \in S$ such that $v \xrightarrow{(aa)}_{STS^{rev}} w$ and $v \xrightarrow{a}_{STS^{rev}} q$ *(**)*. Hence, by the definition of STS^{rev}, $w \xrightarrow{(\overline{a}\overline{a})}_{STS^{rev}} v$ *(***)*.

Let $M_s = \psi(s)$, for $s \in \{v, w, q\}$. By the definition of STS^{rev} and *(*)*, the step $\beta = (a\overline{a})$ is not enabled at M_q. Hence, there is a place p of N such that $M_q(p) < \mathrm{pre}_N(\beta)(p)$ *(†)*. On the other hand, by *(**)* and *(***)*, we have:

$$\mathrm{pre}_N(aa) \leq M_v \qquad \mathrm{pre}_N(\overline{a}\overline{a}) \leq M_w \qquad M_w = M_v + \mathrm{eff}_N(aa) \qquad M_q = M_v + \mathrm{eff}_N(a)\ .$$

Thus $\mathrm{pre}_N(\beta) + \mathrm{pre}_N(\beta) = \mathrm{pre}_N(aa\overline{a}\overline{a}) \leq M_v + M_w = M_v + M_v + \mathrm{eff}_N(aa) = M_q + M_q$, yielding a contradiction with *(†)*. □

In view of Proposition 4.1, when facing the problem of implementing a reverse of non-set step transition system STS using PT-nets, one may consider set reversibility based on STS^{srev}, or mixed reversibility based on STS^{mrev}.[5]

Among these two options, one might prefer STS^{srev} to STS^{mrev} as the latter introduces steps containing both the original and reverse actions. However, as the next example shows, it not always possible to 'replace' a mixed reversibility solution by a set reversibility solution.

Example 4.2. Let us consider a CEST-system $STS = (\{s_0, s_1, \dots\}, \{a, b\}, \to, s_0)$ such that:

$$s_i \xrightarrow{a^j}_{STS} s_i \qquad \text{and} \qquad s_i \xrightarrow{b+a^j}_{STS} s_{i+1} \ \text{ for all } i \geq 0 \text{ and } j \leq i\ .$$

[5]We will discuss split reversibility separately in Section 7.

It is straightforward to see that STS^{mrev} is solvable by the PT-net shown in Figure 3(b). However, STS^{srev} is *not* solvable by any PT-net. If such a PT-net N existed, then it would have distinct reachable markings $M_0, M_1, \ldots$ satisfying, for every $i \geq 0$:

$$M_i \xrightarrow{b}_N M_{i+1} \text{ (*)} \qquad M_i \xrightarrow{a^i}_N M_i \text{ (**)} \qquad M_i \xrightarrow{\overline{a}}_N M_i \text{ (***)} \qquad \neg M_i \xrightarrow{(a\overline{a})}_N \text{ (†)} .$$

We now observe that $M_0 \leq M_1 \leq \cdots$ due to *(*)*. Hence, there is a place p such that $\mathrm{pre}_N(a\overline{a})(p) > M_0(p) = M_1(p) = \cdots$ *(‡)*, due to *(†)* and the finiteness of N. On the other hand, $\mathrm{pre}_N(\overline{a})(p) \leq M_0(p) = M_1(p) = \cdots$ due to *(***)*, and $\mathrm{pre}_N(a)(p) = 0$ due to *(**)* and *(‡)*. As a result, $\mathrm{pre}_N(a\overline{a})(p) \leq M_0(p)$, yielding a contradiction with *(‡)*. ◊

Example 4.2 demonstrated that there are step transition systems which can be treated using mixed reversibility, but not using set reversibility. What is more, the example worked because the step transition system considered was not step-finite. As the next result shows, that was the only reason why set reversibility failed to hold.

Theorem 4.3. Let STS be a CEST-system such that STS^{mrev} is solvable. Then STS^{srev} is solvable if and only if STS is step-finite.

Proof:

Let $STS = (S, T, \rightarrow, s_0)$.

($\Longrightarrow$) Suppose that STS^{srev} is solvable by a PT-net $N = (P, T \cup \overline{T}, F, M_0)$, and that STS is not step-finite. By the finiteness of P and T as well as *SEQ* for STS, there is $a \in T$ and reachable markings $M_1 \leq M_2 \leq \ldots$ such that $M_i \xrightarrow{a^i}_N$, for every $i \geq 1$. Hence, by *SEQ* for CRG_N, there is a marking M_i' such that $M_i \xrightarrow{a}_N M_i'$ and $M_i' \xrightarrow{a^{i-1}}_N$ *(∗)*, for every $i \geq 1$. As a result, $M_i' \xrightarrow{a}_N$ and $M_i' \xrightarrow{\overline{a}}_N$ *(∗∗)*, for every $i \geq 2$.

We now observe that $(M =) M'_{m+2} \xrightarrow{(a\overline{a})}_N$, where $m = max\{F(p,\overline{a}) \mid p \in P\}$. Indeed, otherwise there is $p \in P$ such that $M(p) < F(p,a) + F(p,\overline{a}) \leq F(p,a) + m$ *(†)*. On the other hand, by *(∗∗)*, $M(p) \geq F(p,a)$ and $M(p) \geq F(p,\overline{a})$. Hence, it must be the case that $F(p,a) > 0$. Thus, by *(∗)*, $M(p) \geq (m+1) \cdot F(a,p) = m + F(a,p)$, contradicting *(†)*. As a result, $M \xrightarrow{(a\overline{a})}_N$, yielding a contradiction with our initial assumption.

($\Longleftarrow$) If STS is step-finite, then there is $k \geq 1$ such that $|\alpha| \leq k$, whenever $s \xrightarrow{\alpha}_{STS}$. Moreover, since STS^{mrev} is solvable, there exists a PT-net $N = (P, T \cup \overline{T}, F, M_0)$ such that $STS^{mrev} \simeq_\psi CRG_N$. We then modify N, by adding to P a set of fresh places $P' = \{p_{ab} \mid a \in T \wedge b \in \overline{T}\}$. Each p_{ab} is such that $M_0(p_{ab}) = k$ and has four non-zero connections, $F(a, p_{ab}) = F(p_{ab}, a) = 1$ and $F(b, p_{ab}) = F(p_{ab}, b) = k$. For the resulting PT-net N', we have $STS^{srev} \simeq_{\psi'} CRG_{N'}$, where $\psi'(s) = \psi(s) + \sum_{p \in P'} p^k$, for every $s \in S$. □

We have therefore obtained a full characterisation of step transition systems for which mixed reversibility solutions can be replaced by set reversibility solutions. In addition, the second part of the proof of Theorem 4.3 provides a straightforward construction achieving this.

A direct corollary of the last result is that for a set step transition system it is always possible to replace a mixed reversibility solution by a set reversibility solution.

Theorem 4.4. Let STS be a set CEST-system. If STS^{mrev} is solvable, then STS^{rev} is also solvable.

Proof:
As a set CEST-system, STS is step-finite and $STS^{rev} = STS^{srev}$. Hence the result follows from Theorem 4.3. □

A concluding observation is that all three versions of reversibility which do not involve splitting are worthy of investigation.

5. Mixed reversibility

In this section, we consider the problem of deciding whether the mixed reverse STS^{mrev} of a solvable step transition system STS is also solvable. A specific concern we implicitly address is the size of STS^{mrev} which (in the finite case) can be exponentially larger than that of STS. The aim is therefore to avoid dealing directly with STS^{mrev}. As shown below, this is possible as the checking of feasibility of mixed reversing can be replaced by checking the solvability of the original transition system, and the solvability of its reverse.

Throughout this section we make the following assumptions:

- $STS = (S, T, \to, s_0)$ is a CEST-system and R is a home cover of STS.
- $\overline{STS} = (S, \overline{T}, \{(s', \overline{\alpha}, s) \mid s \xrightarrow{\alpha}_{STS} s'\}, R)$ is a step transition system with multiple initial states.
- $\overline{STS}_r = (S_r, \overline{T}, \to_r, r)$ is a step transition system such that $r \in R$, $S_r = \{s \in S \mid r \in \mathrm{pred}_{STS}(s)\}$, and $\to_r = \to \cap (S_r \times \mathrm{mult}(T) \times S_r)$.

That is, $\overline{STS}$ is obtained by reversing each transition of STS, and considering all the states in the home cover R as the initial states.

Proposition 5.1. Let $r \in R$.

1. $\overline{STS}_r$ is a CEST-system.
2. $s_0 \in \bigcap_{s \in S_r} \mathrm{pred}_{STS}(s)$.
3. $S = \bigcup_{r \in R} S_r$.

Proof:
(1) The only non-trivial property to show is *CE*. For every $\mathcal{A}$-vector α with support $\overline{T}$, let $\widehat{\alpha}$ be the $\mathcal{A}$-vector with support T such that $\widehat{\alpha}(a) = -\alpha(\overline{a})$, for every $a \in T$.

We first observe that, for every $\pi \in \mathrm{paths}_{\overline{STS}_r}(s, s')$, there is $\pi' \in \mathrm{paths}_{STS}(s, s')$ such that $\mathrm{sign}(\pi') = \widehat{\mathrm{sign}(\pi)}$ *(*)*. Hence, we also have that $\alpha \bowtie_{\overline{STS}_r} \beta$ implies $\widehat{\alpha} \bowtie_{STS} \widehat{\beta}$, for all $\mathcal{A}$-vectors α and β with support $\overline{T}$ *(**)*. Thus, $\overline{STS}_r$ satisfies *CE* by *(*)* and *(**)*.

(2) Follows from the fact that STS satisfies *REA*.

(3) Follows from the fact that R is a home cover. □

Theorem 5.2. STS^{mrev} is solvable if and only if both STS and $\overline{STS}$ are solvable.

Proof:
($\Longrightarrow$) By Theorem 3.1(3), STS is solvable. To show that $\overline{STS}$ is solvable, suppose that $N = (P, T, F, M_0)$ is a PT-net such that $STS^{mrev} \simeq_\psi CRG_N$. We will show that $\overline{STS}_r \simeq_{\psi|_{S_r}} CRG_{N_r}$, where, for every $r \in R$, N_r is the PT-net $N|_{\overline{T}}$ with the initial marking set to $\psi(r)$. Note that the enabling and firing of steps over $\overline{T}$ is exactly the same in both N and N_r *(*)*.

We first observe that the initial states of $\overline{STS}_r$ and CRG_{N_r} are related by ψ. Suppose then that $s \in S_r$ is such that $\psi(s) \in \mathrm{reach}_{N_r}$. To show that the executions of steps are preserved by ψ in both directions, we consider two cases, where $\alpha \in \mathrm{mult}(T)$.

Case 1.1: $s \xrightarrow{\overline{\alpha}}_{\overline{STS}_r} s'$. Then $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} s'$ and so, by Lemma 3.2(2), $s \xrightarrow{\overline{\alpha}}_{STS^{mrev}} s'$. Hence, by $STS^{mrev} \simeq_\psi CRG_N$, we have $\psi(s) \xrightarrow{\overline{\alpha}}_N \psi(s')$. Thus, by *(*)*, $\psi(s) \xrightarrow{\overline{\alpha}}_{N_r} \psi(s')$.

Case 1.2: $\psi(s) \xrightarrow{\overline{\alpha}}_{N_r} M$. Then, by *(*)*, $\psi(s) \xrightarrow{\overline{\alpha}}_N M$. Hence, by $STS^{mrev} \simeq_\psi CRG_N$, we have $M \in \psi(S)$ and $s \xrightarrow{\overline{\alpha}}_{STS^{mrev}} \psi^{-1}(M)$. Thus, by Lemma 3.2(2), $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} \psi^{-1}(M)$. Hence $s \xrightarrow{\overline{\alpha}}_{\overline{STS}_r} \psi^{-1}(M)$.

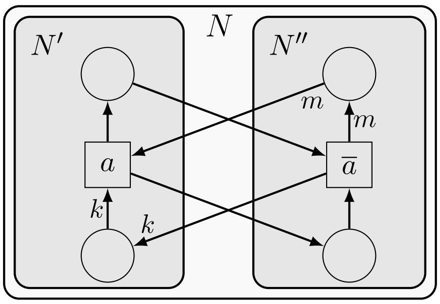


Figure 4. An illustration of the second part of the proof of Theorem 5.2.

($\Longleftarrow$) Since STS is solvable, there is a PT-net $N' = (P', T, F', M'_0)$ such that $STS \simeq_{\psi'} CRG_{N'}$. (Note that $\psi'(s_0) = M'_0$.) Moreover, since $\overline{STS}$ is solvable, there is an umarked PT-net $N'' = (P'', \overline{T}, F'')$ and a mapping $\psi'' : S \to \mathrm{mult}(P'')$ such that $\overline{STS}_r \simeq_{\psi''|_{S_r}} CRG_{N_r}$, where $N_r = (P'', \overline{T}, F'', M_r)$ and $M_r = \psi''(r)$, for every $r \in R$. Clearly, we may assume that $P' \cap P'' = \varnothing$ as the identities of places play no role in the solvability problems of STS and $\overline{STS}$.

Let $N = (P' \cup P'', T \cup \overline{T}, F, M_0)$ be the PT-net with strict reverses (illustrated in Figure 4) such that $M_0 = M'_0 \sqcup \psi''(s_0) = \psi'(s_0) \sqcup \psi''(s_0)$ and, for every $a \in T$:

$$\begin{array}{ll} \mathrm{pre}_N(a) = \mathrm{pre}_{N'}(a) \sqcup \mathrm{post}_{N''}(\overline{a}) & \mathrm{post}_N(a) = \mathrm{post}_{N'}(a) \sqcup \mathrm{pre}_{N''}(\overline{a}) \\ \mathrm{pre}_N(\overline{a}) = \mathrm{pre}_{N''}(\overline{a}) \sqcup \mathrm{post}_{N'}(a) & \mathrm{post}_N(\overline{a}) = \mathrm{post}_{N''}(\overline{a}) \sqcup \mathrm{pre}_{N'}(a)\ . \end{array} \tag{2}$$

Let ψ be a mapping with the domain S which, for every $s \in S$, returns $\psi'(s) \sqcup \psi''(s)$. Note that ψ is well-defined due to Lemma 5.1(3) and $\psi(s_0) = M_0$.

Lemma 5.3. Let STS' be CRG_N with all the transitions labelled by steps of the form $\alpha + \overline{\beta}$, for $\alpha, \beta \neq \varnothing$, deleted.

1. $STS^{rev} \simeq_{\psi} STS'$.
2. STS' satisfies *REA*.
3. $\psi(s \oplus \alpha) = \psi(s) + \mathrm{eff}_N(\alpha)$, for all $s \xrightarrow{\alpha}_{STS}$.

Proof:
[Lemma 5.3] (1) We observe that the initial states of STS^{rev} and STS' are related by ψ. Suppose now that $s \in S$ and $\psi(s) \in \mathrm{reach}_N$. To show that the executions of steps are preserved by ψ in both directions, we consider four cases, where $\alpha \in \mathrm{mult}(T)$.

Case 2.1: $s \xrightarrow{\alpha}_{STS^{rev}} s'$. Then, by $STS \simeq_{\psi'} CRG_{N'}$, we have $\psi'(s) \xrightarrow{\alpha}_{N'} \psi'(s')$ and $\psi'(s) \geq \mathrm{pre}_{N'}(\alpha)$. Moreover, $s' \xrightarrow{\overline{\alpha}}_{STS^{rev}} s$. Hence, by Lemma 5.1(3), there is $r \in R$ such that $s' \xrightarrow{\overline{\alpha}}_{\overline{STS}_r} s$. Thus, by $\overline{STS}_r \simeq_{\psi''|_{S_r}} CRG_{N_r}$, we have $\psi''(s') \xrightarrow{\overline{\alpha}}_{N''} \psi''(s)$ and $\psi''(s) \geq \mathrm{post}_{N''}(\overline{\alpha})$. Hence, by Eq.(2):

$$\psi(s) = (\psi'(s) \sqcup \psi''(s)) \geq (\mathrm{pre}_{N'}(\alpha) \sqcup \mathrm{post}_{N''}(\overline{\alpha})) = \mathrm{pre}_N(\alpha) \ .$$

As a result, $\psi(s) \xrightarrow{\alpha}_N \psi(s) + \mathrm{eff}_N(\alpha)$. Hence $\psi(s) \xrightarrow{\alpha}_N \psi(s')$ as we have, by Eq.(2):

$$\begin{aligned}
\psi(s) + \mathrm{eff}_N(\alpha) &= (\psi'(s) \sqcup \psi''(s)) + \mathrm{post}_N(\alpha) - \mathrm{pre}_N(\alpha) \\
&= (\psi'(s) \sqcup \psi''(s)) + (\mathrm{post}_{N'}(\alpha) \sqcup \mathrm{pre}_{N''}(\overline{\alpha})) - (\mathrm{pre}_{N'}(\alpha) \sqcup \mathrm{post}_{N''}(\overline{\alpha})) \\
&= (\psi'(s) + \mathrm{eff}_{N'}(\alpha)) \sqcup (\psi''(s) - \mathrm{eff}_{N''}(\overline{\alpha})) \\
&= \psi'(s') \sqcup \psi''(s') = \psi(s') \ .
\end{aligned}$$

Case 2.2: $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} s'$. Then $s' \xrightarrow{\alpha}_{STS^{rev}} s$ and so, by Case 2.1, $\psi(s') \xrightarrow{\alpha}_N \psi(s)$. Hence, since N is a PT-net with strict reverses, $\psi(s) \xrightarrow{\overline{\alpha}}_N \psi(s')$.

Case 2.3: $\psi(s) \xrightarrow{\alpha}_N M$. Then, by Eq.(2), we have:

$$\begin{aligned}
\psi'(s) \sqcup \psi''(s) = \psi(s) &\geq \mathrm{pre}_N(\alpha) = \mathrm{pre}_{N'}(\alpha) \sqcup \mathrm{post}_{N''}(\overline{\alpha}) \\
M = \psi(s) + \mathrm{eff}_N(\alpha) &= (\psi'(s) \sqcup \psi''(s)) + (\mathrm{post}_{N'}(\alpha) \sqcup \mathrm{pre}_{N''}(\overline{\alpha})) - (\mathrm{pre}_{N'}(\alpha) \sqcup \mathrm{post}_{N''}(\overline{\alpha})) \ .
\end{aligned}$$

Hence, by $P' \cap P'' = \varnothing$, $\psi'(s) \geq \mathrm{pre}_{N'}(\alpha)$ and $\psi''(s) \geq \mathrm{post}_{N''}(\overline{\alpha})$ as well as:

$$M|_{P'} = \psi'(s) + \mathrm{eff}_{N'}(\alpha) \qquad \text{and} \qquad M|_{P''} = \psi''(s) - \mathrm{eff}_{N''}(\overline{\alpha}) \ .$$

Thus $\psi'(s) \xrightarrow{\alpha}_{N'} M|_{P'}$. Hence, by $STS \simeq_{\psi'} CRG_{N'}$, we obtain $M|_{P'} \in \psi'(S)$ and $s \xrightarrow{\alpha}_{STS^{rev}} s'$, where $\psi'(s') = M|_{P'}$. We still need to show that $\psi(s') = M$. This follows from $\psi''(s') = M|_{P''}$. Indeed, we have $s' \xrightarrow{\overline{\alpha}}_{STS^{rev}} s$ and so, by Lemma 5.1(3), there is $r \in R$ such that $s' \in S_r$. Now, by $\overline{STS}_r \simeq_{\psi''|_{S_r}} CRG_{N_r}$, $\psi''(s') \xrightarrow{\overline{\alpha}}_{N''} \psi''(s)$, which means that $\psi''(s') = \psi''(s) - \mathrm{eff}_{N''}(\overline{\alpha}) = M|_{P''}$.

Case 2.4: $\psi(s) \xrightarrow{\overline{\alpha}}_N M$. Then, by Eq.(2), we have:

$$\psi'(s) \sqcup \psi''(s) = \psi(s) \geq \mathrm{pre}_N(\overline{\alpha}) = \mathrm{pre}_{N''}(\overline{\alpha}) \sqcup \mathrm{post}_{N'}(\alpha)$$
$$M = (\psi'(s) \sqcup \psi''(s)) + (\mathrm{post}_{N''}(\overline{\alpha}) \sqcup \mathrm{pre}_{N'}(\alpha)) - (\mathrm{pre}_{N''}(\overline{\alpha}) \sqcup \mathrm{post}_{N'}(\alpha)) \; .$$

Hence, by $P' \cap P'' = \varnothing$, $\psi'(s) \geq \mathrm{post}_{N'}(\alpha)$ and $\psi''(s) \geq \mathrm{pre}_{N''}(\overline{\alpha})$ as well as:

$$M|_{P'} = \psi'(s) - \mathrm{eff}_{N'}(\alpha) \qquad \text{and} \qquad M|_{P''} = \psi''(s) + \mathrm{eff}_{N''}(\overline{\alpha}) \; .$$

Thus $\psi''(s) \xrightarrow{\overline{\alpha}}_{N''} M|_{P''}$. Hence, by Lemma 5.1(3), there is $r \in R$ such that $s \in S_r$. Thus, by $\overline{STS}_r \simeq_{\psi''|_{S_r}} CRG_{N_r}$, $M|_{P''} \in \psi''(S)$ and $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} s'$, where $\psi''(s') = M|_{P''}$. We still need to show that $\psi(s) = M$. This follows from $\psi'(s') = M|_{P'}$. Indeed, we have $s' \xrightarrow{\alpha}_{STS^{rev}} s$ and so, by $STS \simeq_{\psi'} CRG_{N'}$, we obtain $\psi'(s') \xrightarrow{\alpha}_{N'} \psi'(s)$, which means that $\psi'(s') = \psi'(s) - \mathrm{eff}_{N'}(\alpha) = M|_{P'}$.

(2) The modification of CRG_N does not produce unreachable states since CRG_N satisfies *SEQ*.

(3) Follows from part (1) and the forward determinism of STS and CRG_N. [Lemma 5.3] □

Returning to the proof of $STS^{mrev} \simeq_\psi CRG_N$, suppose that $s \in S$ is such that $\psi(s) \in \mathrm{reach}_N$ and consider two cases, where $\alpha, \beta \in \mathrm{mult}(T)$.

Case 3.1: $s \xrightarrow{\alpha+\beta}_{STS}$ and $s \oplus \alpha \xrightarrow{\overline{\alpha}+\beta}_{STS^{mrev}} s \oplus \beta$. Then we have $s \xrightarrow{\alpha+\beta}_{STS^{rev}}$ as well as:

$$s \xrightarrow{\alpha}_{STS} s \oplus \alpha \qquad s \xrightarrow{\beta}_{STS} s \oplus \beta \qquad s \xrightarrow{\alpha}_{STS^{rev}} s \oplus \alpha \qquad s \xrightarrow{\beta}_{STS^{rev}} s \oplus \beta \; .$$

Hence, by Lemma 5.3(1,3), we have:

$$\psi(s) \xrightarrow{\alpha+\beta}_N \quad \psi(s) \xrightarrow{\alpha}_N \psi(s \oplus \alpha) = \psi(s) + \mathrm{eff}_N(\alpha) \quad \psi(s) \xrightarrow{\beta}_N \psi(s \oplus \beta) = \psi(s) + \mathrm{eff}_N(\beta) \; .$$

Thus $\psi(s) \geq \mathrm{pre}_N(\alpha + \beta)$, and so $\psi(s) + \mathrm{eff}_N(\alpha) \geq \mathrm{pre}_N(\alpha + \beta) + \mathrm{eff}_N(\alpha) = \mathrm{pre}_N(\overline{\alpha} + \beta)$ due to Eq.(2). Hence, again by Eq.(2):

$$\psi(s \oplus \alpha) = \psi(s) + \mathrm{eff}_N(\alpha) \xrightarrow{\overline{\alpha}+\beta}_N \psi(s) + \mathrm{eff}_N(\alpha) + \mathrm{eff}_N(\overline{\alpha} + \beta) = \psi(s) + \mathrm{eff}_N(\beta) = \psi(s \oplus \beta) \; .$$

Case 3.2: $\psi(s) \xrightarrow{\overline{\alpha}+\beta}_N M$. Then $\psi(s) \xrightarrow{\overline{\alpha}}_N \psi(s) + \mathrm{eff}_N(\overline{\alpha}) (= M')$. Hence, by Lemma 5.3(1), $s \xrightarrow{\overline{\alpha}}_{STS^{rev}} \psi^{-1}(M') (= s')$. Thus, by the definition of STS^{rev}, $s' \xrightarrow{\alpha}_{STS} s = s' \oplus \alpha$. We then observe that, by Eq.(2):

$$M' = \psi(s) + \mathrm{eff}_N(\overline{\alpha}) \geq \mathrm{pre}_N(\overline{\alpha} + \beta) + \mathrm{eff}_N(\overline{\alpha}) = \mathrm{pre}_N(\alpha + \beta) \; .$$

Hence $M' \xrightarrow{\alpha+\beta}_N$ and so, by Lemma 5.3(1), $s' \xrightarrow{\alpha+\beta}_{STS^{rev}}$ and, as a consequence, $s' \xrightarrow{\alpha+\beta}_{STS}$ and $s' \xrightarrow{\beta}_{STS}$. Hence, by the definition of STS^{mrev}, $s' \oplus \alpha \xrightarrow{\overline{\alpha}+\beta}_{STS^{mrev}} s' \oplus \beta$. Moreover,

$$\psi(s' \oplus \alpha) = \psi(s') + \mathrm{eff}_N(\alpha) = M' + \mathrm{eff}_N(\alpha) = \psi(s) + \mathrm{eff}_N(\overline{\alpha}) + \mathrm{eff}_N(\alpha) = \psi(s)$$
$$\psi(s' \oplus \beta) = \psi(s') + \mathrm{eff}_N(\beta) = M' + \mathrm{eff}_N(\beta) = \psi(s) + \mathrm{eff}_N(\overline{\alpha}) + \mathrm{eff}_N(\beta) = M \; ,$$

by Lemma 5.3(3) and Eq.(2). □

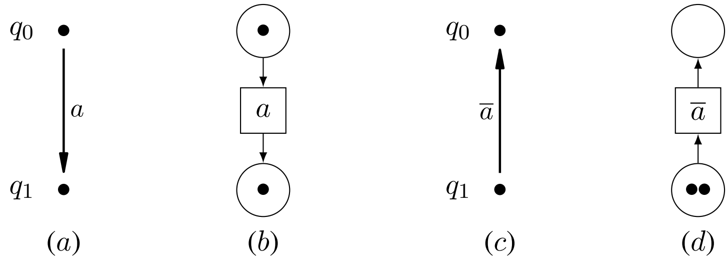


Figure 5. Reversing a solution does not give a solution to reversing (Example 5.4).

As the next example shows, reversing a solution of STS may not lead to a solution of $\overline{STS}$. Hence, in general, one needs to consider finding solutions to both STS and $\overline{STS}$.

Example 5.4. Let us consider STS, a step transition system depicted in Figure 5(a), and its only home state q_1. The PT-net N depicted Figure 5(b) solves STS. However, the direct reverse of N with the initial marking corresponding to q_1, depicted in Figure 5(d), does not solve the step transition system $\overline{STS}_{q_1}$ shown in Figure 5(c). ♢

As the set of all the states of a step transition system is a home set, Theorem 5.2 is *fundamental* as it provides a way of solving mixed reversibility using (much) simpler synthesis problems. In particular, if one is interested whether the mixed reverse CRG_N^{mrev} of the concurrent reachability graph of a PT-net N is solvable when CRG_N has a home state.

Theorem 5.5. If r is a home state of STS, then STS^{mrev} is solvable if and only if both STS and $\overline{STS}_r$ are solvable.

Proof:
Follows directly from Theorems 5.2. □

The above result and the proof of Theorem 5.2 provide a method for *constructing* a PT-net implementing mixed step reversibility provided that one can synthesise PT-nets for two step transition systems using, e.g., theory of regions [1, 12].

The method for checking the solvability of mixed reversibility easily extends to checking direct reversibility of set transition systems.

Theorem 5.6. Let STS be a set transition system and r be a home state of STS. Then STS^{rev} is solvable if and only if both STS and $\overline{STS}_r$ are solvable.

Proof:
($\Longrightarrow$) Let $STS^{rev} \simeq_\psi CRG_N$. Then $STS \simeq_\psi CRG_{N|_T}$ and $\overline{STS}_r \simeq_\psi CRG_{N'}$, where N' is $N|_{\overline{T}}$ with the initial marking set to $\psi(r)$.

($\Longleftarrow$) Follows from Theorems 5.2 and 4.4. □

6. From sequential reversibility to step reversibility

Checking the feasibility of step reversibility is, in general, a difficult task. The next result shows that in certain cases it is possible to proceed more effectively, if one is given a PT-net that solves the original step transition system, over-approximates its reverse containing only spikes, and under-approximates its mixed reverse.

Theorem 6.1. Let $STS = (S, T, \rightarrow, s_0)$ be a CEST-system and $N = (P, T \cup \overline{T}, F, M_0)$ be a PT-net such that:

$$(STS^{spike})^{rev} \lhd CRG_N \lhd STS^{mrev} \qquad \text{and} \qquad STS \simeq CRG_{N|_T} . \tag{3}$$

Then STS^{mrev} is solvable.

Proof:
The states as well as the initial states of $(STS^{spike})^{rev}$, STS^{mrev}, and STS are the same. Moreover, $((STS^{spike})^{rev}|_T)^{seq} = (STS^{mrev}|_T)^{seq} = STS^{seq}$. Similarly, the initial states of CRG_N and $CRG_{N|_T}$ are the same and we have $(CRG_N)|_T = CRG_{N|_T}$. We also observe that all step transition systems in Eq.(3) are CEST-systems, and there is a unique bijection ψ such that:

$$(STS^{spike})^{rev} \lhd_\psi CRG_N \qquad STS^{mrev} \lhd_\psi CRG_N \qquad STS \simeq_\psi CRG_{N|_T} . \tag{4}$$

By the first part of Eq.(3), *SEQ*, and the fact that we may assume that each action in T appears in the labels of the transitions of STS, we have:

$$\text{reach}_N = \text{reach}_{N|_T} \qquad \text{and} \qquad \text{eff}_N(a) = -\,\text{eff}_N(\overline{a}) \quad \text{for every } a \in T . \tag{5}$$

Lemma 6.2. It can be assumed that $\text{pre}_N(\overline{a}) \geq \text{post}_N(a)$ and $\text{post}_N(\overline{a}) \geq \text{pre}_N(a)$, for every $a \in T$.

Proof:
[Lemma 6.2] Suppose that $F(p, \overline{a}) < F(a, p)$, and so also $F(\overline{a}, p) > F(p, a)$. We then modify F to become F' which is the same as F except that $F'(p, \overline{a}) = F(a, p)$ and $F'(\overline{a}, p) = F(p, a)$. Let N' be the resulting PT-net. Clearly, $\text{eff}_N = \text{eff}_{N'}$.

After this modification, which does not affect actions in T, the second part of Eq.(3) is still satisfied after taking N' to play the role of N. However, the first part of Eq.(3) needs to be demonstrated.

We observe that the modification can only restrict the enabling of steps involving $\overline{a}$. Hence, if the first part of Eq.(3) does not hold with N' playing the role of N, then there is $M \in \text{reach}_{N'} \subseteq \text{reach}_N$ and $k \geq 1$ such that $M \xrightarrow{\overline{a}^k}_N M'$ *(*)* and $\neg M \xrightarrow{\overline{a}^k}_{N'}$ *(**)*. By Eq.(5) and *(*)*, we have $M' \xrightarrow{a^k}_N M$, and so $M(p) \geq \text{post}_N(a^k)(p)$ *(***)*.

By construction, *(**)* implies $\text{pre}_{N'}(\overline{a}^k)(p) > M(p)$. Thus, by $\text{pre}_{N'}(\overline{a}^k)(p) = \text{post}_N(a^k)(p)$, we obtain $\text{post}_N(a^k)(p) > M(p)$, yielding a contradiction with *(***)*.

We can apply the above modification as many times as needed, finally concluding that the result holds, as any modification does not invalidate the conditions captured in the formulation of this lemma that were obtained by the previous modifications. [Lemma 6.2] □

We will show that STS^{mrev} is solvable by a PT-net $\widetilde{N} = (\widetilde{P}, T \cup \overline{T}, \widetilde{F}, \widetilde{M_0})$ constructed thus:

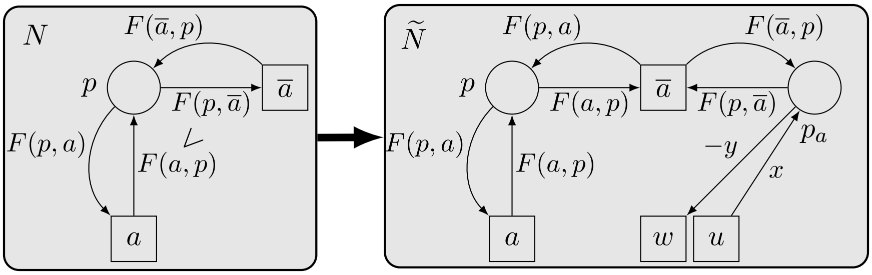


Figure 6. Introducing place p_a in the proof of Theorem 6.1, where u represents any place in $T \cup \overline{T} \setminus \{\overline{a}\}$ for which $x = \text{eff}_N(u)(p) > 0$, and w any place for which $y = \text{eff}_N(w)(p) \leq 0$.

- $\widetilde{P} = \bigcup_{p \in P} P_p$, where, for every $p \in P$,[6] $P_p = \{p\} \cup \{p_a \mid a \in T \wedge F(p,\overline{a}) > F(a,p)\}$ and $\widetilde{M_0}(P_p) = \{M_0(p)\}$.
- The connections in $\widetilde{N}$ are set as follows, where $p \in P$ and $u \in T \cup \overline{T} \setminus \{\overline{a}\}$:
  - $\widetilde{F}(p,\overline{a}) = F(a,p)$ and $\widetilde{F}(\overline{a},p) = F(p,a)$.
  - $\widetilde{F}(p_a,\overline{a}) = F(p,\overline{a})$ and $\widetilde{F}(\overline{a},p_a) = F(\overline{a},p)$.
  - $\text{eff}_N(u)(p) > 0$ implies $\widetilde{F}(p_a,u) = 0$ and $\widetilde{F}(u,p_a) = \text{eff}_N(u)(p)$.
  - $\text{eff}_N(u)(p) \leq 0$ implies $\widetilde{F}(u,p_a) = 0$ and $\widetilde{F}(p_a,u) = -\text{eff}_N(u)(p)$.
  - $\widetilde{F}$ on $(P \times T) \cup (T \times P)$ is as F unless it has been set explicitly above.

In what follows, for every marking M of N, we use $\phi(M)$ to denote the marking of $\widetilde{N}$ such that $\phi(M)(P_p) = \{M(p)\}$, for every $p \in P$. Hence $\phi(M_0) = \widetilde{M_0}$.

We now present a number of straightforward properties of $\widetilde{N}$. We first observe that, by Lemma 6.2, for all $a \in T$, $u \in T \cup \overline{T}$, and $p \in P$,

$$\begin{array}{ll} \text{pre}_{\widetilde{N}}(\overline{a}) \geq \text{post}_{\widetilde{N}}(a) & \text{eff}_{\widetilde{N}}(a) = -\text{eff}_{\widetilde{N}}(\overline{a}) \\ \text{post}_{\widetilde{N}}(\overline{a}) \geq \text{pre}_{\widetilde{N}}(a) & \text{eff}_{\widetilde{N}}(u)(P_p) = \{\text{eff}_N(u)(p)\}\ . \end{array} \tag{6}$$

Therefore, for every marking M of N and every $\kappa \in \text{mult}(T \cup \overline{T})$ such that $M + \text{eff}_N(\kappa) \geq \varnothing$,

$$\phi(M) + \text{eff}_{\widetilde{N}}(\kappa) = \phi(M + \text{eff}_N(\kappa))\ . \tag{7}$$

The construction does not affect the enabling of steps involving just one action as well as steps α over T since $p_a \in P_p$ cannot disable α if it is not also disabled by p. Hence, for all markings M of N, $u \in T \cup \overline{T}$, $k \geq 1$, and $\alpha \in \text{mult}(T)$:

$$M \xrightarrow{u^k}_N \iff \phi(M) \xrightarrow{u^k}_{\widetilde{N}} \qquad \text{and} \qquad M \xrightarrow{\alpha}_N \iff \phi(M) \xrightarrow{\alpha}_{\widetilde{N}}\ . \tag{8}$$

[6]Intuitively, each $p_a \in P_p$ is a (suitably adjusted) copy of p.

Thus, by Eqs.(4,7,8) and $\widetilde{M}_0 = \phi(M_0)$,

$$(STS^{spike})^{rev} \lhd_{\phi\circ\psi} CRG_{\widetilde{N}} \quad \text{and} \quad STS \simeq_{\phi\circ\psi} CRG_{\widetilde{N}|_T} \simeq_{\phi^{-1}} CRG_{N|_T} \,. \tag{9}$$

Lemma 6.3. Let $\alpha, \beta \in \text{mult}(T)$ and $\widetilde{M} = \phi(M)$, for some $M \in \text{mult}(P)$.

1. $\widetilde{M} \xrightarrow{\overline{\alpha}+\beta}_{\widetilde{N}}$ implies $\widetilde{M} - \text{eff}_{\widetilde{N}}(\alpha) \xrightarrow{\alpha+\beta}_{\widetilde{N}} \widetilde{M} + \text{eff}_{\widetilde{N}}(\beta)$.
2. $\widetilde{M} \xrightarrow{\alpha+\beta}_{\widetilde{N}}$ implies $\widetilde{M} + \text{eff}_{\widetilde{N}}(\alpha) \xrightarrow{\overline{\alpha}+\beta}_{\widetilde{N}} \widetilde{M} + \text{eff}_{\widetilde{N}}(\beta)$.

Proof:
[Lemma 6.3] (1) We first observe that, by *SEQ*, $\widetilde{M} - \text{eff}_{\widetilde{N}}(\alpha) = \widetilde{M} + \text{eff}_{\widetilde{N}}(\overline{\alpha}) \in \text{reach}_{\widetilde{N}}$. We then observe that, by $\widetilde{M} \geq \text{pre}_{\widetilde{N}}(\overline{\alpha} + \beta)$, the step $\alpha + \beta$ is enabled at $\widetilde{M} - \text{eff}_{\widetilde{N}}(\alpha)$, and so, by Eq.(6):

$$\widetilde{M} - \text{eff}_{\widetilde{N}}(\alpha) \geq \text{pre}_{\widetilde{N}}(\overline{\alpha}+\beta) - \text{eff}_{\widetilde{N}}(\alpha) = \text{pre}_{\widetilde{N}}(\overline{\alpha}) + \text{pre}_{\widetilde{N}}(\beta) - \text{post}_{\widetilde{N}}(\alpha) + \text{pre}_{\widetilde{N}}(\alpha) \geq \text{pre}_{\widetilde{N}}(\alpha+\beta) \,.$$

Hence, the result holds, as $\widetilde{M} - \text{eff}_{\widetilde{N}}(\alpha) + \text{eff}_{\widetilde{N}}(\alpha + \beta) = \widetilde{M} + \text{eff}_{\widetilde{N}}(\beta)$.

(2) By *SEQ*, $\widetilde{M} \xrightarrow{\alpha}_{\widetilde{N}} \widetilde{M} + \text{eff}_{\widetilde{N}}(\alpha)(= M')$. Suppose that $M' \xrightarrow{\overline{\alpha}+\beta}_{\widetilde{N}}$ does not hold. Then there is $q \in \widetilde{P}$ such that $\text{pre}_{\widetilde{N}}(\overline{\alpha} + \beta)(q) > M'(q)$ (*). Moreover, $\widetilde{M} \geq \text{pre}_{\widetilde{N}}(\alpha + \beta)$. Hence, we have:

$$\text{pre}_{\widetilde{N}}(\overline{\alpha}+\beta)(q) > \widetilde{M}(q) + \text{eff}_{\widetilde{N}}(\alpha)(q) \geq \text{pre}_{\widetilde{N}}(\alpha+\beta)(q) + \text{eff}_{\widetilde{N}}(\alpha)(q) = \text{pre}_{\widetilde{N}}(\beta)(q) + \text{post}_{\widetilde{N}}(\alpha)(q) \,,$$

and so $\text{pre}_{\widetilde{N}}(\overline{\alpha})(q) > \text{post}_{\widetilde{N}}(\alpha)(q)$. Thus there is $a \in \alpha$ and such that $\widetilde{F}(q, \overline{a}) > \widetilde{F}(a, q)$ and so, by the definition of $\widetilde{N}$, $q = p_a$, for some $p \in P$. Now, it follows from the construction of $\widetilde{N}$, that there are $\alpha_0, \alpha_1, \beta_0, \beta_1$ and $k \geq 1$ such that $\alpha = a^k + \alpha_0 + \alpha_1$ and $\beta = \beta_0 + \beta_1$ and $a \notin \alpha_0 + \alpha_1$ and, for $x = \alpha, \beta$, we have:

$$\begin{array}{llll}\text{post}_{\widetilde{N}}(x_1)(p_a) = \text{pre}_{\widetilde{N}}(x_0)(p_a) & = 0 = & \text{pre}_{\widetilde{N}}(\overline{x_1})(p_a) = \text{post}_{\widetilde{N}}(\overline{x_0})(p_a) \\ \text{pre}_{\widetilde{N}}(\overline{x_0})(p_a) & = \text{post}_{\widetilde{N}}(x_0)(p_a) & & \text{pre}_{\widetilde{N}}(x_1)(p_a) = \text{post}_{\widetilde{N}}(\overline{x_1})(p_a) \,.\end{array}$$

By *SEQ*, $\widetilde{M} \xrightarrow{\alpha_1+\beta_1}_{\widetilde{N}} \widetilde{M} + \text{eff}_{\widetilde{N}}(\alpha_1 + \beta_1) \xrightarrow{a^k}_{\widetilde{N}} \widetilde{M} + \text{eff}_{\widetilde{N}}(\alpha_1 + \beta_1 + a^k)$. Thus, by Eq.(9), $\widetilde{M} + \text{eff}_{\widetilde{N}}(\alpha_1 + \beta_1 + a^k) \xrightarrow{\overline{a}^k}_{\widetilde{N}} \widetilde{M} + \text{eff}_{\widetilde{N}}(\alpha_1 + \beta_1)$, and so we have:

$$\begin{array}{rcl}\widetilde{M}(p_a) + \text{eff}_{\widetilde{N}}(\alpha_1 + \beta_1 + a^k)(p_a) & = & \widetilde{M}(p_a) + \text{eff}_{\widetilde{N}}(a^k)(p_a) + \text{eff}_{\widetilde{N}}(\alpha_1 + \beta_1)(p_a) \\ & = & \widetilde{M}(p_a) + \text{eff}_{\widetilde{N}}(a^k)(p_a) - \text{pre}_{\widetilde{N}}(\alpha_1 + \beta_1)(p_a) \\ & \geq & \text{pre}_{\widetilde{N}}(\overline{a}^k)(p_a) \,.\end{array}$$

We therefore have:

$$\begin{array}{rcl}M'(p_a) & = & M(p_a) + \text{eff}_{\widetilde{N}}(a^k)(p_a) - \text{pre}_{\widetilde{N}}(\alpha_1)(p_a) + \text{post}_{\widetilde{N}}(\alpha_0)(p_a) \\ & \geq & \text{pre}_{\widetilde{N}}(\overline{a}^k)(p_a) + \text{pre}_{\widetilde{N}}(\beta_1)(p_a) + \text{post}_{\widetilde{N}}(\alpha_0)(p_a) \\ & = & \text{pre}_{\widetilde{N}}(\overline{a}^k)(p_a) + \text{pre}_{\widetilde{N}}(\beta_1)(p_a) + \text{pre}_{\widetilde{N}}(\overline{\alpha}_0)(p_a) \\ & = & \text{pre}_{\widetilde{N}}(\overline{\alpha})(p_a) + \text{pre}_{\widetilde{N}}(\beta)(p_a) \\ & = & \text{pre}_{\widetilde{N}}(\overline{\alpha} + \beta)(p_a) \,,\end{array}$$

yielding a contradiction with *(*)*. Thus $M' \xrightarrow{\overline{\alpha}+\beta}_{\widetilde{N}}$ holds. Hence we obtain the result as we have $M' + \mathrm{eff}_{\widetilde{N}}(\overline{\alpha}+\beta) = \widetilde{M} + \mathrm{eff}_{\widetilde{N}}(\alpha) + \mathrm{eff}_{\widetilde{N}}(\overline{\alpha}+\beta) = \widetilde{M} + \mathrm{eff}_{\widetilde{N}}(\beta)$. [Lemma 6.3] □

We now conclude that $STS^{mrev} \simeq_{\phi\circ\psi} CRG_{\widetilde{N}}$ holds thanks to Eq.(9) and Lemma 6.3. □

The last result leads to a simple sufficient condition for the solvability of direct reversibility in the case that proper multisets are not involved.

Theorem 6.4. Let STS be a solvable set CEST-system such that $(STS^{seq})^{rev}$ is solvable. Then STS^{rev} is solvable.

Proof:
Referring to the notation and proof of Theorem 6.1, we construct a new net $\widetilde{N}'$, by adding to $\widetilde{N}$ a fresh set of (mutex) places $P' = \{p_{ab} \mid a, b \in T\}$, where each p_{ab} is such that $\widetilde{M}_0(p_{ab}) = 1$ and has four non-zero connections: $\widetilde{F}(a, p_{ab}) = \widetilde{F}(p_{ab}, a) = \widetilde{F}(\overline{b}, p_{ab}) = \widetilde{F}(p_{ab}, \overline{b}) = 1$.

Since all the steps in STS are sets P' ensure that each step enabled at a reachable marking of $\widetilde{N}'$ is a subset of T or a subset of $\overline{T}$. Moreover, the enabling of such steps is not affected by adding P', so we obtain $STS^{rev} \simeq CRG_{\widetilde{N}'}$ as $STS^{mrev} \simeq CRG_{\widetilde{N}}$ holds by Theorem 6.1. □

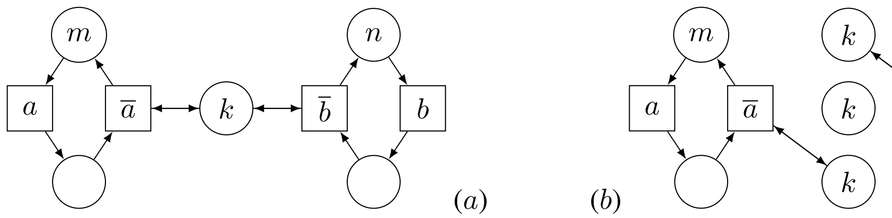


Figure 7. PT-net $N_{n,m}$ with $k = \max(m, n)$ and $m, n \geq 1$ (a); and the same net after applying the construction from Theorem 6.1 (b).

As the next example shows, modifying the original PT-net in Theorem 6.1 is unavoidable.

Example 6.5. Figure 7(a) depicts a family $N_{n,m}$ of PT-nets which satisfy the assumptions of Theorem 6.1. We have $CRG_{N_{n,m}} \not\simeq STS^{mrev}$, where STS is the step reachability graph of the PT-net obtained from $N_{n,m}$ after deleting actions $\overline{a}$ and $\overline{b}$. However, the construction from the proof of Theorem 6.1 yields the PT-net $CRG_{\widetilde{N}_{n,m}}$, shown in Figure 7($b$), satisfying $CRG_{\widetilde{N}_{n,m}} \simeq STS^{mrev}$. ♢

It is not possible to drop Eq.(3) from the formulation of Theorem 6.1. The next example shows a CEST-system which has only one non-singleton step and is reversible in the sequential semantics, but cannot be reversed in step sequence semantics, even with mixed reverses.

Example 6.6. Let us consider a step transition system STS together with a PT-net solving it, shown in Figure 8(a, b). If we erase the spike between the states v_0 and v_2, and add all the reverses (see

Figure 8(c)), then the resulting step transition system is solvable (see Figure 8(d)). However, STS cannot be reversed, as shown below.

Suppose that there is a PT-net N solving STS^{mrev}. Let M_i be the marking of N corresponding to the state v_i, for $i = 0, \ldots, 4$. Then the step $(\overline{a}\overline{a})$ is enabled at M_2, and $\overline{a}$ is not be enabled at M_3 *(*)*.

Let p be any place of N. We first observe that M_4 is a marking, and so $0 \leq M_4(p) = M_2(p) + 2k$, where $k = \mathrm{eff}_N(b)(p)$. Hence $\frac{1}{2} \cdot M_2(p) + k \geq 0$. We then recall that $(\overline{a}\overline{a})$ is enabled at M_2, and so $M_2(p) \geq 2 \cdot F(p, \overline{a})$. Hence $\frac{1}{2} \cdot M_2(p) \geq F(p, \overline{a})$. We therefore have:

$$M_3(p) = M_2(p) + k = \frac{1}{2} \cdot M_2(p) + k + \frac{1}{2} \cdot M_2(p) \geq 0 + F(p, \overline{a}) = F(p, \overline{a}) \; .$$

This means that $\overline{a}$ is a step enabled at M_3, yielding a contradiction with *(*)*. $\diamondsuit$

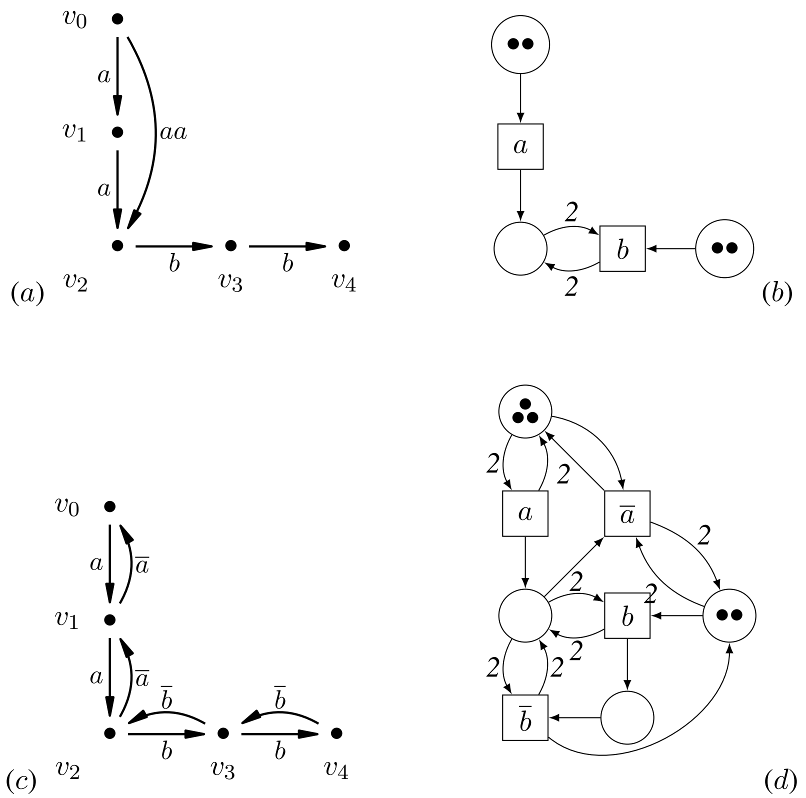


Figure 8. A step transition system STS with one spike (a), and a PT-net solving it (b). STS without the spike between v_0 and v_2 can be reversed (c, d), but STS cannot.

One might expect that, as it was shown to be the case for bounded PT-nets executed under the sequential semantics [3], it is sufficient to use PT-nets with split reverses also for the reversing under the step semantics. This, however, is not the case as demonstrated in the following example.

Example 6.7. Let us consider a step transition system STS together with a PT-net solving it, shown in Figure 9(a, b). Suppose that there is a PT-net N with split reverses such that CRG_N is a split reverse of STS. Moreover, let M_i be the marking of N corresponding to v_i, for $i = 1, \ldots, 6$.

Let p be any place of N. We first observe that the effect of executing the sequences of actions aaa and bb on p is the same, when going from M_1 to M_6. Hence, $3 \cdot \mathrm{eff}_N(a)(p) = 2 \cdot \mathrm{eff}_N(b)(p)$, and so there is an integer k such that $\mathrm{eff}_N(a)(p) = 2k$ and $\mathrm{eff}_N(b)(p) = 3k$. With this observation, and by considering different arrows in STS, we obtain:

$$\begin{array}{lll} M_2(p) = M_1(p) + 2k & M_3(p) = M_1(p) + 3k & M_4(p) = M_1(p) + 5k \\ M_5(p) = M_1(p) + 4k & M_6(p) = M_1(p) + 6k\ . & \end{array}$$

Hence, in particular, we have:

$$M_3(p) \leq M_5(p) \leq M_4(p) \qquad \text{or} \qquad M_3(p) \geq M_5(p) \geq M_4(p)\ . \tag{10}$$

Suppose now that $(\overline{a}_{\langle i\rangle}\overline{b}_{\langle j\rangle})$ is a step reversing (ab) at M_4. Then, by *SEQ* and *CE* holding for the concurrent reachability graphs of PT-nets, $\overline{b}_{\langle j\rangle}$ is also enabled at M_3. On the other hand, $\overline{b}_{\langle j\rangle}$ is not enabled at M_5. Then there must be a place p of N such that $M_5(p) < \mathrm{pre}_N(\overline{b}_{\langle j\rangle})(p)$. But we also have $M_3(p) \geq \mathrm{pre}_N(\overline{b}_{\langle j\rangle})(p)$ and $M_4(p) \geq \mathrm{pre}_N(\overline{b}_{\langle j\rangle})(p)$, as $\overline{b}_{\langle j\rangle}$ is enabled at M_3 and M_4. This, however, produces a contradiction with Eq.(10). ♢

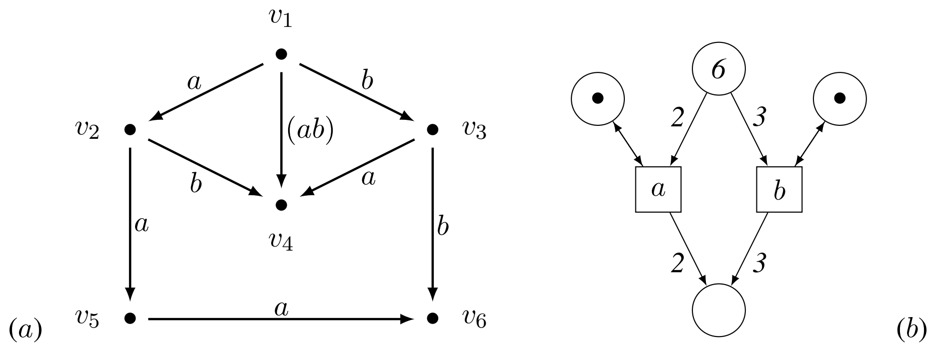


Figure 9. Splitting is not enough to guarantee reversing (Example 6.7). Note that v_1 is the initial state.

Example 6.7 can be used further to show that even allowing inhibitor arcs in N would not help.[7] The reason is that due to the formulas Eq.(10) for the markings M_3, M_4, and M_5, no inhibitor place p could be empty at M_3 and M_4, and contain a token at M_5. It would therefore be useless to block $\overline{b}_{\langle j\rangle}$ at M_5 and still allow the execution of $\overline{b}_{\langle j\rangle}$ at M_3 and M_4. Thus, reversing using PT-nets with inhibitor arcs is also not going to work in the general case, when considering the step semantics. This justifies the need to use test arcs 'stronger' than inhibitor arcs in addition to the splitting of reverse actions. Indeed, a general solution can then be obtained using an extended model of PT-nets, as shown in the next section.

[7] An inhibitor arc between a place p and action t means that if t is enabled at a marking M, then $M(p) = 0$.

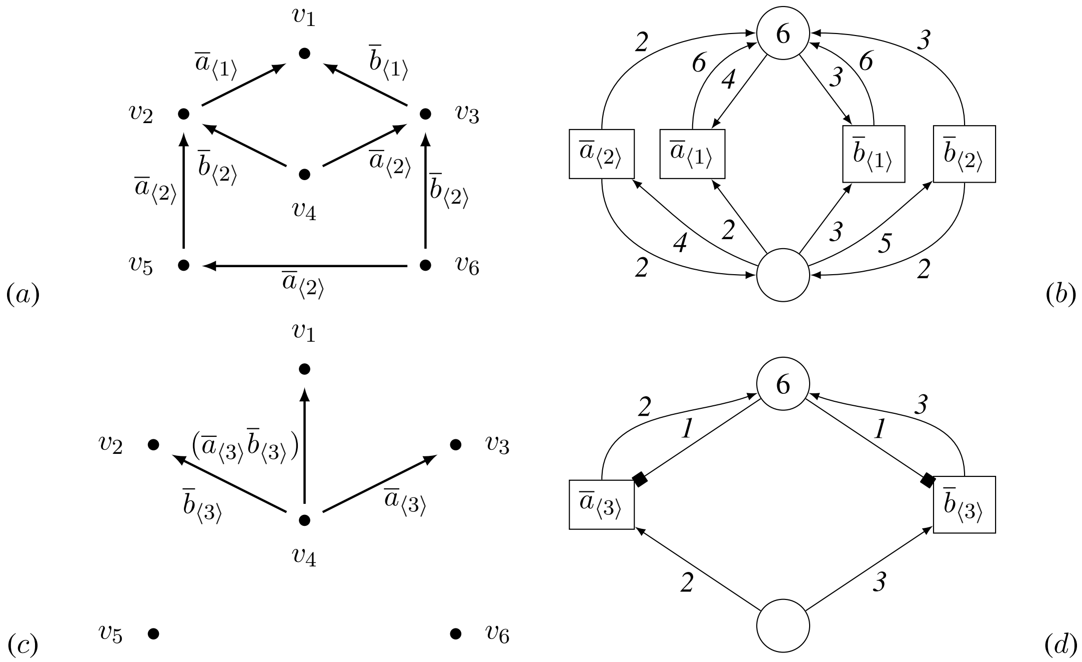


Figure 10. Reversing with splitting: phase one (a, b), and phase two (c, d).

7. A solution combining splitting and weighted read arcs

A PT-*net with weighted read arcs* (or PTR-net) is a tuple $N = (P, T, F, R, M_0)$ such that $N' = (P, T, F, M_0)$ is a PT-net, and $R : P \times T \to \mathbb{N}$ is a partial function defining *read arcs*. All the notations and concepts introduced for N' are applicable to N except that a step α of N is enabled at a marking M if it is enabled at marking M in N' and, in addition, $R(p, t) = M(p)$, whenever $a \in \alpha$ and $p \in P$ are such that $R(p, a)$ is defined. Read arcs are depicted as arrows with square arrowheads and labelled by their weights.

As the read arcs do not affect markings which result from firing steps of actions, the concurrent reachability graphs of PTR-nets satisfy *CE*. Although *SEQ* may fail to hold, it is the case that if α is an enabled step, then each step $\beta \leq \alpha$ is also enabled.

We first show that there is a PT-net with weighted read arcs reversing the reachability graph from Example 6.7.

Example 7.1. Recall the step transition system and the PT-net from Example 6.7. The construction of a solution comes in two phases. In the first phase, splitting is used to reverse all singleton steps. The result, which uses two reverses for a and two reverses for b, is shown in Figure 10(b). Note that although all the singleton steps are indeed reversed, the only non-singleton step (ab) is not. The second phase of the construction adds reverses for a and b which are simultaneously executable at M_4, as shown in Figure 10(d). A solution is then obtained by joining together Figures 9(b), 10(b)

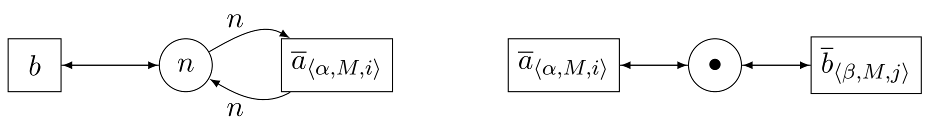
and 10(d), by identifying the places with 6 tokens and the places with 0 tokens. ♢

The solution presented in Example 7.1 inspired the development of a general construction which works for an arbitrary bounded PT-net.

Let $N = (P, T, F, M_0)$ be a bounded PT-net, and let n be an upper limit on the sizes of steps enabled at its reachable markings (such an n always exists as the concurrent reachability graph of N is finite). Moreover, for every marking $M \in \mathrm{reach}_N$, the steps annotating actions incoming to M in the concurrent reachability graph are $\mathrm{in}_N(M) = \{\alpha \mid \exists M' \in \mathrm{reach}_N : M' \xrightarrow{\alpha}_N M\}$. Since CRG_N is a CEST-system, $\alpha \leq \beta \in \mathrm{in}_N(M)$ implies $\alpha \in \mathrm{in}_N(M)$.

We then construct a PTR-net $N' = (P \uplus P', T \uplus T', F \sqcup F', R, M_0 \sqcup M_0')$. A key aspect of the construction is that for each reachable marking M of N, and for each maximal step[8] $\alpha \in \mathrm{in}_N(M)$, we add a set of fresh actions $T_{\alpha,M} = \{\overline{a}_{\langle\alpha,M,i\rangle} \mid a \in \alpha \wedge 1 \leq i \leq \alpha(a)\}$. We then proceed thus:

- For every new action $\overline{a}_{\langle\alpha,M,i\rangle} \in T'$:
  - $\mathrm{pre}_{N'}(\overline{a}_{\langle\alpha,M,i\rangle})|_P = \mathrm{post}_N(a)$ and $\mathrm{post}_{N'}(\overline{a}_{\langle\alpha,M,i\rangle})|_P = \mathrm{pre}_N(a)$.
  - For every $b \in T$, we add a fresh (mutex) place, as in Figure 11(a).
  - For every $\overline{b}_{\langle\beta,M,j\rangle} \in T'$ with $\alpha \neq \beta$, we add a fresh (mutex) place, as in Figure 11(b).
- $P \times T'$ is the domain of R and $R(p, \overline{a}_{\langle\alpha,M,i\rangle}) = M(p)$, for all $p \in P$ and $\overline{a}_{\langle\alpha,M,i\rangle} \in T'$.
- $M_0' \in \mathrm{mult}(P')$ is the marking of the places in P' as indicated in Figure 11.



(a) (b)

Figure 11. Places P' added in the construction of N'.

We then obtain the desired result.

Theorem 7.2. $CRG_{N'}$ is a split reverse of CRG_N.

Proof:
Let $STS = CRG_N$ and $STS' = CRG_{N'}$. We first gather together some immediate facts about N'.

Lemma 7.3.

1. $\overline{a}_{\langle\alpha,M,i\rangle}$ is an indexed reverse of a, for all $\overline{a}_{\langle\alpha,M,i\rangle} \in T'$ and $a \in T$.
2. $\mathrm{eff}_{N'}(\alpha) = \mathrm{eff}_N(\alpha) \sqcup \varnothing_{P'}$, for every $\alpha \in \mathrm{mult}(T)$.

[8] That is, $\alpha \leq \beta \in \mathrm{in}_N(M)$ implies $\alpha = \beta$.

3. $\mathrm{eff}_{N'}(\gamma) = -\,\mathrm{eff}_N(\alpha) \sqcup \varnothing_{P'}$, for all $\gamma \in \mathrm{mult}(T')$ and $\alpha \in \mathrm{mult}(T)$ such that $\overline{\alpha} = \mathrm{noidx}(\gamma)$.

4. $M|_{P'} = M'_0$, for every $M \in \mathrm{reach}_{N'}$.

5. If γ is a step enabled at $M \in \mathrm{reach}_{N'}$, then $\gamma \in \mathrm{mult}(T)$, or there is $\alpha \in \mathrm{in}_N(M)$ such that γ is a set included in $T_{\alpha,M} \subseteq T'$.

Proof:

[Lemma 7.3] (1,2) Follow directly from the definition of N'.

(3) Follows from part (1).

(4) Follows from parts (2) and (3).

(5) By part (4), $M|_{P'} = M'_0$. Hence the result follows from the presence of the weighted read arcs R and the mutex places shown in Figure 11. [Lemma 7.3] □

We will show that $\mathrm{reach}_{N'} = \{M \sqcup M'_0 \mid M \in \mathrm{reach}_N\}$ and $STS^{rev} \simeq_\psi \mathrm{noidx}(STS')$, where $\psi(M) = M \sqcup M'_0$, for every $M \in \mathrm{reach}_N$.

We first observe that $\psi(M_0) = M_0 \sqcup M'_0$ is the initial marking of N'. Suppose that $M \in \mathrm{reach}_N$ is such that $\psi(M) = M \sqcup M'_0 \in \mathrm{reach}_{N'}$. To show that the executions of steps are preserved by ψ in both directions, we consider four cases, after taking into account Lemma 7.3(5).

Case 1: $M \xrightarrow{\alpha}_{STS} M'$. Then, since n in Figure 11(a) is such that $|\alpha| \leq n$, the addition of the new places P' does not block α. Hence α is enabled at $M \sqcup M'_0$. Moreover, by Lemma 7.3(2), $M \sqcup M'_0 \xrightarrow{\alpha}_{STS'} M' \sqcup M'_0$.

Case 2: $M \xrightarrow{\overline{\alpha}}_{STS^{rev}} M'$. Then $M' \xrightarrow{\alpha}_{STS} M$ and $\alpha \in \mathrm{in}_N(M)$. Let β be any maximal step in $\mathrm{in}_N(M)$ such that $\alpha \leq \beta$ (such a step exists since CRG_N is finite). Then there is a subset γ of $T_{\beta,M}$ such that $\mathrm{noidx}(\gamma) = \overline{\alpha}$. By construction, γ is enabled at $M \sqcup M'_0$. Hence, by Lemma 7.3(3), $M \sqcup M'_0 \xrightarrow{\gamma}_{STS'} M' \sqcup M'_0$.

Case 3: $M \sqcup M'_0 \xrightarrow{\alpha}_{STS'} M'$ and $\alpha \in \mathrm{mult}(T)$. Then, by construction and Lemma 7.3(2), α is enabled at M and $M' = (M + \mathrm{eff}_N(\alpha)) \sqcup M'_0$. Moreover, $M \xrightarrow{\alpha}_{STS^{rev}} M + \mathrm{eff}_N(\alpha)$.

Case 4: $M \sqcup M'_0 \xrightarrow{\gamma}_{STS'} M'$, where γ is a subset of $T_{\alpha,M}$ for some $\alpha \in \mathrm{in}_N(M)$. Let $\beta = \mathrm{noidx}(\gamma) \leq \overline{\alpha}$. Then, by construction and Lemma 7.3(3), $M' = (M - \mathrm{eff}_N(\beta)) \sqcup M'_0$, β is enabled at $M - \mathrm{eff}_N(\beta)$, and $M - \mathrm{eff}_N(\beta) \xrightarrow{\beta}_{STS} M$. Hence $M \xrightarrow{\overline{\beta}}_{STS^{rev}} M - \mathrm{eff}_N(\beta)$. □

We have developed a general construction which brings us to the same level of reversibility as in the sequential case. However, we had to pay the (costly) price of using of a non-standard class of read arcs. The construction presented above is far from being optimal. Taking as an example the solution from Example 7.1, we observe that it would introduce 5 reverses of a, 4 reverses of b, and a total of 31 additional places. One can easily see that a large number of them could be avoided, by considering the conditions that force the introduction of each split reversal and those requiring the addition of the new control places.

8. Concluding remarks

In this paper, we continued a study of reversibility in PT-nets, when the step semantics based on executing steps (multisets) of actions rather than single actions is considered, thus capturing *real parallelism*. In a more abstract setting, the (partial) reversal of steps, thus generating *mixed steps* possibly containing both original and reverse action, has been studied in [25]. Here we discussed how such reversing can be done in a concrete operational framework of PT-nets.

In the future work, we plan to develop an effective solution to the synthesis problem for the step transition systems with multiple initial states, and address the optimisation of the general solution based on PTR-nets presented in the last section.

Acknowledgement The authors are indebted to the referees for their very useful comments and suggestions for improvement. This research was supported by the EU Cost Action IC1405 and the Polish National Agency for Academic Exchange through the International Academic Partnerships program. The first author was partially supported by the Spanish project TRACES (TIN2015-67522-C3-3), and by Comunidad de Madrid as part of the program S2018/TCS-4339 (BLOQUES-CM) co-funded by EIE Funds of the European Union. The third author was partially supported by the Polish NCN grant 2017/27/B/ST6/02093.

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