On wake modulation and interaction features of a pair of dual-step circular cylinders in side-by-side arrangements

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ABSTRACT: The proximity interference and the vortex dislocation are fundamental fluid dynamic features that determine a wake pattern of dual-step circular cylinders in a side-by-side arrangement. Each dual-step cylinder consists of two coaxial cylinders with different diameters. A complex wake modulation is expected due to an intrinsic interaction between the vortex shedding of the larger-diameter cylinder (LC) and the smaller-diameter cylinder (SC), and to the cross-wake interaction of side-by-side cylinders in a uniform flow. To understand the physics of fluid wake modulations, three-dimensional direct numerical simulations of flow around a pair of dual-step circular cylinders in side-by-side arrangements are performed and investigated. Six simulation cases are presented with different combinations of diameter ratio (1.33-2.00), gap ratio (1.00-3.17), and Reynolds number (60-200) in a laminar flow regime, providing key insights into the two distinct LC-dominated and SC-governing wake modulation characteristics. For the LC-dominated wake modulation, the wake pattern of side-by-side cylinders is determined by the predominant vortex dynamics in the LC wake. For the SC-governing wake modulation, the LC wake pattern becomes modulated and controlled by the SC wake through the connections of vortex tubes shed downstream of the two nonuniform cylinders. Owing to the proximity interference and vortex dislocation, these new evolutionary wake modulations result in distinct wake patterns, vortex structures, and associated hydrodynamic force features for the pair of side-by-side dual-step circular cylinders.

Key words: dual-step cylinder, side-by-side arrangement, vortex dislocation, proximity interference

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1. Introduction

The complex vortex shedding phenomena of multiple cylindrical structures in engineering practice such as marine risers, pipeline bundles, heat exchange tubes, and wind turbine towers have attracted great attention from the scientific and industrial communities. The mechanisms of vortex dislocation and wake interference have been found to play a significant role in the wake vortex dynamics.

For flow past a long uniform cylinder in a laminar regime, two rows of vortex tubes are alternately shed from the cylinder. However, due to the cylinder end effect, the vortex tubes are asymptotically oblique to the cylinder axis. A vortex dislocation phenomenon, first suggested by Williamson¹, occurs and separates the vortex shedding into cells featured by different oblique angles and vortex shedding frequencies along the cylinder span. Such an intrinsic feature is found to be responsible for the intermittency characteristic in the wake velocity with a low-frequency fluctuation².

For a circular cylinder attached with a ring, for which a ratio of the larger diameter (D) to the smaller diameter (d) is in the range of $1.1 \leq D/d \leq 2$ and an aspect ratio $L/D = 0.5$, where L is the spanwise length of the larger-diameter ring, a forced vortex dislocation was observed³. The frequency of the vortex dislocation (f_{dis}) is found to be the difference between the vortex shedding frequency of the smaller-diameter cylinder (f_s) and that of the larger-diameter ring (f_L), namely $f_{dis} = f_s - f_L$. The forced vortex dislocation is further studied through circular cylinders with local discontinuity in the spanwise direction. For laminar flow past a single-step cylinder (i.e., a nonuniform stepped cylinder comprising two coaxial cylinders with different diameters), the vortex shedding cells have been found in the experiment of Lewis and Gharib⁴. The S-cell shed from the smaller-diameter cylinder (referred to as SC hereafter) has a higher vortex shedding frequency f_s while the L-cell shed from the larger-diameter cylinder (referred to as LC hereafter) has a lower vortex shedding frequency f_L . The vortex dislocation reveals the different vortex connections depending on the diameter ratio. A direct vortex connection mode is found when $D/d < 1.25$, with the in-phase shed vortex tubes of S-cell and L-cell connecting across a wake interface. As $D/d > 1.55$, the vortices shed in different cells become out-of-phase, and an indirect connection mode occurs as observed in the experiment of Lewis and Gharib⁴ and the computational study of Vallès et al.⁵. A vortex linkage, which is named the half-loop connection and connects two counter-rotating S-cell vortex tubes shed from the opposite sides of SC, exists in a buffer zone between S-cell and L-cell. This zone is named the modulation zone where a third frequency,

lower than f_S and f_L , is discovered. Such a modulation zone has been identified as N-cell⁶, with the vortex shedding from the step on the LC side having the lowest vortex shedding frequency f_N . Numerical simulations with $D/d \sim 2$ and Reynolds number based on the larger diameter $Re_D \sim 150$ showed that the connective region between S-cell and N-cell is relatively narrow and stable⁷, while that between N-cell and L-cell is wide⁸ and varies with a phase difference^{9, 10}. A modulation zone was also reported in the turbulent flow past a single-step cylinder¹¹, indicating a similarity in the vortex shedding pattern in laminar and turbulent flow regimes^{4, 12}.

For flow past a dual-step cylinder, the vortex shedding behind LC is highly dependent on the aspect ratio (L/D), diameter ratio (D/d) and Reynolds number (Re). For turbulent flow, the effect of L/D on the vortex shedding at $Re_D = 1050$ and $D/d = 2$ has been studied¹³ with four distinct vortex shedding patterns being identified for the varying L/D from 1 to 17. Two N-cells and one L-cell were identified in LC region when $L/D \geq 15$ while only one L-cell was found in the region of $8 \leq L/D \leq 14$. For $2 \leq L/D \leq 6$, there is no L-cell shedding behind LC; the associated vortex shedding (N-cell) is highly three-dimensional and substantially deformed into a hairpin-like shape. No independent vortex shedding behind LC at a lower aspect ratio ($L/D \leq 2$). In a wider range of $2000 \leq Re_D \leq 5500$ with $1.33 \leq D/d \leq 2$ and $0.5 \leq L/D \leq 5$ ¹⁴, the regular wake structures downstream of LC can only be found in cases with a moderate L/D ($L/D \geq 0.5$) and a lower D/d ($D/d \leq 2$). Recently, a more detailed map of six distinct flow regimes for flow around a dual-step cylinder for $Re_D = 2100$ has been reported¹⁵ with $0.2 \leq L/D \leq 5$ and $1.33 \leq D/d \leq 4$. Within this map, a dual-step cylinder with $L/D = 5$ and $D/d < 2$ generally has a simple and stable N-cell vortex shedding.

As for laminar flow, a numerical study¹⁶ at $Re_D = 150$ for $1.1 \leq D/d \leq 4$ and $0.2 \leq L/D \leq 5$ reveals four distinct types of the wake topology downstream of LC, which are similar to those found in the turbulent flow regime and closely related to the vortex dislocation between S-cell and N-cell. In addition to the half-loop vortex connection between S-cell and N-cell, two new types of connection (double-half-loop and triple-half-loop connections) have been found in the wake of a dual-step cylinder at $Re_D = 200$ for $L/D = 5$ and $D/d \leq 2$ ¹⁷. In the wake of a dual-step cylinder undergoing vortex-induced vibration, a new “X-shaped” connection has been discovered by Ji et al.¹⁸. Moreover, the downwash flow, referring to a swirling flow directing to the middle span and originally discovered on the aft surface of a cylinder with two free ends¹⁹, can also be found near the sharp edges of the dual-step

cylinder¹⁶. As expected, the larger D/d induces a greater downwash effect, indicating a higher base pressure on LC, a longer vortex formation length, and a lower lift fluctuation.

For two side-by-side cylinders, the proximity interference takes place when the two structures are arranged in a relatively close position. Since the pioneering work of Zdravkovich²⁰, several researchers have investigated the proximity interference between a pair of side-by-side uniform circular cylinders in cross flow²¹⁻²⁶. As summarized by Sumner²⁷, different flow regimes appear when the gap ratio $g^* = g/D$ is varied, in which g is the gap between the side-by-side cylinder surface. For $g^* < 0.1-0.2$, the two cylinders behave like a single bluff body with a single vortex street observed in the wake²¹. For $0.2 \leq g^* \leq 1.2$, an asymmetric flow regime with a deflected gap flow is observed^{22, 25}. For $1.2 \leq g^* \leq 5$, the vortex shedding from the cylinders is in-phase or anti-phase synchronized²³. For laminar flow at $Re = 40 \sim 160$, Kang²⁴ observed six wake patterns, including the anti-phase synchronized (AP), in-phase synchronized (IP), flip-flopping (FF), deflected (DF), single bluff-body, and steady wake. These patterns are determined by Re and g^* .

Among these flow patterns, the flip-flopping flow – featured by a biased gap flow intermittently and randomly changing its direction at a low switching frequency – is the most intriguing phenomenon which has attracted interest from many researchers^{23, 26, 28-31}. The deflected gap flow toward one of the side-by-side cylinders generates narrower and wider wakes. The cylinder with the narrower wake is subject to a greater drag force and a higher vortex shedding frequency. Experimental tests at $g^* = 0.75$ and $Re = 2-7 \times 10^3$ conducted by Kim and Durbin²⁵ showed that the switching of the gap flow takes place at a frequency being lower, by three orders of magnitude, than the vortex shedding frequency. From the flow stability standpoint, the FF flow is typically considered as a dynamical system consisting of two quasi-stable asymmetric states²⁵. By performing a linear global stability analysis, Carini *et al.*^{32, 33} found that the FF flow stems from an instability in the gap region out of an in-phase synchronized vortex shedding. This is supported by the Dynamic Mode Decomposition (DMD) analysis in Liu and Jaiman^{34, 35}.

Although previous studies have highlighted the vortex shedding of a stepped cylinder in cross flow and the proximity interference of side-by-side uniform cylinders, the complex wake interaction and modulation of side-by-side dual-step cylinders are still relatively unknown. Such features are crucial for understanding the complicated hydrodynamics and vibration responses of practical engineering

structures, such as marine drilling or production risers assembled with buoyancy modules. The effect of internal flow and heat transfer may also be considered for the riser pipe^{36, 37}.

In this study, we apply the direct numerical simulation to numerically investigate the three-dimensional flow around a pair of dual-step circular cylinders in side-by-side arrangements with different combinations of diameter ratio, gap ratio and Reynolds number. The rest of this paper is organized as follows. Section 2 presents the numerical methodology and the computational domain. Section 3 evaluates and presents hydrodynamic force coefficients, flow energy spectra and evolutions of vortex structures. DMD modes of the wake flow are also discussed to elucidate the origin and dynamic features of the wake interaction and modulation. New key findings are summarized in Section 4. A summary of abbreviations used in this paper is provided in Table 1 as follows.

Table 1. A summary of abbreviations

Abbreviation	Definition
AP	Anti-phase synchronization flow pattern
DF	Deflected flow pattern
FF	Flip-flopping flow pattern
IP	In-phase synchronization flow pattern
LC	Larger-diameter cylinder
SC	Smaller-diameter cylinder
L-cell	Vortex shedding cell from LC
N-cell	Vortex shedding cell near the step (i.e., a modulation zone) between LC and SC
S-cell	Vortex shedding cell from SC
DMD	Dynamic Mode Decomposition
FFT	Fast Fourier Transform
PSD	Power Spectral Density

2. Numerical methodology

2.1 Governing equations

The three-dimensional incompressible flow past a pair of dual-step circular cylinders is governed by the continuity and Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}, \quad (2)$$

where $\mathbf{u}$ is the velocity vector, t the time, ρ the fluid density, p the pressure, and ν the kinematic viscosity. These equations are numerically solved by applying the open-source framework Nektar++³⁸⁻⁴⁰ using the spectral/hp element method⁴¹. The fourth-order polynomials are adopted for achieving an improved numerical stability. A high-order splitting scheme⁴² is employed for the time integration by using an approach proposed by Guermond and Shen⁴³, see www.nektar.info for greater details.

2.2 Computational setup

As shown in Figure 1(a), the computational domain is set as $[-10D, 15D] \times [-10D, 10D] \times [0, 10D]$ in x , y , and z direction, respectively, with the origin locating at the middle point between the two cylinders and at the bottom surface ($z = 0$). For convenience, the back and front cylinders are named cylinder [a] and [b], respectively, hereafter. The inflow boundary with the Dirichlet boundary conditions ($u = U_\infty$; $v = w = 0$) is $10D$ upstream the center of the cylinders, while the outflow boundary with the Neumann boundary conditions ($\partial u / \partial x = \partial v / \partial x = \partial w / \partial x = 0$) is $15D$ downstream. The upper and lower surfaces are set as periodic boundaries, while the front and back surfaces are set as free-slip boundaries ($\partial u / \partial y = \partial w / \partial y = 0$; $v = 0$). The cylinder surfaces are treated as no-slip boundaries with $u = v = w = 0$. For the pressure, $p = 0$ is set at the outflow boundary while $\partial p / \partial n = 0$ is set at other boundaries.

Figure 1(b) shows a sketch of a pair of identical dual-step circular cylinders in a side-by-side configuration. The cylinders have a larger-to-smaller diameter ratio D/d , with a side-by-side gap ratio $g^*_D (= G/D)$ for LC and $g^*_d (= g/d)$ for SC, where G and g are the gap sizes as illustrated in Figure 1(b). The overall aspect ratio is $L/D = 10$, where L represents the total cylinder length. LC has a spanwise length $L_D = 5D$ while SC has a spanwise length $L_d = 2.5D$ for each top and bottom part. For a given free stream velocity U_∞ , $Re_D = U_\infty D / \nu$ for LC and $Re_d = U_\infty d / \nu$ for SC.

Figure 1. Computational set-up: (a) simulation domain, (b) side-by-side dual-step cylinders

A structured mesh is generated by applying the three-dimensional finite element mesh generator Gmsh, as shown in Figure 2. To guarantee accuracy of the simulations, the mesh is refined in the x - y plane near the cylinders in a region of $2D$ in the x direction and $4D$ in the y direction, and $1D$ in the z direction near each step. The total element number is 436160 with the smallest grid size near the cylinder of about $d/64$. The adopted dimensionless time step size is $\Delta t U_\infty / D = 0.008$. The grid dependency of the adopted mesh has been checked in our pervious study³⁸ for flow past two side-by-side dual-step cylinders with $D/d = 1.33$, $Re_d = 100$ ($Re_D = 133.33$) and $g^*_d = 3$ ($g^*_D = 2$). The acceptable simulation accuracy was also demonstrated in Yan et al.³⁸ when comparing our numerical results with published data for flow past a pair of uniform cylinders in a side-by-side arrangement with $g^* = 0.7$ at $Re = 100$.

Figure 2. Computational mesh: (a) x - y plane, (b) close-up view, and (c) y - z plane

3. Results and Discussion

For flow past a pair of dual-step cylinders in side-by-side configurations, two types of vortex interactions, i.e., the vortex dislocation near the steps and the proximity interference between the cylinders, are significant. The vortex dislocation, mostly influenced by the diameter ratio D/d , separates the vortex shedding into cells. These cells may interact due to the proximity interference when the two cylinders have a low to medium gap ratio g^*_D (g^*_d). To investigate an interaction between the vortex dislocation and the proximity interference, six cases with different combinations of D/d and g^*_D (g^*_d) are simulated at different Re , as listed in Table 2. For a comparison purpose, the vortex shedding patterns of two side-by-side uniform cylinders at the same gap ratios and Reynolds numbers of LC and SC are also listed, according to the results of previous studies^{21, 23, 24, 27, 35}. Note that these six simulation cases have been selected based on an individual wake pattern of a pair of side-by-side uniform cylinders, such that the most complicated wake patterns (AP, FF, IP, DF) do coexist and interact downstream of the nonuniform LC and SC in a side-by-side arrangement leading to a new evolution of wake modulation and interaction characteristics.

By comparing the flow patterns of dual-step cylinders with those of uniform cylinders, the wake modulations of LC and SC are investigated. From Table 2, two types of wake modulations can be observed: (i) the LC-dominated modulation in Cases 1-3 where the flow pattern of dual-step cylinders is mainly determined by the vortex shedding of LC; (ii) the SC-governing modulation in Cases 4-6

where the vortex shedding of SC plays an important role in determining the wake pattern of dual-step cylinders. For instance, the flow pattern of dual-step cylinders in Case 3 is FF being the same as that of side-by-side uniform cylinders at $g^*_D = 1.0$ and $Re_D = 200$. On the other hand, in Case 4, the FF flow, taking place in the wake of dual-step cylinders, is different from IP and AP flows of uniform cylinders at $g^*_D = 1.5$ and $Re_D = 120$ (IP) and at $g^*_d = 3.17$ and $Re_d = 72$ (AP), respectively, according to the results in previous studies^{23, 24, 27, 35}. A detailed explanation about these two kinds of wake modulation will be discussed in the following.

Table 2. Computational cases of uniform flow past two side-by-side dual-step cylinders. Flow patterns of side-by-side dual-step cylinders are listed, together with those of side-by-side uniform cylinders at the gap ratios corresponding to their dual-step counterparts (AP: anti-phase synchronized vortex shedding; IP: in-phase synchronized vortex shedding; FF: flip-flopping gap flow; DF: stably deflected gap flow).

Cases	D/d	g^*_d	Re_d	g^*_D	Re_D	Uniform Cylinders		Dual-step Cylinders
						at g^*_d and Re_d	at g^*_D and Re_D	
1	1.33	3.00	100	2.00	133.33	AP	AP	AP
2	1.67	3.00	100	1.40	166.67	AP	FF	FF
3	2.00	3.00	100	1.00	200.00	AP	FF	FF
4	1.67	3.17	72.0	1.50	120.00	AP	IP	FF
5	1.67	3.17	60.0	1.50	100.00	AP	IP	DF
6	1.67	2.75	72.0	1.25	120.00	AP	IP	AP

3.1 LC-dominated modulation

The LC-dominated modulation has been identified as a consequence of the wake instability stemming from the LC region by Yan et al.⁴⁴, where the three-dimensional DMD analysis reveals the origin of this wake modulation exclusively in the LC region. In this study, we further explore the effect of diameter ratio on the wake modulation features.

3.1.1 Flow regime

The drag and lift force coefficients of LC and SC are calculated as follow:

$$C_{D,L} = \frac{2(F_{D,L} + F_{D,step})}{\rho U_{\infty}^2 DL_D}, \quad C_{D,S} = \frac{2F_{D,S}}{\rho U_{\infty}^2 dL_d}, \quad (3)$$

$$C_{L,L} = \frac{2(F_{L,L} + F_{L,step})}{\rho U_{\infty}^2 DL_D}, \quad C_{L,S} = \frac{2F_{L,S}}{\rho U_{\infty}^2 dL_d}, \quad (4)$$

where $C_{D,L}$ ($C_{D,S}$) and $C_{L,L}$ ($C_{L,S}$) are the drag and lift coefficients of LC (SC), $F_{D,L}$ ($F_{D,S}$) and $F_{L,L}$ ($F_{L,S}$) are the drag and lift forces on LC (SC), $F_{D,step}$ and $F_{L,step}$ are the summation of total drag and lift forces on the two step surfaces (at $Z/D = 2.5$ and 7.5), respectively.

Figure 3 presents the drag and lift coefficients of LC in Cases 1-3, together with the power spectral density (PSD) of the normalized energy along the spanwise probe lines in the wake of cylinder [a]. The drag and lift coefficients of SC are similar to those of LC; for the sake of conciseness, they are not shown here. Figures 4 and 5 display the representative snapshots from the associated supplementary video files which help in understading the variations of hydrodynamic force coefficients.

In Case 1 with $D/d = 1.33$, the C_D curves of cylinder [a] and [b] overlap, and the C_L curves are synchronized in an anti-phase manner, as shown in Figure 3(a). Although the LC force coefficients reveal the AP flow characteristics as reported in Kang²⁴, they further manifest the beating-type features with time-varying amplitudes. As shown in Figure 3(d), the beating frequency $f_{beat} = 0.057$ is equal to the difference between the vortex shedding frequency of LC at $f_N = 0.184$ and that of SC at $f_S = 0.241$, reflecting a nonlinear interaction of N-cell and S-cell vortices^{3, 16}. Moreover, the beating frequency mainly exists near the steps, implying that such a wake modulation is caused by the vortex dislocation with a frequency $f_{dis} \approx f_{beat}$.

For the FF flow behind LC in Case 2 ($D/d = 1.67$) and Case 3 ($D/d = 2$), the time histories of force coefficients are much more complicated. The C_D and C_L curves of cylinder [a] and [b] are neither synchronized nor showing any clear periodicity. As reported in previous studies^{24, 25, 33}, corresponding to the intermittent and random switching of the gap flow, the drag force predominance and the vortex shedding frequencies of the cylinders change alternately. That is, the cylinder with a narrower wake has a greater drag and a higher vortex shedding frequency. The vortex shedding frequencies are $f_N = 0.186$ for the N-cell and $f_S = 0.152$ for the S-cell in Case 2, and they are $f_N = 0.152$ and $f_S = 0.370$ in

Case 3. The nonlinear interaction of N-cell and S-cell vortices generates the vortex dislocation at the frequencies $f_{dis} = f_S - f_N = 0.120$ and 0.218 in Cases 2 and 3, respectively, see Figure 3 (h) and (i). However, apart from frequencies of the vortex dislocation, there are some very low frequency components in the spectra (see Figure 3 (e) and (f)), i.e., the flip-flopping frequencies $f_{FF} = 0.053$ for Case 2 and $f_{FF} = 0.054$ for Case 3. Due to the existence of the flip-flopping flow, the vortex dislocation is no longer periodic but showing a significant temporal variation. This will be discussed later on.

Figure 3. Time histories of drag (C_D) and lift (C_L) coefficients of LC in (a) Case 1, (b) Case 2 and (c) Case 3. Subfigures (d-i) show PSD spectra of the energy density E_{uu} and E_{vv} , normalized by the total power density, in the streamwise and transverse directions, respectively. (d) and (g) are for $tU_\infty/D = 205-290$ in Case 1, (e) and (h) are for $tU_\infty/D = 240-400$ in Case 2, and (f) and (i) are for $tU_\infty/D = 416-510$ in Case 3. E_{uu} and E_{vv} (see Ref ⁴⁵ for definitions) are calculated based on the velocities probed at

241 $[x/D = 5, y/D = 2]$ for Case 1, $[x/D = 5, y/D = 1.7]$ for Case 2, and $[x/D = 5, y/D = 1.5]$ for Case 3.

242

243 **Figure 4.** Development history of z -vorticity ω_z of case 1 at $tU_\infty/D = 128-194$ in (a) $z/D = 5$ and (b)
 244 $z/D = 9.5$; and of case 2 at $tU_\infty/D = 282-313$ in (c) $z/D = 5$ and (d) $z/D = 9.5$, and at $tU_\infty/D = 313-349$
 245 (e) $z/D = 5$ (f) $z/D = 9.5$. Visualization is provided through a supplementary video (Multimedia View).

246

247 **Figure 5.** Development history of z -vorticity ω_z of case 3 at $tU_\infty/D = 416-511$ in (a) $z/D = 5$ and (b)

248 $z/D = 9.5$, at $tU_\infty/D = 635-704$ in (c) $z/D = 5$ and (d) $z/D = 9.5$, and at $tU_\infty/D = 705-770$ (e) $z/D = 5$ (f)
249 $z/D = 9.5$. Visualization is provided through a supplementary video (Multimedia View).

250 3.1.2 Vortex structures

251 To gain better understanding of wake modulation features of dual-step cylinders in side-by-side
252 configurations, the vortex structures are now analyzed. Owing to the less significant discontinuity in
253 the z direction, i.e., the lower diameter ratio, Case 1 has a relatively simpler vortex shedding pattern.
254 Based on the snapshot series in Figure 6, a regular vortex modulation in the wake of cylinder [b] is
255 observed, where the blue (marked by positive numbers) and grey (marked by negative numbers) tubes
256 represent vortices shed from the free-stream side and the gap side, respectively. Between the S-cell
257 and N-cell vortices, a regular cycle of the vortex connection is recognized. For example, in Figure 6(a),
258 the vortex pair +1 and -1 is nearly parallel to the cylinders, and the direct connections between the S-
259 cell and N-cell are observed. However, the vortex tubes +2 and -2 significantly bend in the x - y plane
260 and become oblique toward the middle span due to the relatively low streamwise velocity in the wake
261 of LC. Further, the vortex tube -2 directly connects with the vortex tube -3, forming an ‘X-shape’
262 connection¹⁸ between S-cell and N-cell. The +3 tube only exists in the wake of SC and connects with
263 the +4 tube, generating a half-loop connection. The vortex tube -4 is straight and connects with the
264 vortex tube -3 through the streamwise vortices, as shown in Figure 6(b-c). As time marches on, this
265 vortex connection process repeats, as demonstrated by the vortex series 5-8 and 9-12, exhibiting a
266 striking similarity to the instantaneous 1-4 vortex structures. The vortex dislocation frequency matches
267 the frequency spectra as shown in Figure 3. In the 1-4 cycle, four vortex pairs are shed from SC while
268 three vortex pairs are shed from LC due to the absence of +3 vortex tube and the coalesce of -2 and -
269 3 tubes in N-cell. This is in line with the frequency ratios $f_S/f_{dis} = 0.241 / 0.057 \approx 4$ and $f_N/f_{dis} = 0.184$
270 $/ 0.057 \approx 3$. Owing to the different vortex shedding frequency of N-cell and S-cell, the phase lag
271 between the vortex shedding changes continuously with time, and the phase realignment occurs in
272 every fundamental dislocation cycle.

Figure 6. Instantaneous vortex structures, represented by the iso-surface $Q = 0.2$, for Case 1 at tU_∞/D = (a) 138.92, (b) 141.52, (c) 144.12, (d) 146.72, (e) 149.32, (f) 151.92, (g) 154.52, (h) 157.12, (i) 159.72, (j) 162.32, (k) 164.92, (l) 167.52, (m) 170.12, (n) 172.72, (o) 175.32, and (p) 177.92.

The vortex shedding in Cases 2 and 3 manifests obvious cell patterns^{6, 16}: the S-cell behind SC has a much higher shedding frequency than the N-cell behind LC. As a result, the direct half-loop and ‘X-shape’ connections exist in the wake of both cylinders. However, different from the periodic vortex dislocation at $D/d = 1.33$, the vortex shedding is aperiodic and irregular, especially in the wider wake, owing to the unstable flip-flopping gap flow. As stated in our pervious study with $D/d = 2.0$ ⁴⁴, the vortex dislocations in the wider and narrower wakes evolve differently. For example, the half-loop connection happens more frequently in the narrower wake than in the wider wake, and this feature can be attributed to the frequency difference between S-cell and N-cell in the wider and narrower wakes. The vortex dislocation frequency is lower ($f_{dis} = 0.169$) in the wider wake ($tU_\infty/D = 636.22-695.5$) and higher ($f_{dis} = 0.218$) in the narrower wake ($tU_\infty/D = 706.94-760.5$), see Ref⁴⁴ for more details. Further, the vortex reconnection and the multiple connection dominate the wider wake region because of the

less N-cell vortices in the wider wake with a lower vortex shedding frequency. These features can also be observed in Case 2, as shown in Figure 7, where the vortex structures are visualized by the instantaneous iso-surface $Q = 0.2$. Snapshots (a-h) are extracted during a period when the gap flow is deflected to cylinder [a]. That is, cylinder [a] has a narrower wake than cylinder [b]. As expected, more multiple connections and highly distorted N-cell vortices are observed in the wider wake (see red and green dashed lines for vortex connections in the narrower and wider wake in Figure 7). However, since the diameter ratio is lower than 2, the less half-loop connections and the more direct connections are observed in the narrower wake.

Figure 7. Instantaneous vortex structures, represented by the iso-surface $Q = 0.2$ of Case 2 at $tU_\infty/D = 325.28-349.20$. Subfigures (a-h) show vortex structures when cylinder [a] has a narrower wake while subfigures (i-p) show the vortex structures when cylinder [b] has a narrower wake. Grey and blue tubes stand for vortices shed from the free-stream side of cylinders [a] and [b], respectively.

3.1.3 Dynamic Mode Decomposition analysis

In this section, the Dynamic Mode Decomposition (DMD) method⁴⁶⁻⁴⁸ is employed to decompose the complex flip-flopping flow system and extract essential flow characteristics in Case 3. The DMD procedure described in Kutz et al.⁴⁷ is adopted, and readers are referred to Kutz et al.⁴⁷ for greater details. The three-dimensional DMD analysis is performed based on the snapshots of the z -direction vorticity ω_z extracted from period I ($tU_\infty/D = 416.22-509.82$), II ($tU_\infty/D = 636.22-695.5$) and III ($tU_\infty/D = 706.94-760.5$) in a three-dimensional region $[-0.5, 8] \times [-3, 3] \times [0, 10]$ ($x \times y \times z$). The three periods have been carefully selected with the gap flow alternately switching six times (three times toward each side) in period I, or stably biasing to cylinder [b] in period II and cylinder [a] in period III. The time interval between two sequential snapshots is $\Delta tU_\infty/D = 0.52$, which generates more than 100 snapshots for each period.

As mentioned in Bagheri⁴⁹, when the sampled data series contain non-exponential behaviors, the DMD results show clustered complex modes describing the dynamics. This phenomenon can be observed in the DMD spectra shown in Figure 8(a) and (e) where three clusters of DMD modes appear, representing three major dynamics in period I. Except for the mean flow mode shown in Figure 8(b-d), the first cluster is the DMD modes with the frequency $f_{DMD} < 0.06$, colored in purple in the DMD spectra. The representative mode at $f_{DMD} = 0.033$ is shown in Figure 8(f-h) where the z -vorticity is mainly concentrated within the gap flow region downstream of LC. The slice extracted at $z/D = 5.5$ shows a typical pitchfork bifurcation⁵⁰ which is related to the deflected gap flow. Such bifurcation mode was also found to be an intrinsic feature of the flip-flopping flow in previous studies^{33, 35}. In addition, the DMD frequency $f_{DMD} = 0.033$ is close to the flip-flopping frequency $f_{FF} = 0.054$ as recognized in Section 3.1.1.

The second cluster is colored by green and cyan in the DMD spectra with $f_{DMD} = 0.15-0.21$, describing the vortex shedding behavior of N-cell, i.e., the Hopf bifurcations⁵⁰ in LC region. The representative modes are shown in Figure 8(i-j) and 8(k-l) with the DMD frequency $f_{DMD} = 0.156$ and 0.209 , respectively. These two frequencies are close to the vortex shedding frequency $f_N = 0.152$ of the LC wake and the vortex dislocation frequency $f_{dis} = 0.218$ near the steps, as shown in Figure 3(i). However, one can notice that these two Hopf bifurcation modes not only have different frequencies, but also reveal their different synchronization patterns. Specifically, the mode at $f_{DMD} = 0.156$, as

shown in Figure 8(i-j), is symmetric with respect to the middle plane, corresponding to the in-phase synchronized vortex shedding. However, the mode at $f_{DMD} = 0.209$ as shown in Figure 8(k-l) is anti-symmetric, corresponding to the anti-phase synchronized vortex shedding. This is consistent with the findings of Chen et al.⁵¹ in which the IP pattern has a lower vortex shedding frequency than the AP pattern due to the confined space between the two gap vortices in AP flow.

Similarly, the coexistence of AP and IP modes also appears in the third cluster, as colored in red with $f_{DMD} = 0.36-0.39$, describing the vortex shedding behavior of S-cell. At the DMD frequencies $f_{DMD} = 0.377$ and 0.386 , two DMD modes as shown in Figure 8(m-n) and 8(o-p) are significant in the wake of SC. Despite the vorticity distribution behind the two SCs are not perfectly aligned, they can still be recognized as the IP and AP Hopf bifurcation, respectively. Note that, because the gap between SC is greater than the gap between LC, the pushing effect between the AP synchronized gap vortices in S-cell is less significant. As a result, the frequency difference (0.009) between IP and AP modes of S-cell is less than that (0.053) of N-cell.

From the above analyses, it is worth emphasizing that frequencies of DMD modes are closely related to the represented physical features. For instance, the DMD mode for the pitchfork bifurcation has a frequency of the switching gap flow, while the DMD mode for the Hopf bifurcation has a frequency of the vortex shedding. Such features were also reported in Liu and Jaiman^{34,35}, where a DMD analysis was employed for the FF flow behind two uniform cylinders. Further, the vorticity distributions of these DMD modes are reasonably similar to those in Mizushima and Ino⁵⁰ where a strict bifurcation analysis was performed. Therefore, it is confirmed that the DMD mode shown in Fig. 8(f-h) is related to the pitchfork bifurcation while the DMD modes in Fig. 8(i-p) are related to the Hopf bifurcation.

As stated in Carini³³, the interaction between the pitchfork and Hopf bifurcations is believed to be the origin of the flip-flopping flow. However, the pitchfork bifurcation is only found in the LC region and it cannot be observed in the SC region (see Figure 8(f-h)). This suggests that the flip-flopping flow behavior of S-cell vortices is essentially originated from the LC region⁴⁴. This proves to be the predominance of N-cell vortices in the LC-dominated wake modulation.

Figure 8. DMD analysis of Case 3 in period I ($tU_\infty/D = 416.22-509.82$). (a) and (e) show DMD spectra of the eigenvector μ and the amplitude b , where μ_r and μ_i are real and imaginary parts of the eigenvector, respectively, and f is the frequency. DMD modes are represented by iso-surface at $\omega_z = \pm 5 \times 10^{-4}$ and x - y slices with ω_z contours, at the frequencies (b-d) $f_{DMD} = 0$ or $f_{DMD} = \ln(\mu_i)/2\pi$, (f-h) $f_{DMD} = 0.033$, (i-j) $f_{DMD} = 0.156$, (k-l) $f_{DMD} = 0.209$, (m-n) $f_{DMD} = 0.377$, and (o-p) $f_{DMD} = 0.386$. In the subfigures, cylinder

362 [a] is on the right while cylinder [b] is on the left.

363 According to the above DMD analysis, we notice that the vortex shedding of the flip-flopping
364 flow of two side-by-side cylinders is out-of-synchronization, showing the co-existence of AP and IP
365 flow patterns. Recall that the even distribution of vorticity in Figure 8(i-l) is related to how the gap
366 flow switches its direction six times in period I, an effort should be made to verify if AP and IP modes
367 still coexist when no switching occurs within a period. Therefore, we further perform DMD analysis
368 on the periods II and III when the gap flow is consistently deflected toward cylinder [b] and cylinder
369 [a], respectively. Here, only the DMD modes related to N-cell are presented for the sake of conciseness.
370 As seen in Figure 9, when the gap flow is consistently deflected toward cylinder [b], there are also two
371 DMD modes with frequencies close to the vortex shedding frequency of N-cell, $f_N = 0.152$. The first
372 DMD mode at $f_{DMD} = 0.128$ concentrates within the wider wake, see Figure 9(b). By elaborating the
373 contours of ω_z in the near wake, this represents an IP mode, owing to the z -vorticity around the
374 cylinders with the same negative sign, see Figure 9(c). On the contrary, the second DMD mode with
375 $f_{DMD} = 0.186$ concentrates within the narrower wake. Due to the opposite-sign vorticity behind the
376 cylinders (see Figure 9(g)), this mode is recognized as an AP mode. The opposite scenario takes place
377 when the gap flow consistently deflects toward cylinder [a] in period III, as shown in Figure 9(i-p).
378 Hence, these results highlight that AP and IP modes are intrinsic to the flip-flopping flow.

Figure 9. DMD analysis of Case 3 in period II (a-h, $tU_\infty/D = 636.22-695.5$) and III (i-p, $tU_\infty/D = 706.94-760.5$). The first column shows DMD spectra of the eigenvector μ and the amplitude b , where μ_r and μ_i are real and imaginary parts of the eigenvector, respectively, and f is the frequency. DMD modes are represented by iso-surface at $\omega_z = \pm 5 \times 10^{-4}$, x - y slices at $z/D = 5$ and y - z slices at $x/D = 1.1$ with ω_z contours, at the frequencies (b-d) $f_{DMD} = 0.128$, (f-h) $f_{DMD} = 0.186$, (j-l) $f_{DMD} = 0.140$, and (n-p) $f_{DMD} = 0.188$. In the subfigures, cylinder [a] is on the right while cylinder [b] is on the left.

3.2 SC-governing modulation

For the SC-governing modulation, the flow pattern downstream of LC is altered and controlled by the vortex shedding of SC.

3.2.1 Flow regime

As listed in Table 2, the flow patterns of two side-by-side uniform cylinders at the specified gap ratios g^*_D are all the IP flow. However, according to the force coefficient time histories and energy spectra in Figure 10 and the vortex shedding patterns in Figures 11-13, none of the flow pattern of LC in Cases 4-6 turns out to be an IP flow^{23, 24, 27}, due to the wake modulation with S-cell vortices. Like Cases 2 and 3, Case 4 ($g^*_D = 1.5$, $Re_D = 120$) shows the FF pattern, according to the spectral analysis in Figure 10(d) and (g). The vortex shedding shows a typical cell pattern with a higher frequency $f_S = 0.283$ for S-cell and a lower frequency $f_N = 0.182$ for N-cell. A weak vortex dislocation frequency $f_{dis} = 0.1$ is found near the steps while the flip-flopping frequency is recognized as the lowest frequency $f_{FF} = 0.01$. Case 5 ($g^*_D = 1.5$, $Re_D = 100$) shows the stably deflected (DF) gap flow (toward cylinder [b]) with different drag and lift forces on each of the dual-step cylinders. The vortex shedding of the pair of SCs is synchronized at $f_S = 0.188$ while that of LCs are asynchronized with $f_N = 0.067$ for cylinder [a] and $f_N = 0.107$ for cylinder [b], as shown in Figure 10(e) and (h). The beating frequency $f_{beat} = 0.040$ represents the nonlinear interaction of the two LC wakes. For Case 6 ($g^*_D = 1.25$, $Re_D = 120$), the flow pattern turns out to be an AP one and the vortex shedding of the two cylinders is symmetric. The S-cell has a higher vortex shedding frequency at $f_S = 0.263$ whereas the N-cell has a lower frequency at $f_N = 0.126$. Other frequencies are due to the vortex dislocation ($f_{dis} = 0.137$) and the beating ($f_{beat} = 0.011$). Note that the amplitudes of drag and lift coefficients are small in Figure 10(c) because the shear layers of the cylinders are stable in the near wake with vortices rolling up far away from the cylinders, as shown in the supplementary video.

Figure 10. Time histories of drag (C_D) and lift (C_L) coefficients of LC in (a) Case 4, (b) Case 5 and (c) Case 6. Subfigures (d) to (i) show PSD spectra of the energy density E_{uu} and E_{vv} , normalized by the total power density, in the streamwise and transverse directions, respectively. (d) and (g) are for $tU_\infty/D = 360-385$ in Case 4, (e) and (h) are for $tU_\infty/D = 320-470$ in Case 5, and (f) and (i) are for $tU_\infty/D = 120-216$ in Case 6. E_{uu} and E_{vv} are calculated based on the velocities probed at $[x/D = 5, y/D = 1.75]$ for Case 4, $[x/D = 5, y/D = 1.75]$ for Case 5, and $[x/D = 5, y/D = 1.625]$ for Case 6.

Figure 11. Development history of z -vorticity ω_z of case 4 at $tU_\infty/D = 80$ -150 in (a) $z/D = 5$ and (b) $z/D = 9.5$, at $tU_\infty/D = 150$ -220 in (c) $z/D = 5$ and (d) $z/D = 9.5$, and at $tU_\infty/D = 220$ -290 (e) $z/D = 5$ (f) $z/D = 9.5$. Visualization is provided through a supplementary video (Multimedia View).

Figure 12. Development history of z -vorticity ω_z of case 5 at $tU_\infty/D = 110$ -170 in (a) $z/D = 5$ and (b) $z/D = 9.5$, at $tU_\infty/D = 170$ -270 in (c) $z/D = 5$ and (d) $z/D = 9.5$, and at $tU_\infty/D = 270$ -319 (e) $z/D = 5$ (f) $z/D = 9.5$.

424 $z/D = 9.5$. Visualization is provided through a supplementary video (Multimedia View).

425

426 **Figure 13.** Development history of z -vorticity ω_z of case 6 at $tU_\infty/D = 2-70$ in (a) $z/D = 5$ and (b) z/D
 427 $= 9.5$, at $tU_\infty/D = 70-140$ in (c) $z/D = 5$ and (d) $z/D = 9.5$, and at $tU_\infty/D = 140-170$ (e) $z/D = 5$ (f) $z/D =$
 428 9.5 . Visualization is provided through a supplementary video (Multimedia View).

429 3.2.2 Vortex structures

430 The vortex structures in Case 4 are similar to those in Cases 2 and 3; hence, they are herein not
 431 presented. When Re is slightly reduced as in Case 5, the N-cell vortices behind cylinder [a] and [b] are
 432 shed in different ways, as the flow is consistently asymmetric. As shown in Figure 14(a-e), in the
 433 narrower wake, the N-cell vortices are shed in parallel to the cylinders and maintain their shapes in a
 434 relatively long distance downstream. By contrast, the N-cell vortices in the wider wake region deform
 435 into a hairpin-like shape in the near wake region, and they gradually raise their fronts in the $+Y$
 436 direction. Due to the higher streamwise flow velocity in the outer region away from the cylinders, the
 437 hairpin-like vortices are accelerated downstream. As a result, the streamwise distance between the two
 438 adjacent vortex tubes becomes more elongated in the wider wake than in the narrower wake. The direct
 439 connection between N-cell and S-cell vortices dominates in the narrower wake, although the vortex
 440 tubes are highly deformed. However, in the wider wake, the half-loop connection, consisting of two

sequential counter-rotating S-cell vortices, is dominant owing to the +Y deflected N-cell vortices.

When the gap ratio is reduced in Case 6, as compared to Case 4, the AP flow occurs in both SC and LC wakes. The vortex shedding is anti-phase synchronized and symmetric with respect to the middle plane. The hairpin-like N-cell vortices can be observed in both free-stream sides, with the dominant half-loop connections, while the vortex shedding of the gap shear layers is rather weak, as shown in the supplementary video.

Figure 14. Instantaneous vortex structures represented by the iso-surface $Q = 0.2$ for Case 5 at $tU_\omega/D = 363.12-373.52$, (a-e) cylinder [b], (f-j) cylinder [a]; and Case 6 at $tU_\omega/D = 140.0-150.4$, (k-o) cylinder [b], (p-t) cylinder [a]. Grey and blue tubes denote vortices with $\omega_z < 0$ and $\omega_z > 0$, respectively.

3.2.3 Dynamic Mode Decomposition analysis

To ascertain the absence of IP pattern in the wake of dual-step cylinders, the wake modulations between S-cell and N-cell vortices are further investigated by carrying out the DMD analysis for Cases 4-6. The snapshot series of z -vorticity ω_z in a three-dimensional region $[-0.5, 8] \times [-3, 3] \times [0, 10]$ ($x \times y \times z$) have been extracted from the flow fields during $tU_\infty/D = 80$ -290 for Case 4, $tU_\infty/D = 360$ -390 for Case 5 and $tU_\infty/D = 140$ -169 for Case 6, with a time interval $\Delta tU_\infty/D = 0.52$. Attention is placed on the identification of associated wake modulations, elucidating similarity and differences in the flow pattern evolutions in Cases 4-6. Prior to that, it is necessary to recognize that both IP and FF flows of two side-by-side uniform cylinders begin with the AP flow. The different evolution of AP flow determines the fully developed flow pattern. If the AP flow gradually dissipates, as shown in the Appendix, the IP flow occurs when the flow is fully developed. If the AP flow persists, the FF flow will appear as discussed in Section 3.1.3.

For Case 4, the flow development is divided into the developing and fully developed phases. In the developing phase ($tU_\infty/D = 80$ -150), the vortex shedding modes for both N-cell and S-cell are anti-phase synchronized, see Figure 15(b-e). These modes show a great similarity with the Hopf bifurcation mode previously reported^{50, 52}, and they can be identified as the AP vortex shedding. Nevertheless, when the flow is fully developed, the gap flow deflects toward cylinder [b] during $tU_\infty/D = 220$ -290. As a result, two modes appear in the LC wake, with one showing the features of AP mode (Figure 16(i-j)) while the other reflecting the IP mode (Figure 16(k-l)). Due to the asymmetric wake of LC, the strengths of the wake modes are unequal with respect to the middle plane. Different from its larger diameter counterpart, only the SC shows the AP mode in the wake (Figure 16(m-p)), in spite of the unequal modal strengths downstream of the cylinders. Due to the coexistence of AP and IP modes in the LC wake, the FF flow takes place as demonstrated by the pitchfork bifurcation shown in Figure 16(g). However, according to the two-dimensional simulation at the same $Re = 120$ and $g^* = 1.5$ in the Appendix, the flow pattern in the fully developed phase becomes IP because of the dissipated AP mode. When comparing the flow patterns downstream of dual-step versus uniform cylinders, the main reason which distinguishes the former from the latter is that the stably existed AP vortex shedding of the S-cell maintains the AP mode in the LC wake through the vortex connections for dual-step cylinders. The coexisting AP and IP modes lead to the FF flow. For this reason, this feature is termed as the SC-

governing wake modulation.

481

482 **Figure 15.** DMD analysis of Case 4 in the developing phase ($tU_\infty/D = 80-150$). (a) DMD spectra of
 483 the eigenvector μ , where μ_r and μ_i are real and imaginary parts of the eigenvector, respectively. DMD
 484 modes are represented by iso-surface at $\omega_z = \pm 5 \times 10^{-4}$ and x - y slices with ω_z contours, at the frequencies
 485 (b-c) $f_{DMD} = 0.159$, $f_{DMD} = \ln(\mu_i)/2\pi$, and (d-e) $f_{DMD} = 0.275$. In the subfigures, cylinder [a] is on the
 486 right while cylinder [b] is on the left.

Figure 16. DMD analysis of Case 4 in the fully developed phase ($tU_\infty/D = 220-290$). DMD spectra of (a) the eigenvector μ , where μ_r and μ_i are real and imaginary parts of the eigenvector, respectively, and (e) the amplitude b and the frequency f . DMD modes are represented by iso-surface at $\omega_z = \pm 5 \times 10^{-4}$ and x - y slices with ω_z contours, at the frequencies at (b-d) $f_{DMD} = 0, f_{DMD} = \ln(\mu_i)/2\pi$; (f-h) $f_{DMD} = 0.018$; (i-j) $f_{DMD} = 0.189$; (k-l) $f_{DMD} = 0.135$; (m-n) $f_{DMD} = 0.283$; (o-p) $f_{DMD} = 0.257$. In the subfigures, cylinder

[a] is on the right while cylinder [b] is on the left.

For Case 5, the DMD mode of N-cell behind cylinder [a] shows some similarity with its counterpart behind cylinder [b], owing to the stably biased gap flow. Cylinder [b] shows a typical Hopf bifurcation mode (see Figure 17 (c)), corresponding to the periodic vortex shedding downstream of LC, while cylinder [a] displays a non-typical mode (see Figure 17 (e)) related to the hairpin-like N-cell vortices. The modal frequencies are $f_{DMD} = 0.092$ and 0.055 , respectively, which are comparable to the vortex shedding frequencies at $f_N = 0.107$ in the narrower wake and $f_N = 0.071$ in the wider wake, as shown in Figure 14 (e) and (h). From the wake patterns, the two modes appear to be weakly coupled, showing a significant strength behind one cylinder and a negligible distribution behind the other.

For Case 6, both N-cell and S-cell vortices show the anti-phase synchronized modes. The wake mode of S-cell experiences a smaller deformation as in a typical AP flow, see Figure 17(g-h), while that of N-cell is stable but with the hairpin-like vortices on both free-stream sides.

From the DMD modes of Cases 5 and 6, it is found that the IP flow pattern, which is expected for the side-by-side uniform cylinder cases at the same conditions of LC, is absent from the side-by-side dual-step cylinders. This can be attributed to the influence of AP flow on SC, which modulates the flow pattern of LC through the vortex connections.

509

510 Figure 17. DMD analysis of Case 5 (a-e) and Case 6 (f-j). (a) and (f) show DMD spectra of the
 511 eigenvector μ , where μ_r and μ_i are real and imaginary parts of the eigenvector, respectively. DMD
 512 modes are represented by iso-surface at $\omega_z = \pm 5 \times 10^{-4}$ and x-y slices with ω_z contours, at the frequencies
 513 at (b-c) $f_{DMD} = 0.092$, $f_{DMD} = \ln(\mu_i)/2\pi$, (d-e) $f_{DMD} = 0.055$, (g-h) $f_{DMD} = 0.126$, (i-j) $f_{DMD} = 0.063$. In the
 514 subfigures, cylinder [a] is on the right while cylinder [b] is on the left.

4. Conclusions

Three-dimensional direct numerical simulations of a uniform flow past a pair of side-by-side dual-step circular cylinders with different diameter ratios and gap ratios in a laminar flow regime have been performed by using Nektar++. Through the analyses of vortical structures, hydrodynamic forces and Dynamic Mode Decomposition (DMD) of the associated complex flow characteristics, two distinct wake modulation and interaction features have been identified and discussed, including the LC-dominated and SC-governing modulations. The origin of these fundamental vortex dynamics has been elucidated. Key main findings and conclusions are summarized as follows.

- a) For the LC-dominated modulation, the wake pattern of side-by-side dual-step cylinders is controlled by the predominant vortex dynamics of the LC wake. The diameter ratio plays an important role in determining the wake pattern. In Case 1 with $D/d = 1.33$, the wake pattern represents a symmetric, periodic, and anti-phase synchronized vortex shedding with a repeatable vortex dislocation cycle observed through the vortex evolution. The vortical structures are insignificantly deformed until being far away from the cylinders. However, in Case 2 with $D/d = 1.67$ and Case 3 with $D/d = 2.00$, the LC flow pattern affects the SC vortex dynamics through the vortex connections. The vortex tubes in these two cases experience a greater deformation with inconsistency in the vortex dislocation periods. The vortex dislocation occurring in the wider wake reveals a more frequent occurrence of multiple vortex connection and reconnection, while that in the narrower wake displays a more frequent occurrence of direct and half-loop connections.
- b) For the LC-dominated modulation, the DMD modes reveal the coexistence of anti-phase and in-phase synchronized vortex shedding in the LC region as an intrinsic feature of the flip-flopping flow. The DMD results reveal that the pitchfork bifurcation is only found in the N-cell vortex tube, indicating that the flip-flopping flow in the S-cell vortex tube is essentially originated from the N-cell vortex tube.
- c) For the SC-governing modulation, the flow pattern of N-cell vortex is modulated because of the presence of S-cell vortex. It is shown by the two-dimensional DMD modes of two uniform side-by-side cylinders that the evolution of anti-phase synchronized flow determines the fully developed flow pattern. With the gradually dissipated anti-phase synchronized flow, the in-

phase synchronized flow dominates the wake. Nevertheless, if the former flow pattern persists, the flip-flopping flow will eventually take place. Hence, the flow development mechanism of the SC-governing modulation is identified by the remaining anti-phase synchronized N-cell vortex resulting from the vortex connection with the S-cell vortex in the stable anti-phase synchronized flow pattern.

- d) For the SC-governing modulation, the S-cell vortex acts as a ‘source’ of the anti-phase vortex shedding which continuously modulates with the in-phase synchronized N-cell vortex. Consequently, the N-cell flow pattern becomes flip-flopping in Case 4 ($g^*_D = 1.5$, $Re_D = 120$), stably deflected in Case 5 ($g^*_D = 1.5$, $Re_D = 100$), and anti-phase synchronized in Case 6 ($g^*_D = 1.25$, $Re_D = 120$), owing to the different modulation ‘efficiency’ of the vortex connections depending on the gap ratio (Case 4 vs. 6) and Reynolds number (Case 4 vs. 5).

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Appendix

Two-dimensional flow past two side-by-side uniform cylinders with the same configuration of the LC in Case 4 is simulated to investigate how the wake modes develop. The simulation adopts a two-dimensional mesh being the same as the x - y plane projection of the three-dimensional mesh mentioned in Section 2. To analyze the flow development in detail, we separate the evolution process into three consecutive periods, including the developing ($tU_\infty/D = 800$ -874), transient ($tU_\infty/D = 875$ -1000) and fully developed ($tU_\infty/D = 1001$ -1100) phases, and we perform DMD analysis on each period.

As shown in Figure 18, in the developing phase, the vortex shedding is anti-phase synchronized, with a typical AP mode appearing in the wake, see Figure 19(b). The frequency of this mode ($f_{DMD} = 0.20$) matches the vortex shedding frequency recognized by the FFT analysis of the lift coefficient ($f_{FFT} = 0.20$). However, there is another mode matching this frequency, as shown in Figure 19(c). According

to the distribution of z -vorticity, the vortex shedding of this mode is in-phase synchronized and tends to pair up in a far downstream region, coinciding with the well-known feature of IP flow: the binary vortex street²³. In the transient phase, the AP mode separates into two modes at the same frequency ($f_{DMD} = 0.20$), with each of them deflecting toward one side of the wake. Also, the IP mode appears at a lower frequency ($f_{DMD} = 0.19$), see Figure 19(h). Finally, when the flow is fully developed, the AP mode is totally dissipated, enabling the remaining IP mode to match with the vortex shedding frequency $f_{DMD} = 0.19$, as shown in Figure 19(j).

From the above discussion, it is known that the IP vortex shedding is created during the developing phase, appears in the transient phase, and then dominates the wake in the fully developed phase. At the same time, the AP vortex shedding, that is used to dominating the flow in the developing phase, is weakened in the transient phase, and finally disappears in the fully developed phase.

The cases with $g^* = 1.5$ and $Re = 100$ as well as $g^* = 1.25$ and $Re = 120$ are similar to the above scenario, and therefore they are omitted for conciseness. Finally, we confirm that the development of IP flow is featured by the dissipation of AP vortex shedding.

Figure 18. Flow development of the case with $g^* = 1.5$ and $Re = 120$. (a) time histories of drag and lift coefficients, (b-d) instantaneous distribution of z -vorticity ω_z at $tU_\infty/D = 850$ (b), 955 (c), and 1025 (d). The vertical dashed lines divide the flow development into the developing, transient and fully

590 developed phases.

591
592 **Figure 19.** DMD analysis of the case with $g^* = 1.5$, $Re = 120$ in the developing phase (a-d, $tU_\infty/D =$
593 $800-874$), transient phase (e-h, $tU_\infty/D = 875-1000$) and fully-developed phase (i-l, $tU_\infty/D = 1001-1100$).
594 (a), (e) and (i) show FFT spectra of lift coefficients. DMD modes are represented by the ω_z contours,
595 at the frequencies (b) $f_{DMD} = 0.20$, (c) $f_{DMD} = 0.20$, (d) $f_{DMD} = 0.40$, (f) $f_{DMD} = 0.20$, (g) $f_{DMD} = 0.20$ (h)
596 $f_{DMD} = 0.19$, (j) $f_{DMD} = 0.19$, (k) $f_{DMD} = 0.38$, and (l) $f_{DMD} = 0.56$.

597 Data availability

598 The data that support the findings of this study are available from the corresponding author upon
599 reasonable request.

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