Stirling numbers and periodic points

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Abstract

We introduce the notion of ‘almost realizability’, an arithmetic generalization of ‘realizability’ for integer sequences, which is the property of counting periodic points for some map. We characterize the intersection between the set of Stirling sequences (of both the first and the second kind) and the set of almost realizable sequences.

1 Introduction, definitions, and examples

Denote the sets of non-negative integers and prime numbers by $\mathbb{N}$ and $\mathbb{P}$, respectively. Write $S^{(1)}(n, k)$ for the (signless) Stirling numbers of the first kind, defined for any $n \geq 1$ and $0 \leq k \leq n$ to be the number of permutations

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of $\{1, \dots, n\}$ with exactly k cycles. Write $S_{\pm}^{(1)}(n, k)$ for the (signed) Stirling numbers of the first kind, defined by the relation

$$(x)_n = x(x-1)(x-2) \cdots (x-n+1) = \sum_{k=0}^n S_{\pm}^{(1)}(n, k) x^k$$

for all $n \geq 1$ and $0 \leq k \leq n$. The two are related by

$$(-1)^{n-k} S_{\pm}^{(1)}(n, k) = S^{(1)}(n, k)$$

for all $n \geq 1$ and $0 \leq k \leq n$. Finally, let $S^{(2)}(n, k)$ for $n \geq 1$ and $1 \leq k \leq n$ denote the Stirling numbers of the second kind, so $S^{(2)}(n, k)$ counts the number of ways to partition a set comprising n elements into k non-empty subsets.

Definition 1.1. For each $k \geq 1$ define sequences

$$S_k^{(1)} = (S^{(1)}(n+k-1, k))_{n \geq 1}$$

and

$$S_k^{(2)} = (S^{(2)}(n+k-1, k))_{n \geq 1}.$$

The properties we wish to discuss depend entirely on the offset $(k-1)$ chosen for the index n , so for definiteness we mention two examples:

$$(1.1) \quad S_3^{(1)} = (1, 6, 35, 225, \dots)$$

and

$$(1.2) \quad S_3^{(2)} = (1, 6, 25, 90, \dots).$$

A natural combinatorial property that a sequence of non-negative integers may have is that it counts periodic points for iterates of a map. For our purposes, a map can simply be regarded as a permutation of $\mathbb{N}$ with the property that there are only finitely many cycles of each length (though one may equally well ask that the map be a C^∞ diffeomorphism of the annulus, by a result of Windsor [7]).

Definition 1.2. An integer sequence $A = (A_n)$ is called *realizable* if either of the two equivalent conditions holds:

- (a) There is a map $T: X \rightarrow X$ on a set X with the property that

$$A_n = \text{Fix}_{(X, T)}(n) = |\{x \in X \mid T^n x = x\}|$$

for all $n \geq 1$; or

(b) The sequence A satisfies both the *Dold condition*

$$(1.3) \quad n | (\mu * A)(n) = \sum_{d|n} \mu(n/d) A_d$$

for all $n \geq 1$ (from [1]) and the *sign condition*

$$(\mu * A)(n) = \sum_{d|n} \mu(n/d) A_d \geq 0$$

for all $n \geq 1$.

Here μ denotes the classical Möbius function, and $*$ denotes Dirichlet convolution. The equivalence between the two definitions is just a consequence of the fact that the set of points of period n under a map is the disjoint union of the points living on closed orbits of length d for $d|n$, and each closed orbit of length d has exactly d points on it. We refer to work of Pakapongpun, Puri and Ward [3, 4, 6] for more on this equivalence and some of its functorial consequences.

Example 1.3. The sequences $S_3^{(1)}$ and $S_3^{(2)}$ are not realizable. To see this, notice that the Dold condition applied at $n = 2$ shows that if the sequence A is realizable, then $A_2 - A_1$ must be even, while (1.1) and (1.2) do not have this property.

Every rational number x can be written in a unique way as a quotient of two coprime integers with positive denominator. Let us denote the denominator in this representation of x as $\text{Denom}(x)$. If additionally $x \neq 0$, then there exists a unique sequence $(\alpha_p)_{p \in \mathbb{P}}$ of integers, where $\alpha_p \neq 0$ only for finitely many primes p , such that

$$|x| = \prod_{p \in \mathbb{P}} p^{\alpha_p}.$$

We then define the p -adic norm of x to be

$$|x|_p = p^{-\alpha_p}.$$

Ignoring the sign condition for the moment, an integer sequence A fails to satisfy the Dold condition if and only if

$$\text{Denom} \left(\frac{1}{n} (\mu * A)(n) \right) = \prod_{p \in \mathbb{P}} \max \left\{ 1, \left| \frac{1}{n} (\mu * A)(n) \right|_p \right\} > 1$$

for some $n \geq 1$. This gives a *measure of failure* for a sequence A to be realizable, as follows.

Definition 1.4. For a sequence A of non-negative integers, we write

$$\text{Fail}(A) = \begin{cases} \text{lcm}(\{\text{Denom}((\frac{1}{n}\mu * A)(n)) \mid n \geq 1\}) & \text{if this is finite;} \\ \infty & \text{if not.} \end{cases}$$

The sequence A is said to be *almost realizable* if $\text{Fail}(A) < \infty$ and it satisfies the sign condition.

A realizable sequence A has $\text{Fail}(A) = 1$, and if A is almost realizable, then

$$\text{Fail}(A) \cdot A = (\text{Fail}(A) \cdot A_n)_{n \geq 1}$$

is realizable. That is, in such a case the failure to be realizable can be ‘repaired’ simply by multiplying by a single number. The motivation for this (admittedly odd) notion comes from a result of Moss and Ward, where it arose in an unexpected setting.

Example 1.5 (Moss and Ward [2]). The Fibonacci sequence $F = (F_n) = (1, 1, 2, \dots)$ is not almost realizable, but the Fibonacci sequence sampled along the squares (F_{n^2}) is almost realizable with $\text{Fail}((F_{n^2})) = 5$, the discriminant of F .

2 Main results

Our results establish when—and to what extent—the Stirling numbers satisfy Definitions 1.2 and 1.4. In order to expose the interaction between the arithmetic properties of Stirling numbers and the properties of realizability and almost realizability, we move the proofs of the purely combinatorial lemmas into Section 3.

Theorem 2.1 (Stirling numbers of the first kind). *For $k \geq 1$ the sequence $S_k^{(1)}$ is not almost realizable.*

Theorem 2.2 (Stirling numbers of the second kind). *For $k \leq 2$ the sequence $S_k^{(2)}$ is realizable. For $k \geq 3$ the sequence $S_k^{(2)}$ is not realizable, but is almost realizable with $\text{Fail}(S_k^{(2)}) \mid (k-1)!$ for all $k \geq 1$.*

Proof of Theorem 2.2 for $k \leq 2$. We have

$$S_1^{(2)} = (1, 1, 1, 1, \dots)$$

which is realized by the identity map on a singleton. For $k = 2$ we have

$$S_2^{(2)} = (1, 3, 7, 15, \dots) = (2^n - 1).$$

This is easily seen to be realizable, either by checking the conditions or by noticing that if T is the map $z \mapsto z^2$ on $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then $\text{Fix}_{(X,T)}(n) = 2^n - 1$ for all $n \geq 1$. $\square$

The non-trivial parts of the arguments below concern proving that the Dold condition is—or is not—satisfied, and to what extent. The sequences we are concerned with grow very rapidly, so the sign condition is not a concern, but for completeness we include a simple argument that verifies the sign condition for $S_k^{(2)}$.

Lemma 2.3. *For $k \geq 1$ we have $(\mu * S_k^{(2)})(n) \geq 0$ for all $n \geq 1$.*

The proof of Theorem 2.1 for $k \geq 3$ involves finding a prime index at which the Dold congruence fails—that is, finding a witness to non-realizability like the prime 2 in Example 1.3. In order to do this, we need some information on the properties of $S^{(1)}$ modulo a prime.

Lemma 2.4. *For a prime p and $k \in \mathbb{N}$ we have*

$$(2.1) \quad S^{(1)}(p+k-1, k) \equiv 1 \pmod{p}$$

if and only if $k \equiv j \pmod{p^2}$ for some

$$j \in \{(p-2)p+1, (p-2)p+2, \dots, (p-1)p\}.$$

For an odd prime p and $k \in \mathbb{N}$ we have

$$(2.2) \quad S_{\pm}^{(1)}(p+k-1, k) \equiv 1 \pmod{p}$$

if and only if $k \equiv j \pmod{p^2}$ for some

$$j \in \{(p-2)p+1, (p-2)p+2, \dots, (p-1)p\}.$$

Proof of Theorem 2.1. Let p be a prime and $k \in \mathbb{N}$. If $k \leq (p-2)p$, then we certainly have $k \notin \{(p-2)p+1, \dots, (p-1)p\} \pmod{p^2}$. By Lemma 2.4 we have

$$S^{(1)}(p+k-1, k) \not\equiv 1 \pmod{p}.$$

Since $S^{(1)}(k, k) = 1$, we deduce that $\left| \frac{1}{p} (S^{(1)}(p+k-1, k) - S^{(1)}(k, k)) \right|_p = p$, so the set of denominators among the expressions $\frac{1}{n}(\mu * S^{(1)})(n)$ contains infinitely many primes. Thus $S^{(1)}$ is not almost realizable.

The argument for the signed Stirling numbers is similar, and shows that they fail the Dold congruence in the same strong way. $\square$

The structure of the proof that $S_k^{(2)}$ is not realizable for $k \geq 3$ is very similar to that of Theorem 2.1, but the details differ. To do this, once again some information on the properties of $S_k^{(2)}$ modulo a prime is needed.

Lemma 2.5. *For a prime p and $k \in \mathbb{N}$ we have*

$$(2.3) \quad S^{(2)}(p+k-1, k) \equiv 1 \pmod{p}$$

if and only if $k \equiv j \pmod{p^2}$ for some $j \in \{1, \dots, p\}$.

Proof of Theorem 2.2 for $k \geq 3$: non-realizability. If A is a realizable sequence, then the Dold condition (1.3) requires $A_p \equiv A_1 \pmod{p}$. So it is enough to show that for $k \geq 3$ there is a prime p for which

$$S^{(2)}(p+k-1, k) \not\equiv 1 \pmod{p}.$$

By Lemma 2.5, it is enough to find a prime p with $k \not\equiv \{1, \dots, p\} \pmod{p^2}$. This in turn certainly follows from the (*a priori* stronger) claim that

$$(2.4) \quad \mathbb{N}_{\geq 3} = \{n \in \mathbb{N} \mid n \geq 3\} \subseteq \bigcup_{p \in \mathbb{P}} (p, p^2],$$

since $k \in (p, p^2]$ precludes $k \in \{1, \dots, p\} \pmod{p^2}$. For $k \geq 3$ we therefore seek a prime with $p < k \leq p^2$, or equivalently with $\sqrt{k} \leq p < k$. For $k = 3$ we have $\sqrt{3} \leq 2 < 3$. For $k \geq 4$ we have $2 \leq \sqrt{k} \leq \frac{k}{2}$, and by Bertrand's postulate there is a prime p with $\frac{k}{2} < p < k$, hence with $\sqrt{k} \leq p < k$, and so we have $p < k \leq p^2$, which proves (2.4). $\square$

Proof of Theorem 2.2 for $k \geq 3$: almost realizability. Assume that $k \geq 3$. The statement we wish to prove is that $(k-1)!S_k^{(2)}$ is realizable, and we do this by starting with one of the closed formulas for the Stirling numbers of the second kind,

$$S^{(2)}(n, k) = \frac{1}{k!} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} j^n$$

for $n \in \mathbb{N}$ and $1 \leq k \leq n$. It follows that

$$\begin{aligned} (k-1)!S^{(2)}(n, k) &= \frac{1}{k} \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} j^n = \sum_{j=1}^k (-1)^{k-j} \frac{j}{k} \frac{\binom{k}{j}}{j!} j^{n-1} \\ &= \sum_{j=1}^k (-1)^{k-j} \frac{(k-1)_{j-1}}{(j-1)!} j^{n-1} \\ &= \sum_{j=1}^k (-1)^{k-j} \binom{k-1}{j-1} j^{n-1}, \end{aligned}$$

where we have used the notation $(x)_n$ for the falling factorial as usual, which means that $(k-1)!S^{(2)}(n, k)$ is an integral linear combination of the functions $n \mapsto j^{n-1}$. Write $\nu_p(n)$ for the power of p dividing n , so $|n|_p = p^{-\nu_p(n)}$. We claim that

$$(2.5) \quad \sum_{d|n} \mu(d)(k-1)!S^{(2)}\left(\frac{n}{d} + k - 1, k\right) \equiv 0 \pmod{p^{\nu_p(n)}}$$

for any prime p dividing n . To see this, we first write

$$\sum_{d|n} \mu(d)(k-1)!S^{(2)}\left(\frac{n}{d} + k - 1, k\right) = \sum_{\substack{d|(n/p) \\ p \nmid d}} \mu(d)C(n, k, d, p)$$

where

$$C(n, k, d, p) = \left((k-1)!S^{(2)}\left(\frac{n}{d} + k - 1, k\right) - (k-1)!S^{(2)}\left(\frac{n}{pd} + k - 1, k\right) \right).$$

Write $\frac{n}{d} = mp^a$ where $a = \nu_p(n)$, for some $m \in \mathbb{N}$ for d a divisor of $\frac{n}{p}$ not divisible by p . Then,

$$\begin{aligned} C(n, k, d, p) &= \sum_{j=1}^k (-1)^{k-j} \binom{k-1}{j-1} \left(j^{\frac{n}{d}-1+k-1} - j^{\frac{n}{pd}-1+k-1} \right) \\ &= \sum_{j=1}^k (-1)^{k-j} \binom{k-1}{j-1} \left(j^{mp^a+k-2} - j^{mp^{a-1}+k-2} \right) \\ (2.6) \quad &= \sum_{j=1}^k (-1)^{k-j} \binom{k-1}{j-1} j^{mp^{a-1}+k-2} \left(j^{mp^{a-1}(p-1)} - 1 \right). \end{aligned}$$

Now $p^a \mid \left(j^{mp^{a-1}(p-1)} - 1 \right)$ by Euler's theorem if $p \nmid j$, and $p^a \mid j^{p^{a-1}}$ if $p \mid j$, so (2.6) vanishes modulo p^a . That is, we have proved the claim (2.5).

By taking the prime decomposition of n , we deduce that

$$n \mid (\mu * (k-1)!S_k^{(2)})(n)$$

for all $n \geq 1$, so $(k-1)!S_k^{(2)}$ satisfies the Dold condition. Almost realizability follows by Lemma 2.3. $\square$

3 Combinatorial Proofs

We assemble here the proofs of the purely combinatorial lemmas used in the previous section.

Proof of Lemma 2.3. We start with a simple observation from Puri's thesis [5]: if (A_n) is an increasing sequence of non-negative real numbers with

$$(3.1) \quad A_{2n} \geq nA_n$$

for all $n \geq 1$, then we claim that

$$(3.2) \quad (\mu * A)_n \geq 0$$

for all $n \geq 1$. To see this, notice that we have

$$(\mu * A)_{2n} = \sum_{d|2n} \mu(2n/d) A_d \geq A_{2n} - \sum_{k=1}^n A_k \geq A_{2n} - nA_n \geq 0,$$

since the largest proper divisor of $2n$ is n . Similarly, in the odd case we have

$$(\mu * A)_{2n+1} \geq A_{2n+1} - \sum_{k=1}^n A_k \geq A_{2n+1} - nA_n \geq 0,$$

since the largest proper divisor of $2n+1$ is smaller than n , proving (3.2).

For $k \leq 2$ we know that $S_k^{(2)}$ is realizable, and so has $(\mu * S_k^{(2)})(n) \geq 0$ for all $n \geq 1$.

So it is enough to verify that $S_k^{(2)}$ satisfies (3.1) for $k \geq 3$. By definition, we know that $S^{(2)}(n+k-1, k)$ is the number of partitions of a set with $n+k-1$ elements into k non-empty subsets, and $S^{(2)}(2n+k-1, k)$ is the number of partitions of a set with an additional n elements into k non-empty subsets. We can adjoin each of these n elements to any one of the subsets in a partition of the set with $n+k-1$ elements, showing that

$$S^{(2)}(2n+k-1, k) \geq n \cdot S^{(2)}(n+k-1, k)$$

for all $n \geq 1$, as required. $\square$

Proof of Lemma 2.4. Assume that $n \geq 2$, and make use of the definition of $S^{(1)}(n, k)$ as the number of permutations of $\{1, \dots, n\}$ with exactly k cycles. Let Σ be a set with $p+k-1$ elements, and assume that we have partitioned Σ into k disjoint non-empty cycles $\Sigma_1, \Sigma_2, \dots, \Sigma_k$. We claim that any one of these cycles Σ_j has length no more than p , because each of the other $k-1$ subsets in the partition has at least one element, and so

$$|\Sigma_j| = p+k-1 - \left| \bigsqcup_{i \neq j} \Sigma_i \right| \leq p+k-1 - (k-1) = p.$$

For $\ell = 1, \dots, p$ let

$$j_\ell = |\{i \mid 1 \leq i \leq k, |\Sigma_i| = \ell\}|$$

denote the number of subsets in the partition with exactly ℓ elements. It follows that

$$\sum_{\ell=1}^p j_\ell = k$$

since Σ is partitioned into k non-empty subsets, and

$$\sum_{\ell=1}^p \ell j_\ell = |\Sigma| = p + k - 1.$$

In order to count the ways in which Σ can be partitioned into k cycles, notice first that we can begin by partitioning the set into j_1 singletons, j_2 subsets of cardinality 2, j_3 of cardinality 3, and so on in

$$\frac{(p + k - 1)!}{(1!)^{j_1} j_1! \cdot (2!)^{j_2} j_2! \cdot (3!)^{j_3} j_3! \cdots (p!)^{j_p} j_p!}$$

ways and then form a cycle from one of the subsets of cardinality j in $(j-1)!$ ways by fixing the first element in the subset to be the starting point of the cycle, and then order the remaining elements in the cycle in any one of $(j-1)!$ ways. It follows that we can partition Σ into k disjoint cycles with j_ℓ subsets of cardinality ℓ for $1 \leq \ell \leq p$ in

$$\frac{(p + k - 1)!}{1^{j_1} j_1! \cdot 2^{j_2} j_2! \cdot 3^{j_3} j_3! \cdots p^{j_p} j_p!}$$

ways. This gives

$$(3.3) \quad S^{(1)}(p + k - 1, k) = \sum_{\substack{(j_1, \dots, j_p) \in \mathbb{N}^p \\ j_1 + j_2 + \dots + j_p = k \\ j_1 + 2j_2 + \dots + pj_p = p + k - 1}} \frac{(p + k - 1)!}{1^{j_1} j_1! \cdot 2^{j_2} j_2! \cdot 3^{j_3} j_3! \cdots p^{j_p} j_p!}.$$

We will need some simple counting arguments, as follows:

- (a) If $j_\ell \geq p$ for some $\ell \in \{2, \dots, p\}$, then we have a partition of Σ with at least p cycles of length at least 2 and $k - j_\ell$ cycles of length at least 1. This requires that $|\Sigma| \geq 2j_\ell + (k - j_\ell) \geq k + j_\ell \geq k + p$, a contradiction. So we have $j_\ell < p$ for $\ell \in \{2, \dots, p\}$.
- (b) If $j_1 = k$, then Σ is partitioned into k singletons, contradicting $|\Sigma| = p + k - 1$. Thus $j_1 \leq k - 1$.
- (c) Finally, if $j_p > 0$, then $j_p = 1$, $j_1 = k - 1$ and $j_\ell = 0$ for $\ell \in \{2, \dots, p - 1\}$. Indeed, if $j_p \geq 1$, then we have a partition of Σ with j_p

cycles of length p and $k - j_p$ cycles of length at least 1. This implies that

$$k + p - 1 = |\Sigma| \geq p j_p + (k - j_p) = k + j_p(p - 1),$$

which is possible only if $j_p = 1$ and all the cycles of length less than p are singletons.

Now assume that $(j_1, j_2, \dots, j_{p-1}, j_p) \neq (k - 1, 0, \dots, 0, 1)$, so $j_p = 0$ by (c) and the corresponding term in (3.3) takes the form

$$\frac{(p + k - 1)!}{1^{j_1} j_1! \cdot 2^{j_2} j_2! \cdot 3^{j_3} j_3! \cdots (p - 1)^{j_{p-1}} j_{p-1}!},$$

in which the values of ℓ and of $j_\ell!$ for $\ell \in \{2, \dots, p - 1\}$ are not divisible by p , since ℓ and j_ℓ are smaller than p (the latter by (a)). Since $j_1 \leq k - 1$, we know that $|(p + k - 1)!|_p < |j_1!|_p$ since the product of p consecutive integers is divisible by p . It follows that the term in (3.3) corresponding to any $(j_1, j_2, \dots, j_{p-1}, j_p) \neq (k - 1, 0, \dots, 0, 1)$ vanishes modulo p . It follows that $S^{(1)}(p + k - 1, k)$ is congruent modulo p to the term corresponding to $(j_1, j_2, \dots, j_{p-1}, j_p) = (k - 1, 0, \dots, 0, 1)$, giving

$$(3.4) \quad S^{(1)}(p + k - 1, k) = \frac{(p + k - 1)!}{(k - 1)! p} = \frac{1}{p} (p + k - 1)_p,$$

where we have used the usual notation $(x)_n = x(x - 1)(x - 2) \cdots (x - n + 1)$ for the falling factorial.

We now wish to simplify the expression in (3.4). Let $m = \lceil \frac{k}{p} \rceil$ so that $k = mp - i$ for some $i \in \{0, \dots, p - 1\}$. Then

$$\begin{aligned} \frac{1}{p} (p + k - 1)_p &= \frac{1}{p} \prod_{j=0}^{p-1} (k + j) \\ &= \frac{1}{p} \prod_{j=0}^{p-1} (mp - i + j) \\ &= \frac{1}{p} \left(\prod_{j=p-i}^{p-1} ((m - 1)p + j) \right) \cdot mp \cdot \left(\prod_{j=1}^{p-i-1} (mp + j) \right) \\ &\equiv m \cdot \left(\prod_{j=p-i}^{p-1} j \right) \cdot \left(\prod_{j=1}^{p-i-1} j \right) \equiv -m \pmod{p} \end{aligned}$$

by Wilson's theorem. We deduce that

$$S^{(1)}(p + k - 1, k) = - \left\lceil \frac{k}{p} \right\rceil.$$

It follows that in order for the congruence (2.1) to hold, we must have $m = np + p - 1$ for some $n \in \mathbb{N}$, which is equivalent to

$$k = mp - i = np^2 + (p - 1)p - i$$

with $i \in \{0, 1, \dots, p - 1\}$. This shows the first part of the lemma.

Finally, (2.2) follows from the fact that

$$\begin{aligned} S_{\pm}^{(1)}(p + k - 1, k) &= (-1)^{p+k-1-k} S^{(1)}(p + k - 1, k) \\ &= (-1)^{p-1} S^{(1)}(p + k - 1, k) \\ &= S^{(1)}(p + k - 1, k) \end{aligned}$$

for an odd prime p . □

Proof of Lemma 2.5. The proof proceeds in two stages, the first of which is to find a binomial congruence for $S_k^{(2)}$. We let Σ be a set with $p + k - 1$ elements, and note first that if Σ is partitioned into k non-empty subsets $\Sigma_1, \dots, \Sigma_k$, then

$$|\Sigma_j| = p + k - 1 - \left| \bigsqcup_{i \neq j} \Sigma_i \right| \leq p + k - 1 - (k - 1) = p$$

for $j = 1, \dots, k$. For $\ell = 1, \dots, p$ let

$$j_\ell = |\{i \mid 1 \leq i \leq k, |\Sigma_i| = \ell\}|$$

denote the number of subsets in the partition with exactly ℓ elements. By construction

$$(3.5) \quad \sum_{\ell=1}^p j_\ell = k$$

since Σ is partitioned into k non-empty subsets, and

$$(3.6) \quad \sum_{\ell=1}^p \ell j_\ell = |\Sigma| = p + k - 1.$$

Since we are not interested in the arrangement of elements within each subset of the partition, the number of ways to choose this partition is given by

$$\frac{(p + k - 1)!}{(1!)^{j_1} j_1! \cdot (2!)^{j_2} j_2! \cdot (3!)^{j_3} j_3! \cdots (p!)^{j_p} j_p!}.$$

Clearly the conditions (3.5) and (3.6) are the only requirements for a partition into k non-empty subsets, so we can use these conditions to parameterise all such partitions. Thus we can write

$$(3.7) \quad S^{(2)}(p+k-1, k) = \sum_{\substack{(j_1, \dots, j_p) \in \mathbb{N}^p \\ j_1 + j_2 + \dots + j_p = k \\ j_1 + 2j_2 + \dots + pj_p = p+k-1}} \frac{(p+k-1)!}{(1!)^{j_1} j_1! \cdot (2!)^{j_2} j_2! \cdot (3!)^{j_3} j_3! \cdot \dots \cdot (p!)^{j_p} j_p!}$$

We will need some simple counting arguments, as follows:

- (a) If $j_\ell \geq p$ for some $\ell \in \{2, \dots, p\}$, then there is a corresponding partition of Σ with at least p subsets of cardinality at least 2 and $k-p$ subsets of cardinality at least 1. This requires that Σ has cardinality at least

$$2p + (k-p) > p+k-1,$$

a contradiction. So $j_\ell < p$ for all $\ell \in \{2, \dots, p\}$.

- (b) If $j_1 = k$, then Σ is partitioned into k singletons, which requires

$$|\Sigma| = k < p+k-1,$$

a contradiction. So $j_1 \leq k-1$.

- (c) If $j_p \geq 1$, then Σ is partitioned into j_p subsets of cardinality p and $k-j_p$ subsets of cardinality at least 1. This requires

$$|\Sigma| = p+k-1 \geq pj_p + (k-j_p) = k + j_p(p-1),$$

which is possible only if $j_p = 1$ and all the subsets of cardinality less than p are singletons.

These observations allow the expression (3.7) to be simplified modulo p as follows. For a summand with

$$(j_1, j_2, \dots, j_{p-1}, j_p) \neq (k-1, 0, \dots, 0, 1)$$

we must have $j_p = 0$ by (c) and the corresponding summand has the form

$$\frac{(p+k-1)!}{(1!)^{j_1} j_1! \cdot (2!)^{j_2} j_2! \cdot (3!)^{j_3} j_3! \cdot \dots \cdot ((p-1)!)^{j_{p-1}} j_{p-1}!},$$

where for each $\ell \in \{2, \dots, p-1\}$ the values of $\ell!$ and $j_\ell!$ are not divisible by p (since $\ell, j_\ell < p$). We also have $j_1 \leq k-1$ by (b), so $|(p+k-1)!|_p < |j_1!|_p$, and hence any summand corresponding to

$$(j_1, j_2, \dots, j_{p-1}, j_p) \neq (k-1, 0, \dots, 0, 1)$$

vanishes modulo p . Thus the only term that contributes to (3.7) modulo p is the term corresponding to

$$(j_1, j_2, \dots, j_{p-1}, j_p) = (k-1, 0, \dots, 0, 1),$$

which is

$$\frac{(p+k-1)!}{(k-1)!p!} = \binom{p+k-1}{p}.$$

It follows that

$$(3.8) \quad S^{(2)}(p+k-1, k) \equiv \binom{p+k-1}{p} \pmod{p}.$$

There are well-known ways to express the right-hand side of (3.8). For completeness we give an argument in the spirit of counting cycles and the Dold congruence which lie behind all our results. Write $k+p-1$ as $ap+b$ with $0 \leq b < p$ and $a = \lfloor \frac{k+p-1}{p} \rfloor$ and use these numbers a and b to decompose $\Sigma = \{1, 2, \dots, ap+b\}$ into a groups of p numbers and one group of b remaining numbers (see Figure 1).

Figure 1: Defining the cycles in σ .

Define a map $\sigma: \Sigma \rightarrow \Sigma$ by

$$\sigma(j) = \begin{cases} j & \text{for } j > ap; \\ j+1 & \text{for } j \leq ap, p \nmid j; \\ j-p+1 & \text{for } p \mid j. \end{cases}$$

The map σ is a permutation of Σ with cycle type given by a cycles of length p and b cycles of length 1 (see Figure 2 for an illustration of this). Thus σ^p is the identity map.

Now $\binom{ap+b}{p}$ is the number of subsets of Σ with cardinality p . Each of the sets $\{1, \dots, p\}, \{p+1, \dots, 2p\}, \dots, \{ap-p+1, \dots, ap\}$ are fixed by σ , so they each have an orbit of length 1 under the action of σ on the set of subsets of Σ of cardinality p . All other subsets have orbits of length p since σ does not fix them while σ^p is the identity. Thus $\binom{ap+b}{p} - a$ is divisible by p , and so

$$S^{(2)}(p+k-1, k) \equiv \binom{p+k-1}{p} \equiv a = \left\lfloor \frac{k}{p} \right\rfloor \pmod{p}.$$

This gives (2.3), and hence proves the lemma. $\square$

Figure 2: The permutation σ for $p = 5$, $a = 3$, and $b = 2$.

4 The Sequence of Repair Factors

Implicit in Definition 1.4 is the ‘repair factor’ $\text{Fail}(A)$ for an almost realizable sequence A . It follows that Theorem 2.2 gives rise to a well-defined sequence $(\text{Fail}(S_k^{(2)}))_{k \geq 1}$. We know that $\text{Fail}(S_k^{(2)})$ divides $(k-1)!$, and some calculations are shown in Table 1. Here we list hypothetical values of $\text{Fail}(S_k^{(2)})$ computed as the least common multiples of the denominators of $\frac{1}{n}(\mu * S_k^{(2)})(n)$ for $n \in \{1, \dots, 3000\}$, and then tested for the realizability property for $n \leq 50000$. In order to distinguish them from the actual values, we denote them by $\text{Fail}'(S_k^{(2)})$. We have included columns showing the prime factorization, and showing $\text{Fail}'(S_k^{(2)})/\text{Rad}((k-1)!)$ (here $\text{Rad}(m)$ denotes the greatest square-free divisor, or radical, of m) as a possible aid towards trying to answer the natural question: Is there a closed expression for $\text{Fail}(S_k^{(2)})$?

In fact a closed formula for $\text{Fail}(S_k^{(2)})$ seems unlikely to be accessible. However, based on numerical computations, four conjectures concerning the values of $\text{Fail}(S_k^{(2)})$ emerge.

Conjecture 1. *If $p < k$ is a prime, then $|\text{Fail}(S_k^{(2)})|_p < 1$.*

Conjecture 2. *If $p \in [\sqrt{k}, k)$ is a prime, then $|\text{Fail}(S_k^{(2)})|_p = \frac{1}{p}$.*

Conjecture 3. *The repair factor asymptotically involves non-trivial powers of prime divisors of $(k-1)!$, in the sense that*

$$\lim_{k \rightarrow \infty} \frac{\text{Fail}(S_k^{(2)})}{\text{Rad}((k-1)!)} = \infty.$$

Table 1: Hypothetical values of the repair factor for $S_k^{(2)}$ for $1 \leq k \leq 40$, according to numerical computations.

k	$\text{Fail}'(S_k^{(2)})$	Factorization of $\text{Fail}'(S_k^{(2)})$	$\frac{\text{Fail}'(S_k^{(2)})}{\text{Rad}((k-1)!)}$
1	1	1	1
2	1	1	1
3	2	2^1	1
4	6	$2^1 \cdot 3^1$	1
5	12	$2^2 \cdot 3^1$	2
6	60	$2^2 \cdot 3^1 \cdot 5^1$	2
7	30	$2^1 \cdot 3^1 \cdot 5^1$	1
8	210	$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$	1
9	840	$2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1$	4
10	2520	$2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1$	12
11	1260	$2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1$	6
12	13860	$2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1$	6
13	13860	$2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1$	6
14	180180	$2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1$	6
15	90090	$2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1$	3
16	30030	$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1$	1
17	240240	$2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1$	8
18	4084080	$2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1$	8
19	6126120	$2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1$	12
20	116396280	$2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1$	12
21	58198140	$2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1$	6
22	58198140	$2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1$	6
23	29099070	$2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1$	3
24	669278610	$2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1$	3
25	892371480	$2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1$	4
26	4461857400	$2^3 \cdot 3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1$	20
27	2230928700	$2^2 \cdot 3^1 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1$	10
28	20078358300	$2^2 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1$	90
29	20078358300	$2^2 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1$	90
30	582272390700	$2^2 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1$	90
31	291136195350	$2^1 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1$	45
32	9025222055850	$2^1 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1$	45
33	144403552893600	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1$	720
34	48134517631200	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1$	240
35	24067258815600	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1$	120
36	24067258815600	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1$	120
37	36100888223400	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1$	180
38	1335732864265800	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1 \cdot 37^1$	180
39	333933216066450	$2^1 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1 \cdot 37^1$	45
40	333933216066450	$2^1 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1 \cdot 23^1 \cdot 29^1 \cdot 31^1 \cdot 37^1$	45

Conjecture 4. For each prime p and positive integer j , the following holds:

- (a) if $j = 1$, then $\text{Fail}(S_{p+1}^{(2)}) = p \cdot \text{Fail}(S_p^{(2)})$;
- (b) if $j > 1$, then $\text{Fail}(S_{p^j+1}^{(2)}) \mid p^{j-1} \cdot \text{Fail}(S_{p^j}^{(2)})$;

(c) if $j > 1$, then $|\text{Fail}(S_{p^j}^{(2)})|_p = p^{-1}$;

(d) we have $|\text{Fail}(S_{p^{j+1}}^{(2)})|_p = p^{-j}$.

The reverse of Conjecture 1 is clearly true. Indeed, if p is a prime divisor of $\text{Fail}(S_k^{(2)})$, then $p|(k-1)!$ and so $p < k$. It is easy to show that the statement of Conjecture 1 holds for a prime $p \in [\sqrt{k}, k)$, because in that case

$$p < k \leq p^2$$

and so

$$\left| \frac{1}{p} (S^{(2)}(p+k-1, k) - S^{(2)}(k, k)) \right|_p > 1$$

by Lemma 2.5. As a result, $|\text{Fail}(S_k^{(2)})|_p < 1$. We also point out that the statement of Conjecture 2 is satisfied for primes $p \in (\frac{k-1}{2}, k)$ as

$$\frac{1}{p} = |(k-1)!|_p \leq |\text{Fail}(S_k^{(2)})|_p < 1.$$

This also proves the last part of Conjecture 4 for $j = 1$.

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