Synthesis of Net Systems with Inhibitor Arcs from Step Transition Systems

Marta Pietkiewicz-Koutny

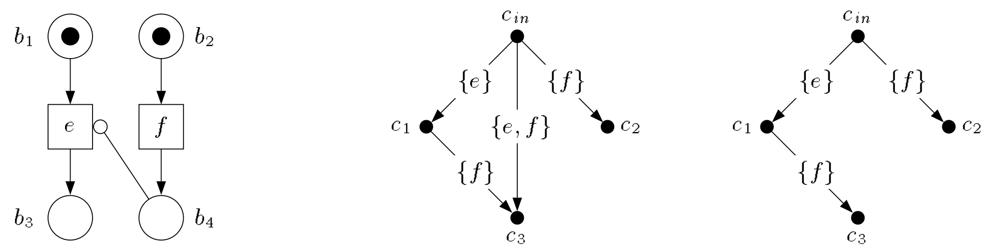
Department of Computing Science, University of Newcastle, Newcastle upon Tyne NE1 7RU, U.K.

Abstract. We here consider transition systems of Elementary Net Systems with Inhibitor Arcs. There are basically two different types of non-interleaving semantics of such Petri nets, the *a-posteriori* and *a-priori* semantics. The synthesis problem for Elementary Net Systems with Inhibitor Arcs executed under the a-priori semantics (ENI) was solved in [7]. The aim of this paper is to completely characterise transition systems which can be generated by Elementary Net Systems with Inhibitor Arcs executed under the a-posteriori semantics (ENI_{apost}). This is achieved by adapting the notion of a step transition system, i.e. one in which arcs are labelled by sets of events executed concurrently. In developing the model, we follow the standard approach in which the relationship between nets and their transition systems is established via the notion of a region. We define, and show consistency of, two behaviour preserving translations between nets and transition systems. We then compare transition systems which are generated by ENI_{apost} and ENI net systems (called respectively TSENI_{apost} and TSENI transition systems).
Keywords: causality/partial order theory of concurrency, analysis and synthesis, structure and behaviour of nets.

1 Introduction

Elementary Net Systems with Inhibitor Arcs are an extension of the Elementary Net Systems of [6], where in addition to the flow relation there is an inhibitor relation between some conditions and events. An example of such a net is shown in figure 1(a). The meaning of all the elements of $\mathcal{N}$ is standard except for the inhibitor arc (with the small circle at the end) between condition b_4 and event e which indicates that e can only be fired if b_4 is empty. This has a clear interpretation if one considers purely interleaving net semantics: $\mathcal{N}$ can execute e or f or ef (i.e. e followed by f). However, when we consider a non-interleaving semantics based on step sequences, then one is faced with the problem whether or not the concurrent step $\{e, f\}$ should be allowed. Basically, both interpretations are possible, as discussed in [2]. The one in which it is possible to execute $\{e, f\}$ is called there the *a-priori* semantics, and that in which this is disallowed is called the *a-posteriori* semantics. In the a-priori semantics, one can interpret the events as not instantaneous, taking some time to complete. For example, when the event f in figure 1(a) is executed a token is not placed in b_4 immediately, giving a chance to execute e at the same time as f. In the a-posteriori semantics, the occurrence of events is understood as taking zero time. Under this semantics, the execution of f from figure 1(a) places a token in b_4 at the same moment as the token of b_2 is removed, blocking immediately any event for which b_4 is an inhibitor condition. Now e and f cannot be executed at the same time. Which of the two semantics should be applied depends on the properties of events which the net is supposed to model, and on the properties of the enabling mechanism (see [2, 4] for details). Whereas the a-posteriori semantics is consistent with the *causal partial order* model of concurrency, the a-priori semantic requires more expressive model. Essentially, in addition to causality one also needs *weak causality* [4]. Elementary Net Systems with Inhibitor Arcs with the a-priori semantics (ENI-systems) and their transition systems (TSENI) were investigated in [7].

In this paper, we will be interested in Elementary Net Systems with Inhibitor Arcs executed under the a-posteriori semantics (*ENI_{apost}-systems*). The first part, sections 2-7, provides a complete characterisation of the class of transition systems generated by ENI_{apost}-systems which we call *Transition Systems Modelling Elementary Nets with Inhibitor Arcs under the a-posteriori semantics ($TSENI_{apost}$)*. For the elementary net system with inhibitor arcs in figure 1(a), $\mathcal{N}$, the corresponding TSENI transition system is shown in figure 1(b) and the TSENI_{apost} transition system in figure 1(c). In section 3, we formulate some important



(a) $\mathcal{N}$ (b) $TS_{\mathcal{N}}$ (a-priori) (c) $TS_{\mathcal{N}}$ (a-posteriori)

Fig. 1. Elementary net system with inhibitor arcs $\mathcal{N}$ and the transition systems it generates.

properties of the TSENI_{apost} transition systems. Like other classes of transition systems (see [5]), TSENI_{apost} transition systems enjoy the 'splitting' property. This property states that if a non-singleton step u is enabled at state s and its execution leads to state r then for every partition v, w of u there is a state q such that:

$$s \overset{v}{\Leftrightarrow\!\!\rightarrow} q \overset{w}{\Leftrightarrow\!\!\rightarrow} r.$$

This property does not hold for the TSENI transition systems. For example, we can take $TS_{\mathcal{N}}$ in figure 1(b) with $u = \{e, f\}$, $v = \{f\}$, $w = \{e\}$ and $s = c_{in}$. As a consequence, TSENI transition systems are not covered by any of the classes of transition systems generated by ordinary Petri nets.

In the second part of this paper, section 8, we will compare the TSENI_{apost} and TSENI transition systems. We will give (in section 8.1) sufficient conditions for building, for any $TS \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$, a transition system called $sat(TS)$ such that $sat(TS) \in \mathrm{TSENI} \setminus \mathrm{TSENI}_{apost}$ and the nets associated with them by the process of synthesis are isomorphic ($\mathcal{N}_{TS} \cong \mathcal{N}_{sat(TS)}$). Similarly, we will formulate (in section 8.2) sufficient conditions to create, for any $TS \in \mathrm{TSENI} \setminus \mathrm{TSENI}_{apost}$, a transition system called $prun(TS)$ such that $prun(TS) \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$ and $\mathcal{N}_{TS} \cong \mathcal{N}_{prun(TS)}$. In both cases, we discuss the possibility of weakening the present conditions (see section 9).

2 Transition Systems

In this section, we introduce TSENI$_{apost}$ transition systems which will later be shown to be the class of transition systems generated by ENI$_{apost}$-systems. We approach the final definition gradually, by introducing the seven axioms characterising TSENI$_{apost}$ transition systems. We prove the properties of TSENI$_{apost}$ transition systems if they differ from the ones introduced and proved for TSENI transition systems in [7]. Otherwise, we state them without proofs.

Let $\mathcal{E}$ be a non-empty set of *events* fixed throughout this paper. A *transition system* is a quadruple $TS = (S, U, T, s_{in})$ where:

TS1 S is a non-empty finite set of *states*.
TS2 $U \subseteq 2^{\mathcal{E}}$ is a finite set of *steps*; every $u \in U$ is finite and non-empty.
TS3 $T \subseteq S \times U \times S$ is the *transition relation*.
TS4 $s_{in} \in S$ is the *initial state*.

We assume that TS satisfies the following three axioms:

A1 For every $(s, u, s') \in T$, $s \neq s'$.
A2 For every $u \in U$, there are $s, s' \in S$ such that $(s, u, s') \in T$.
A3 For every $s \in S \setminus \{s_{in}\}$, there are $(s_0, u_0, s_1), (s_1, u_1, s_2), \ldots, (s_{n-1}, u_{n-1}, s_n) \in T$ such that $s_0 = s_{in}$ and $s_n = s$.

The first axiom excludes transition systems with self-loops, while the second ensures that all the steps in U are indeed used as labels of transitions in TS. Note that we do not require that U be subset closed as this will be a property dealt with later, in proposition 11. The last of the three axioms implies that all the states in TS are *reachable* from the initial state.

Throughout the rest of this section, the transition system TS will be fixed. We will use $s \overset{u}{\Leftrightarrow\!\!\to} s'$ to denote $(s, u, s') \in T$, and respectively call s the *source* and s' the *target* of this transition. Moreover, $E_{TS} = \bigcup_{u \in U} u$ will denote all the events appearing in steps labelling transitions in TS.

We now introduce a notion central to the whole approach as it links nodes of transition system (global states) with conditions in the corresponding net (local states).

Definition 1. A set of states $r \subseteq S$ is a *region* if the following two conditions are satisfied:

1. If $s \overset{u}{\Leftrightarrow\!\!\to} s'$ and $s \in r$ and $s' \notin r$ then there is $e \in u$ such that
   (a) if $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Leftrightarrow\!\!\to} s''$ then $s'' \in r$,
   (b) if $q \overset{v}{\Leftrightarrow\!\!\to} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.
2. If $s \overset{u}{\Leftrightarrow\!\!\to} s'$ and $s \notin r$ and $s' \in r$ then there is $e \in u$ such that
   (a) if $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Leftrightarrow\!\!\to} s''$ then $s'' \notin r$,
   (b) if $q \overset{v}{\Leftrightarrow\!\!\to} q'$ and $e \in v$ then $q \notin r$ and $q' \in r$. □

The event $e \in u$ which satisfies the conditions in definition 1 is unique. Such an event will be called *r-crossing* in u. The set of *non-trivial* regions (i.e. those different from S and $\emptyset$)

will be denoted by R_{TS}. Moreover, for every state $s \in S$, we will denote by R_s the set of non-trivial regions containing s,

$$R_s = \{r \in R_{TS} \mid s \in r\}.$$

The sets of pre-regions, ${}^\circ u$, and post-regions, u°, of a step $u \in U$ are defined as:

$$\begin{aligned} {}^\circ u &= \{r \in R_{TS} \mid \exists (s,u,s') \in T : s \in r \wedge s' \notin r\} \\ \text{and } u^\circ &= \{r \in R_{TS} \mid \exists (s,u,s') \in T : s \notin r \wedge s' \in r\}. \end{aligned}$$

We will use ${}^\circ e$ and e° instead of respectively ${}^\circ\{e\}$ and $\{e\}^\circ$, for every $e \in E_{TS}$. Being a pre- or post-region of a step u is a global property, in the following sense:

Proposition 1. If $s \stackrel{u}{\Leftrightarrow\!\!\!\to} s'$ then

1. $r \in {}^\circ u$ implies $s \in r$ and $s' \notin r$,
2. $r \in u^\circ$ implies $s \notin r$ and $s' \in r$. □

We say that a step $u \in U$ is *enabled* at a state $s \in S$ if there is $s' \in S$ such that $s \stackrel{u}{\Leftrightarrow\!\!\!\to} s'$. We will denote this by $s \stackrel{u}{\Leftrightarrow\!\!\!\to}$. We say that a step $u \in U$ *leads* to a state $s' \in S$ if there is $s \in S$ such that $s \stackrel{u}{\Leftrightarrow\!\!\!\to} s'$. We will denote this by $\stackrel{u}{\Leftrightarrow\!\!\!\to} s'$.

In what follows, we will assume that the transition system TS satisfies a fourth axiom:

A4 If $s \stackrel{u}{\Leftrightarrow\!\!\!\to}$ and $e \in u$ then $s \stackrel{\{e\}}{\Leftrightarrow\!\!\!\to}$.

Essentially, (A4) expresses a rather natural property that a step u cannot be enabled at a state if any of its events is disabled. This will later be generalised to a stronger property that none of the non-empty subsets of u is disabled (proposition 11). The axioms introduced so far are shared by the TSENI_{apost} and TSENI transition systems (see [7]).

Corollary 1. For every $e \in E_{TS}$, $\{e\} \in U$. □

The above corollary ensures that ${}^\circ e$ and e° are defined for all $e \in E_{TS}$.

The sets of pre- and post-regions of a step can be represented as the union of sets of respectively pre- and post-regions of events it comprises.

Proposition 2. If $u \in U$ then ${}^\circ u = \bigcup_{e \in u} {}^\circ e$ and $u^\circ = \bigcup_{e \in u} e^\circ$. □

The next two results state some basic properties of TS. The first asserts that event e appearing in definition 1 is always unique. Intuitively, this corresponds to the property of Petri nets that the sets of tokens consumed by concurrently executed events are disjoint. The second re-establishes some of the properties of regions formulated in [6], for the Elementary Transition Systems, and re-proved for the TSENI transition systems in [7]. They hold for the TSENI_{apost} transition systems as well.

Proposition 3. There exists exactly one event $e \in u$ which satisfies definition 1(1) (or 1(2)). □

Proposition 4. The following hold:

1. $r \subseteq S$ is a region if and only if $S \setminus r$ is a region.
2. If $u \in U$ then $u^\circ = \{S \setminus r \mid r \in {}^\circ u\}$.
3. If $s \overset{u}{\Leftrightarrow} s'$ then $R_s \setminus R_{s'} = {}^\circ u$ and $R_{s'} \setminus R_s = u^\circ$.
   Moreover, ${}^\circ u \subseteq R_s$ and $u^\circ \cap R_s = \emptyset$ and $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. □

The next proposition states the property, shared by both TSENI$_{apost}$ and TSENI transition systems, which ensures that the synthesied nets are contact-free (see section 6).

Proposition 5. Let $s \in S$ and $e \in E_{TS}$ be such that ${}^\circ e \subseteq R_s$. Then $e^\circ \cap R_s = \emptyset$. □

All the notions that we have introduced so far were essentially related to the ordinary arcs appearing in ENI$_{apost}$-systems. The next definition is different in that it attempts to capture, for each event e, those regions (conditions in the corresponding net) which are linked to e by means of an inhibitor arc. We start with an auxiliary definition. Let $e \in E_{TS}$ be an event, and $r \in R_{TS}$ be a non-trivial region. Then

$$\mathcal{B}_r^e = \{(s, \{e\}, s') \in T \mid s \in r \wedge s' \in r\}$$

is the set of all the transitions labelled by $\{e\}$ which are inside r. Having introduced $\mathcal{B}_r^e$, the set of *inhibitor-regions* (I-regions) of e is defined as follows:

$$\overset{\square}{e} = \{r \in R_{TS} \mid \mathcal{B}_r^e = \emptyset \wedge \mathcal{B}_{S\setminus r}^e \neq \emptyset\}.$$

We can extend the last notion to any set of events $u \in U$, as follows:

$$\overset{\square}{u} = \bigcup_{e \in u} \overset{\square}{e}\,.$$

Proposition 6. If $s \overset{\{e\}}{\Leftrightarrow} s'$ then $r \in \overset{\square}{e}$ implies $s, s' \notin r$. □

To characterise fully TSENI$_{apost}$ transition systems we will need the notion of a potential step in TS. The set of all *potential steps* SV_{TS} is defined as follows:

$$SV_{TS} = V_{TS} \cap \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall e, f \in u : (e \neq f \Rightarrow e^\circ \cap \overset{\square}{f} = \emptyset \ \wedge \ f^\circ \cap \overset{\square}{e} = \emptyset)\},$$

where

$$V_{TS} = \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall e, f \in u : (e \neq f \Rightarrow ({}^\circ e \cup e^\circ) \cap ({}^\circ f \cup f^\circ) = \emptyset)\}.$$

SV_{TS} comprises sets of events which share neither pre- nor post-regions. Moreover, a post-region of an event from $u \in SV_{TS}$ cannot be an I-region of some other event from u. The above definition of the set of potential steps in TS is more restrictive than the one used for TSENI transition systems. There the conditions involving I-regions were not needed and the set of all potential steps of a transition system TS was defined as V_{TS}.

We will assume from now on that the transition system TS satisfies an additional axiom which was not used for TSENI transition systems.

A5 If $\overset{u}{\Leftrightarrow} s$ and $e \in u$ then $\overset{\{e\}}{\Leftrightarrow} s$.

The new axiom (A5) will be necessary to prove that the definition of the set of potential steps of TS is consistent with the definition of U.

Proposition 7. $U \subseteq SV_{TS}$.

Proof. Let $u \in U$ and $e \neq f \in u$. By (A2), there is $s \overset{u}{\Longleftrightarrow} s'$.

Suppose that $r \in {}^\circ e \cap {}^\circ f$. This and (A4) and proposition 1(1) implies that there are $s^e, s^f \notin r$ such that $s \overset{\{e\}}{\Longleftrightarrow} s^e$ and $s \overset{\{f\}}{\Longleftrightarrow} s^f$. By proposition 2, we have $r \in {}^\circ u$, so, by proposition 1(1), $s \in r$ and $s' \notin r$. Hence, by proposition 3, there is a unique $g \in u$ such that $s \overset{\{g\}}{\Longleftrightarrow} s''$ and $s'' \notin r$, for some s''. But this produces a contradiction with the already established properties of e and f. That $e^\circ \cap f^\circ = \emptyset$ can be proved similarly.

Now, we prove that $e^\circ \cap {}^\circ f = \emptyset$ (the case $f^\circ \cap {}^\circ e = \emptyset$ is symmetric). Let $r \in e^\circ \cap {}^\circ f$. From (A4) it follows that $s \overset{\{e\}}{\Longleftrightarrow} s^e$ and $s \overset{\{f\}}{\Longleftrightarrow} s^f$, for some $s^e, s^f \in S$. On the one hand, by $r \in {}^\circ e$ and proposition 1(1), $s \in r$. On the other hand, by $r \in f^\circ$ and proposition 1(2), $s \notin r$. We obtained a contradiction.

Finally, we prove that $e^\circ \cap \overset{\Box}{f} = \emptyset$ (the case $f^\circ \cap \overset{\Box}{e} = \emptyset$ is symmetric). Let $r \in e^\circ \cap \overset{\Box}{f}$. From (A5) it follows that $s^e \overset{\{e\}}{\Longleftrightarrow} s'$ and $s^f \overset{\{f\}}{\Longleftrightarrow} s'$, for some $s^e, s^f \in S$. On the one hand, by $r \in e^\circ$ and proposition 1(2), $s' \in r$. On the other hand, by $r \in \overset{\Box}{f}$ and proposition 6, $s' \notin r$. We obtained a contradiction. □

It is straightforward to show that a step can be executed at a state only if the I-regions of the former do not comprise the latter. Due to the new axiom (A5) we can also prove that a step can only lead to a state which is not contained by its I-regions.

Proposition 8. If $s \overset{u}{\Longleftrightarrow} s'$ then $\overset{\Box}{u} \cap R_s = \emptyset$ and $\overset{\Box}{u} \cap R_{s'} = \emptyset$.

Proof. Suppose that $r \in \overset{\Box}{u} \cap R_s \neq \emptyset$. Then there is $e \in u$ such that $r \in \overset{\Box}{e}$. Hence, by proposition 6, if $p \overset{\{e\}}{\Longleftrightarrow} p'$ then $p, p' \notin r$. In particular, by (A4) and $s \overset{u}{\Longleftrightarrow} s'$ and $e \in u$, we have $s \notin r$. On the other hand, by $r \in R_s$, we have $s \in r$, a contradiction.

Suppose now that $r \in \overset{\Box}{u} \cap R_{s'} \neq \emptyset$. Then there is $e \in u$ such that $r \in \overset{\Box}{e}$. Hence, by proposition 6, if $p \overset{\{e\}}{\Longleftrightarrow} p'$ then $p, p' \notin r$. By axiom (A5) and $s \overset{u}{\Longleftrightarrow} s'$ and $e \in u$, we have $s' \notin r$. On the other hand, by $r \in R_{s'}$, we have $s' \in r$, a contradiction. □

We now can define the desired class of transition systems. A transition system TS is a $TSENI_{apost}$ *transition system* if it satisfies, in addition to (A1)-(A5), the following two axioms:

A6 For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.
A7 Let $s \in S$ and $u \in SV_{TS}$ be such that, for every $e \in u$, ${}^\circ e \subseteq R_s$ and $\overset{\Box}{e} \cap R_s = \emptyset$. Then $s \overset{u}{\Longleftrightarrow}$.

The first of the last two axioms is usually called the *state separation property* [1,6]. It essentially means that TS is deterministic, by excluding transition systems like TS_1 shown in figure 2. It was used for the TSENI transition systems as well. The second axiom is a variation of the *forward closure property* in [6] or the *event/state separation property* in [1]. It was used for the TSENI transition systems, but there u was a set of events from V_{TS}. Axiom (A7) excludes transition systems like TS_2 in figure 2 (to make TS_2 a valid TSENI_{apost} system one must add transition $s_{in} \overset{\{a,b\}}{\Longleftrightarrow} s_3$).

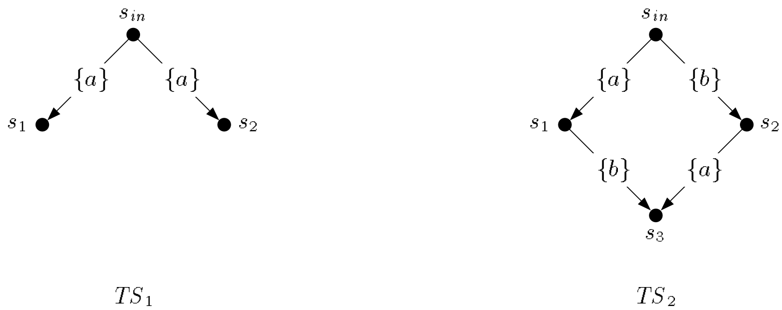


Fig. 2. Transition systems which are neither TSENI nor TSENI$_{apost}$ transition systems.

3 Properties of TSENI$_{apost}$ Transition Systems

We now formulate some properties of a TSENI$_{apost}$ transition system $TS = (S, U, T, s_{in})$. The properties shared with TSENI transition systems are given without proofs (which can be found in [7]).

Proposition 9. For every $e \in E_{TS}$, ${}^\circ e$ and e° are non-empty sets and ${}^\circ e$, e° and $\overset{\square}{e}$ are mutually disjoint sets. $\square$

Proposition 10. For every $u \in U$, ${}^\circ u$ and u° are non-empty disjoint sets. $\square$

The next result implies that the set of steps U is subset closed, if we only ignore the empty subset.

Proposition 11. If $s \overset{u}{\Leftrightarrow}$ and $\emptyset \neq v \subset u$ then $s \overset{v}{\Leftrightarrow}$. $\square$

As we already mentioned, axiom (A6) excludes non-deterministic transition systems. Formally, we have the following result.

Proposition 12. If $s \overset{u}{\Leftrightarrow} s'$ and $s \overset{u}{\Leftrightarrow} s''$ then $s' = s''$. $\square$

It is worth noting that, unlike TSENI transition systems, TSENI$_{apost}$ enjoy the 'splitting' property which is true of other classes of transition systems considered in the literature [5].

Proposition 13. If $s \overset{u}{\Leftrightarrow} s'$ then for every non-empty $v \subset u$ there exists $s'' \in S$ such that $s \overset{v}{\Leftrightarrow} s''$ and $s'' \overset{u \setminus v}{\Leftrightarrow} s'$.

Proof. From proposition 11 it follows that $v, u \setminus v \in U$ and $s \overset{v}{\Leftrightarrow} s''$ for some $s'' \in S$. By proposition 7, we have $U \subseteq SV_{TS}$. Hence, $u \setminus v \in SV_{TS}$. To prove that $s'' \overset{u \setminus v}{\Leftrightarrow}$ we need to show that the conditions in the axiom (A7) hold. First we show that for every $e \in u \setminus v$, ${}^\circ e \subseteq R_{s''}$. From proposition 2 it follows that ${}^\circ u = \bigcup\limits_{e \in u} {}^\circ e$, for every $u \in U$. Hence, for every $e \in u \setminus v$,

$$ {}^\circ e \subseteq {}^\circ (u \setminus v) \overset{u \in SV_{TS}}{=} {}^\circ u \setminus {}^\circ v \overset{prop.\ 4(3)}{=} (R_s \setminus R_{s'}) \setminus (R_s \setminus R_{s''}) \subseteq R_{s''}. $$

Next we need to prove that for every $e \in u \setminus v$, $\overset{\square}{e} \cap R_{s''} = \emptyset$. To the contrary, suppose there is $r \in \overset{\square}{e} \cap R_{s''} \neq \emptyset$ for some $e \in u \setminus v$. Since $r \in \overset{\square}{e}$ there exist $p, p' \in S$ such that $p \overset{\{e\}}{\Leftrightarrow} p'$ and $p, p' \notin r$. From $s \overset{u}{\Leftrightarrow} s'$ and (A4) we have $s \overset{\{e\}}{\Leftrightarrow}$ which, together with proposition 6, gives $s \notin r$. Since $s \overset{v}{\Leftrightarrow} s''$, $s \notin r$ and $s'' \in r$ (by $r \in R_{s''}$) we can apply definition 1(2) and obtain that there is $f \in v$ such that if $q \overset{\{f\}}{\Leftrightarrow} q'$ then $q \notin r$ and $q' \in r$. From (A4) we have $s \overset{\{f\}}{\Leftrightarrow} s^f$ for some $s^f \in S$. Hence, $s \notin r$ and $s^f \in r$. As a result, $r \in f^\circ$. Since $r \in \overset{\square}{e}$, we have $r \in \overset{\square}{e} \cap f^\circ \neq \emptyset$. But this produces a contradiction with $u \in SV_{TS}$, as $e, f \in u$ and $e \neq f$ ($e \in u \setminus v$ and $f \in v$). Hence $\overset{\square}{e} \cap R_{s''} = \emptyset$, for every $e \in u \setminus v$. Thus all the conditions in axiom (A7) are satisfied for s'' and $u \setminus v$. Hence $s'' \overset{u \setminus v}{\Leftrightarrow} s'''$, for some $s''' \in S$.

We finally need to prove that $s' = s'''$. From proposition 4(3), $s \overset{u}{\Leftrightarrow} s'$, $s \overset{v}{\Leftrightarrow} s''$ and $s'' \overset{u \setminus v}{\Leftrightarrow} s'''$ we have:

$$\begin{aligned} R_{s'} &= (R_s \setminus {}^\circ u) \cup u^\circ, \\ R_{s''} &= (R_s \setminus {}^\circ v) \cup v^\circ, \\ R_{s'''} &= \Big(R_{s''} \setminus {}^\circ(u \setminus v)\Big) \cup (u \setminus v)^\circ. \end{aligned}$$

It is then easy to verify, using $v \subset u \in SV_{TS}$ and proposition 10, that:

$$\begin{aligned} R_{s'''} &= \Big(\Big((R_s \setminus {}^\circ v) \cup v^\circ\Big) \setminus {}^\circ(u \setminus v)\Big) \cup (u \setminus v)^\circ \\ &= \Big(\Big((R_s \setminus {}^\circ v) \cup v^\circ\Big) \setminus ({}^\circ u \setminus {}^\circ v)\Big) \cup (u^\circ \setminus v^\circ) \\ &= (R_s \setminus {}^\circ u) \cup u^\circ \\ &= R_{s'}. \end{aligned}$$

Hence $R_{s'''} = R_{s'}$ and, by (A6), we obtain $s''' = s'$. □

Corollary 2. If $\overset{u}{\Leftrightarrow} s$ and $\emptyset \neq v \subset u$ then $\overset{v}{\Leftrightarrow} s$.

Proof. Follows directly from proposition 13. □

Corollary 3. Let $u \in U$ and $|u| = n$. If $s \overset{u}{\Leftrightarrow} s'$ then for every enumeration of the events from u, $(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$, there exist transitions

$$(s_0, \{e_{i_1}\}, s_1), (s_1, \{e_{i_2}\}, s_2), \ldots, (s_{n-1}, \{e_{i_n}\}, s_n)$$

in T such that $s_0 = s$ and $s_n = s'$.

Proof. Follows easily from proposition 13. □

An *event sequence* of TS is a sequence $\sigma = e_1 e_2 \ldots e_n$ of events from E_{TS} for which there are states $s_0, s_1, \ldots, s_n$ satisfying $(s_0, \{e_1\}, s_1), (s_1, \{e_2\}, s_2), \ldots, (s_{n-1}, \{e_n\}, s_n) \in T$. We will denote it by $s_0 \overset{\sigma}{\rightsquigarrow} s_n$, and call s_0 the *source* and s_n the *target* of σ. We will say that an event sequence σ is *enabled* at a state $s \in S$ if there is $s' \in S$ such that $s \overset{\sigma}{\rightsquigarrow} s'$. We will denote this by $s \overset{\sigma}{\rightsquigarrow}$.

Corollary 4. Let $u \in V_{TS}$ (where $|u| = n$), and $(e_{i_1}, e_{i_2}, \dots, e_{i_n})$ and $(e_{j_1}, e_{j_2}, \dots, e_{j_n})$ be enumerations of the events from u. Let $\sigma_1 = e_{i_1} e_{i_2} \dots e_{i_n}$ and $\sigma_2 = e_{j_1} e_{j_2} \dots e_{j_n}$ be event sequences enabled at s, $s \overset{\sigma_1}{\rightsquigarrow} s_1$ and $s \overset{\sigma_2}{\rightsquigarrow} s_2$. Then $s_1 = s_2$.

Proof. Follows from the fact that $u \in V_{TS}$, proposition 4(3) and axiom (A6). □

4 Inhibitor Nets

A *net with inhibitor arcs* (see [4]) is a tuple $N = (B, E, F, I)$ such that B and $E \subseteq \mathcal{E}$ are finite disjoint sets, $F \subseteq (B \times E) \cup (E \times B)$ and $I \subseteq B \times E$. The meaning and graphical representation of B (conditions), E (events) and F (flow relation) is the same as in the standard net theory. An *inhibitor* arc $(b, e) \in I$ means that e can be enabled only if b is not marked (in the diagrams, it is represented by an edge ending with a small circle). We denote, for every $x \in B \cup E$,

$$\begin{aligned} {}^\bullet x &= \{y \mid (y, x) \in F\} && \text{(pre-elements)}, \\ x^\bullet &= \{y \mid (x, y) \in F\} && \text{(post-elements)}, \\ \text{and } \overset{\blacksquare}{x} &= \{y \mid (x, y) \in I \cup I^{-1}\} && \text{(I-elements)}. \end{aligned}$$

The dot-notation extends in the usual way to sets, for example, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$. It is assumed that for every $e \in E$,

$$e^\bullet \neq \emptyset \neq {}^\bullet e \qquad \text{and} \qquad {}^\bullet e \cap e^\bullet = {}^\bullet e \cap \overset{\blacksquare}{e} = e^\bullet \cap \overset{\blacksquare}{e} = \emptyset. \tag{1}$$

An *elementary net system with inhibitor arcs* (ENI$_{apost}$-system) is a tuple

$$\mathcal{N} = (B, E, F, I, c_{in})$$

such that $N_{\mathcal{N}} = (B, E, F, I)$ is the (underlying) net with inhibitor arcs and $c_{in} \subseteq B$ is the *initial case* (in general, any subset of B is a *case*). We will assume that $\mathcal{N}$ is fixed until the end of this section.

The concurrency semantics of ENI$_{apost}$-systems will be based on steps of simultaneously executed events. We first define valid steps. A non-empty set of events $u \subseteq E$ is a *valid step*, denoted $u \in SV_{\mathcal{N}}$, if for all $e \neq f \in u$,

$$({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset \text{ and } e^\bullet \cap \overset{\blacksquare}{f} = \emptyset \text{ and } f^\bullet \cap \overset{\blacksquare}{e} = \emptyset. \tag{2}$$

We recall that for ENI-systems the set of valid steps $V_{\mathcal{N}}$ was defined using only the first out of the three constraints of (2):

$$V_{\mathcal{N}} = \Big\{ u \subseteq E \mid u \neq \emptyset \ \wedge\ \forall e, f \in u : \ \Big(e \neq f \Rightarrow ({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset \Big) \Big\}.$$

The transition relation of $N_{\mathcal{N}}$, denoted by $\rightarrow_{N_{\mathcal{N}}}$, is given by:

$$\rightarrow_{N_{\mathcal{N}}} = \{(c, u, c') \in 2^B \times SV_{\mathcal{N}} \times 2^B \mid c \setminus c' = {}^\bullet u \ \wedge\ c' \setminus c = u^\bullet \ \wedge\ \overset{\blacksquare}{u} \cap c = \emptyset\}. \tag{3}$$

The *state space* of $\mathcal{N}$, denoted by $C_{\mathcal{N}}$, is the least subset of 2^B containing c_{in} such that if $c \in C_{\mathcal{N}}$ and $(c, u, c') \in \rightarrow_{N_{\mathcal{N}}}$ then $c' \in C_{\mathcal{N}}$. The *transition relation* of $\mathcal{N}$, denoted by $\rightarrow_{\mathcal{N}}$, is then defined as $\rightarrow_{N_{\mathcal{N}}}$ restricted to $C_{\mathcal{N}} \times SV_{\mathcal{N}} \times C_{\mathcal{N}}$. The set of *active steps* of $\mathcal{N}$ is given by $U_{\mathcal{N}} = \{u \mid \exists c, c' : (c, u, c') \in \rightarrow_{\mathcal{N}}\}$. We will use $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$ to denote that $(c, u, c') \in \rightarrow_{\mathcal{N}}$. Also, $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$ if $(c, u, c') \in \rightarrow_{\mathcal{N}}$, for some c'. Similarly, we will write $\overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c$ if $(c', u, c) \in \rightarrow_{\mathcal{N}}$, for some c'.

The above definition of the operational semantics of $\mathcal{N}$ is what is referred to as the *a-posteriori* semantics in [2].

Proposition 14. The following hold:

1. Let $c \in C_{\mathcal{N}}$ and $u \in SV_{\mathcal{N}}$. Then $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$ if and only if ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$.
2. If $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$ then $c' = (c \setminus {}^\bullet u) \cup u^\bullet$ and $\overset{\blacksquare}{u} \cap c' = \emptyset$.

Proof. (1) Suppose $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$. Then there is $c' \in C_{\mathcal{N}}$ such that $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$. From (3), ${}^\bullet u \subseteq c$ and $u^\bullet \cap c = \emptyset$ and $\overset{\blacksquare}{u} \cap c = \emptyset$.
Suppose now that ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$. Define $c' = (c \setminus {}^\bullet u) \cup u^\bullet$. It is easy to show that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. Hence, by (3), $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$ and thus $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$.
(2) The first part follows easily from (3). We need to prove that $\overset{\blacksquare}{u} \cap c' = \emptyset$. Suppose there is $b \in \overset{\blacksquare}{u} \cap c'$. Then either $b \in c' \setminus c$ or $b \in c' \cap c$. In the first case $b \in u^\bullet$, and since $b \in \overset{\blacksquare}{u}$ there exist $e, f \in u$ such that $b \in e^\bullet$ and $b \in \overset{\blacksquare}{f}$, and we obtain a contradiction with $u \in SV_{\mathcal{N}}$ (if $e \neq f$) or with (1) (if $e = f$). In the second case $b \in c$, and we obtain a contradiction with $\overset{\blacksquare}{u} \cap c = \emptyset$. □

Notice that by using stronger definition for a valid step, $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$ means not only that $\overset{\blacksquare}{u} \cap c = \emptyset$ (which was true for ENI-systems), but that $\overset{\blacksquare}{u} \cap c' = \emptyset$ is satisfied as well.

To compare solutions to the synthesis problem (in section 8), we will need net isomorphism up to the names of conditions. Let $\mathcal{N}_i = (B_i, E, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be net systems with inhibitor arcs (ENI$_{apost}$-systems or ENI-systems) with the same sets of events. $\mathcal{N}_1$ and $\mathcal{N}_2$ are *isomorphic* if there exists a bijection $f : B_1 \to B_2$ satisfying, for every $b \in B_1$ and $e \in E$, the following conditions:

$$\begin{array}{ll} 1. & (b, e) \in F_1 \Leftrightarrow (f(b), e) \in F_2, \\ 2. & (e, b) \in F_1 \Leftrightarrow (e, f(b)) \in F_2, \\ 3. & (b, e) \in I_1 \Leftrightarrow (f(b), e) \in I_2, \\ 4. & b \in c^1_{in} \quad\;\; \Leftrightarrow f(b) \in c^2_{in}. \end{array}$$

We will denote this by $\mathcal{N}_1 \cong \mathcal{N}_2$.

5 Transition Systems of ENI$_{apost}$-systems

The construction of a transition system for a given ENI$_{apost}$-system is straightforward.

Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI$_{apost}$-system. Then

$$TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \rightarrow_{\mathcal{N}}, c_{in})$$

is the *transition system generated* by $\mathcal{N}$.

Theorem 1. $TS_{\mathcal{N}}$ is a TSENI$_{apost}$ transition system.

Proof. Clearly, $TS_{\mathcal{N}}$ is a transition system. We need to prove that it satisfies (A1)-(A7).
(A1) Suppose $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$ and $c = c'$. Then, by (3), $u^\bullet = {}^\bullet u = \emptyset$, contradicting (1).
(A2) and (A3) follow directly from the definition of $C_{\mathcal{N}}$ and $U_{\mathcal{N}}$.
(A4) Suppose $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$ and $e \in u$. By proposition 14(1), ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$. We also have ${}^\bullet e \subseteq {}^\bullet u$, $e^\bullet \subseteq u^\bullet$ and $\overset{\blacksquare}{e} \subseteq \overset{\blacksquare}{u}$, so ${}^\bullet e \subseteq c$ and $(e^\bullet \cup \overset{\blacksquare}{e}) \cap c = \emptyset$. Thus, from proposition 14(1) it follows that $c \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$.
(A5) Suppose $\overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c$ and $e \in u$. Then there is $c' \in C_{\mathcal{N}}$ such that $c' \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c$. From proposition 14(2) we have $c = (c' \setminus {}^\bullet u) \cup u^\bullet$. From proposition 14(1) we have ${}^\bullet u \subseteq c'$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c' = \emptyset$, and as a result ${}^\bullet(u \setminus \{e\}) \subseteq c'$, $(u \setminus \{e\})^\bullet \cap c' = \emptyset$ and $(u \setminus \overset{\blacksquare}{\{e\}}) \cap c' = \emptyset$. Since $u \setminus \{e\} \in SV_{\mathcal{N}}$ we can apply proposition 14(1) to obtain $c' \overset{u\setminus\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$. Let $c'' \in C_{\mathcal{N}}$ be such that $c' \overset{u\setminus\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c''$. From proposition 14(2), $c'' = (c' \setminus {}^\bullet(u \setminus \{e\})) \cup (u \setminus \{e\})^\bullet$. It can be easily verified that ${}^\bullet e \subseteq c''$, $e^\bullet \cap c'' = \emptyset$ and $\overset{\blacksquare}{e} \cap c'' = \emptyset$ (by $\overset{\blacksquare}{e} \cap c' = \emptyset$ and $\overset{\blacksquare}{e} \cap (u \setminus \{e\})^\bullet \overset{u \in SV_{\mathcal{N}}}{=} \emptyset$). Hence $c'' \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c^e$, for some $c^e \in C_{\mathcal{N}}$. From proposition 14(2) we have $c^e = (c'' \setminus {}^\bullet e) \cup e^\bullet$. It is then easy to verify that $c^e = c$. Hence we have proved that $\overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c$, for every $e \in u$.

Before proving (A6) and (A7) we show that, for every $b \in B$, $r_b = \{c \in C_{\mathcal{N}} \mid b \in c\}$ is (possibly trivial) region in $TS_{\mathcal{N}}$. Moreover,

$$\emptyset \neq r_b \neq C_{\mathcal{N}} \Rightarrow r_b \in R_{TS_{\mathcal{N}}}. \tag{4}$$

Suppose $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$, where $c \in r_b$ and $c' \notin r_b$. Then $b \in c$ and $b \notin c'$. By (3), $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. Hence $b \in {}^\bullet u$ and $b \notin u^\bullet$, and we can choose $e \in u$ such that $b \in {}^\bullet e$. We now observe that if $d \overset{v}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} d'$ and $e \in v$ then $d \in r_b$ and $d' \notin r_b$ (since, by (3), $b \in d$ and $b \notin d'$). Moreover, if $v \subseteq u \setminus \{e\}$ and $c \overset{v}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c''$ then $c'' \in r_b$, since by (2), $b \notin v^\bullet \cup {}^\bullet v$.
Thus the first part of definition 1 is satisfied; the second part can be shown in a similar way. Hence r_b is a region in $TS_{\mathcal{N}}$. Clearly, if $\emptyset \neq r_b \neq C_{\mathcal{N}}$ then r_b is a non-trivial region and (4) holds.
(A6) Suppose that $c \neq c' \in C_{\mathcal{N}}$. Without loss of generality, we may assume that there is $b \in c \setminus c'$. Hence $c \in r_b$ and $c' \notin r_b$. Thus, by (4) and $r_b \in R_c \setminus R_{c'}$, (A6) holds.
(A7) Suppose that $c \in C_{\mathcal{N}}$ and $u \in SV_{TS_{\mathcal{N}}}$ are such that, for every $e \in u$, ${}^\circ e \subseteq R_c$ and $\overset{\square}{e} \cap R_c = \emptyset$. We first show that $c \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$, for every $e \in u$.
Let $e \in u$. Since $e \in E_{TS_{\mathcal{N}}}$ and (A4) and (A2) hold, there are $d, d' \in C_{\mathcal{N}}$ such that $d \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} d'$.
Consider any $b \in {}^\bullet e$. Then $b \in d$ and $b \notin d'$, and so $d \in r_b$ and $d' \notin r_b$. Hence, by (4), $r_b \in R_{TS_{\mathcal{N}}}$ and $r_b \in {}^\circ e$. From ${}^\circ e \subseteq R_c$ we have $r_b \in R_c$ which means $b \in c$. As a result, ${}^\bullet e \subseteq c$.
Consider now any $b \in e^\bullet$. Then $b \notin d$ and $b \in d'$, and so $d \notin r_b$ and $d' \in r_b$. Hence, by (4), $r_b \in e^\circ$. This and $e^\circ \cap R_c = \emptyset$ (follows from ${}^\circ e \subseteq R_c$ and proposition 5) means that $r_b \notin R_c$, and so $b \notin c$. Hence $e^\bullet \cap c = \emptyset$.
Suppose that $b \in \overset{\blacksquare}{e} \cap c \neq \emptyset$. Then $c \in r_b$. By (3) and $\overset{\blacksquare}{e} \cap e^\bullet = \emptyset$, $b \notin d$ and $b \notin d'$. Thus $d \notin r_b$ and $d' \notin r_b$. As a result, by (4), $r_b \in R_{TS_{\mathcal{N}}}$ and $d, d' \in C_{\mathcal{N}} \setminus r_b$. Hence $\mathcal{B}^e_{C_{\mathcal{N}} \setminus r_b} \neq \emptyset$.

Suppose now that $f \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} f'$ belongs to $\mathcal{B}^{e}_{r_b}$. This means $f, f' \in r_b$ and we have $b \in f$ and $b \in f'$. But this and (3) contradict $b \in \overset{\blacksquare}{e}$. Hence $\mathcal{B}^{e}_{r_b} = \emptyset$ and, as a result, $r_b \in \overset{\square}{e}$. Since $\overset{\square}{e} \cap R_c = \emptyset$, $r_b \notin R_c$ which means $b \notin c$, a contradiction with $b \in \overset{\blacksquare}{e} \cap c$. Hence $\overset{\blacksquare}{e} \cap c = \emptyset$ which, together with ${}^\bullet e \subseteq c$ and $e^\bullet \cap c = \emptyset$, yields $c \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$.

We proved that $c \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$, for every $e \in u$. Moreover, we have already shown that $b \in {}^\bullet e$ implies $r_b \in {}^\circ e$, $b \in e^\bullet$ implies $r_b \in e^\circ$, and $b \in \overset{\blacksquare}{e}$ together with $r_b \neq \emptyset$ implies $r_b \in \overset{\square}{e}$, for all $e \in u$. This and $u \in SV_{TS_{\mathcal{N}}}$ means that $u \in SV_{\mathcal{N}}$. Hence $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$. □

6 ENI$_{apost}$-systems of TSENI$_{apost}$ Transition Systems

The reverse translation, from TSENI$_{apost}$ transition systems to ENI$_{apost}$-systems, is based on the pre- post- and I-regions of events appearing in a transition system.
Let $TS = (S, U, T, s_{in})$ be a TSENI$_{apost}$ transition system. The net system *associated* with TS is defined as

$$\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$$

where F_{TS} and I_{TS} are defined thus:

$$\begin{aligned} F_{TS} &= \{(r,e) \in R_{TS} \times E_{TS} \mid r \in {}^\circ e\} \;\cup\; \{(e,r) \in E_{TS} \times R_{TS} \mid r \in e^\circ\}, \\ I_{TS} &= \{(r,e) \in R_{TS} \times E_{TS} \mid r \in \overset{\square}{e}\}. \end{aligned} \tag{5}$$

Directly from the definition of $\mathcal{N}_{TS}$ we obtain that, for every $e \in E_{TS}$,

$${}^\circ e = {}^\bullet e \text{ and } e^\circ = e^\bullet \text{ and } \overset{\square}{e} = \overset{\blacksquare}{e}\,. \tag{6}$$

The proof of the next theorem is omitted as it is similar to the proof of the corresponding property of ENI-systems (see [7]).

Theorem 2. $\mathcal{N}_{TS}$ is an ENI$_{apost}$-system. □

The above construction produces a net which is saturated both with places and inhibitor arcs.

7 Consistency of the Two Translations

In this section, we show that the ENI$_{apost}$-system associated with a TSENI$_{apost}$ transition system TS generates a transition system which is isomorphic to TS.

Proposition 15. Let $TS = (S, U, T, s_{in})$ be a TSENI$_{apost}$ transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI$_{apost}$-system associated with it.

1. $C_{\mathcal{N}} = \{R_s \mid s \in S\}$.
2. $\rightarrow_{\mathcal{N}} = \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.

Proof. Note that from the definition of $C_{\mathcal{N}}$, every $c \in C_{\mathcal{N}}$ is reachable from c_{in} in $\mathcal{N}$; and that from axiom (A3), every $s \in S$ is reachable from s_{in} in TS.

We first show that if $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$ and $c = R_s$, for some $s \in S$, then there is $s' \in S$ such that $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$ and $c' = R_{s'}$. We have that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. This means ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{e} \cap c = \emptyset$, for all $e \in u$. This and (6) implies that ${}^\circ e \subseteq c$ and $\overset{\square}{e} \cap c = \emptyset$, for all $e \in u$. Hence ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for all $e \in u$. Moreover, by $u \in SV_{\mathcal{N}}$ and (6), we have $u \in SV_{TS}$. Hence from (A7) it follows that $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$, for some $s' \in S$. Then, by proposition 4(3), $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. At the same time, from proposition 14(2), $c' = (c \setminus {}^\bullet u) \cup u^\bullet$. Hence, by (6) and proposition 2 and $c = R_s$, $c' = R_{s'}$.
As a result, we have shown (note that $c_{in} = R_{s_{in}} \in \{R_s \mid s \in S\}$) that $C_{\mathcal{N}} \subseteq \{R_s \mid s \in S\}$ and $\rightarrow_{\mathcal{N}} \subseteq \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.

We now will prove that $\{R_s \mid s \in S\} \subseteq C_{\mathcal{N}}$. By definition, $R_{s_{in}} \in C_{\mathcal{N}}$. What needs to be shown is that if $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$ and $R_s \in C_{\mathcal{N}}$ then $R_{s'} \in C_{\mathcal{N}}$. By propositions 4(3) and 8, we have ${}^\circ u \subseteq R_s$ and $(u^\circ \cup \overset{\square}{u}) \cap R_s = \emptyset$. So, using (6) and proposition 2, ${}^\bullet u \subseteq R_s$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap R_s = \emptyset$. Moreover, from proposition 7 and (6) we obtain that u is a valid step in $\mathcal{N}$. Hence, by proposition 14(1), we have $R_s \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}}$. This implies $(R_s \setminus {}^\bullet u) \cup u^\bullet \in C_{\mathcal{N}}$. On the other hand, by proposition 4(3) and $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$, we have $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. Hence, by (6) and proposition 2, $R_{s'} \in C_{\mathcal{N}}$.

What remains to be shown is that $\{(R_s, u, R_{s'}) \mid (s, u, s') \in T\} \subseteq \rightarrow_{\mathcal{N}}$. Suppose $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$. From propositions 4(3) and 8 it follows that $R_s \setminus R_{s'} = {}^\circ u$, $R_{s'} \setminus R_s = u^\circ$ and $\overset{\square}{u} \cap R_s = \emptyset$. We have already proved that $C_{\mathcal{N}} = \{R_s \mid s \in S\}$. So there are $c, c' \in C_{\mathcal{N}}$ such that $c = R_s$ and $c' = R_{s'}$. From (6) and proposition 2 it follows that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. Since $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$, from proposition 7 and (6), it follows that u is a valid step. Hence, by (3), $c \overset{u}{\Leftrightarrow\!\!\rightarrow}_{\mathcal{N}} c'$. □

The proof of the following theorem is omitted as it is similar to the proof of the corresponding property of ENI-systems (see [7]).

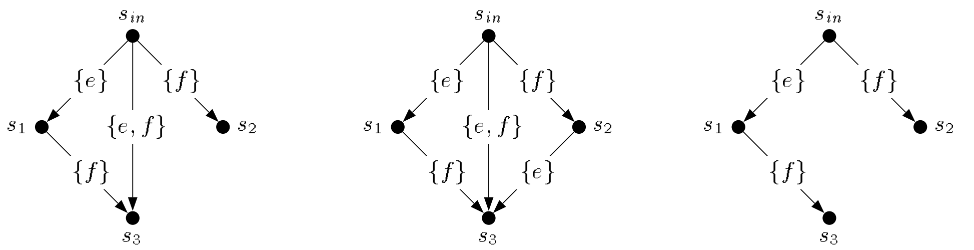
Theorem 3. Let $TS = (S, U, T, s_{in})$ be a TSENI$_{apost}$ transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI$_{apost}$-system associated with it. Then $TS_{\mathcal{N}}$ is isomorphic to TS. □

8 Comparison between TSENI$_{apost}$ and TSENI Transition Systems

We recall from [7] the set of axioms which characterise TSENI transition systems. A transition system TS is a *TSENI transition system* if it satisfies the following six axioms:

A1* For every $(s, u, s') \in T$, $s \neq s'$.
A2* For every $u \in U$, there are $s, s' \in S$ such that $(s, u, s') \in T$.
A3* For every $s \in S \setminus \{s_{in}\}$, there are $(s_0, u_0, s_1), (s_1, u_1, s_2), \ldots, (s_{n-1}, u_{n-1}, s_n) \in T$ such that $s_0 = s_{in}$ and $s_n = s$.
A4* If $s \overset{u}{\Leftrightarrow\!\!\rightarrow}$ and $e \in u$ then $s \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}$.
A5* For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.
A6* Let $s \in S$ and $u \in V_{TS}$ be such that, for every $e \in u$, ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$. Then $s \overset{u}{\Leftrightarrow\!\!\rightarrow}$.

To compare TSENI and TSENI_{apost} transition systems we observe that neither class is a proper subset of the other, and that there are transition systems which satisfy the axioms of both TSENI and TSENI_{apost} class. This is illustrated in figure 3.



$TS_1 \in \text{TSENI} \setminus \text{TSENI}_{apost}$ $TS_2 \in \text{TSENI} \cap \text{TSENI}_{apost}$ $TS_3 \in \text{TSENI}_{apost} \setminus \text{TSENI}$

Fig. 3. Comparison between TSENI and TSENI_{apost} transition systems.

8.1 Part 1

In this section, we consider a transition system $TS = (S, U, T, s_{in}) \in \text{TSENI}_{apost} \setminus \text{TSENI}$ and investigate whether it is possible to find a TSENI transition system whose associated net would be isomorphic to that of TS.

Proposition 16. If $TS \in \text{TSENI}_{apost} \setminus \text{TSENI}$ then there exists $u \in V_{TS} \setminus SV_{TS}$ and $s \in S$ such that for every $e \in u$, ${}^{\circ}e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$.

Proof. Since $TS \in \text{TSENI}_{apost} \setminus \text{TSENI}$ we have that all axioms (A1)-(A7) are satisfied for TS and as a consequence (A1*)-(A5*) are satisfied as well. The only axiom which makes TS fail to be a TSENI transition system is (A6*). Hence, (A7) is satisfied and (A6*) is not satisfied for TS. We introduce some symbols for the subformulae appearing in (A7) and (A6*), where $u \subseteq E_{TS}$ and $s \in S$ are such that (A6*) fails to hold:

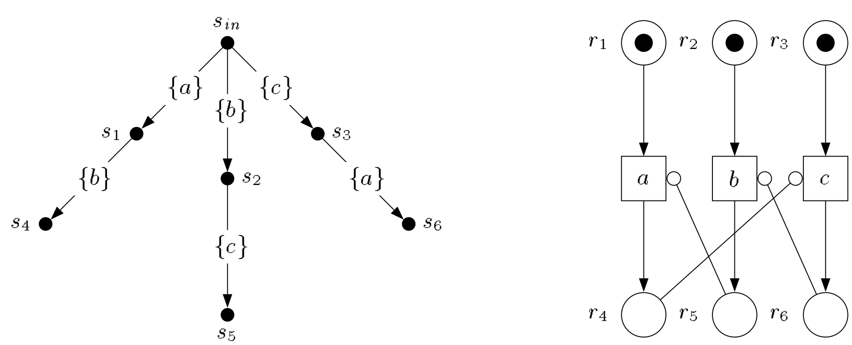
$$
\begin{array}{ll}
\alpha & u \in SV_{TS} \\
\beta & u \in V_{TS} \\
\gamma & \forall e \in u: \ {}^{\circ}e \subseteq R_s \wedge \overset{\square}{e} \cap R_s = \emptyset \\
\delta & s \overset{u}{\not\rightarrow}
\end{array}
$$

(A6*) is false, so β is true, γ is true and δ is false. (A7) is true, so $\alpha \wedge \gamma \Rightarrow \delta$ is true, which means $\alpha \wedge \gamma$ is false. Since γ is true, α is false. So $\beta \wedge \neg\alpha \wedge \gamma$ is true. □

From proposition 16 it follows that in $TS \in \text{TSENI}_{apost} \setminus \text{TSENI}$ there is a set of events $u \subseteq E_{TS}$ and a state $s \in S$ such that u is not enabled as a step at s according to the a-posteriori axioms (A1)-(A7), but it would be enabled at s under the a-priori axioms (A1*)-(A6*). This suggests that by adding to TS an appropriate transition, for every $s \in S$ and $u \subseteq E_{TS}$ which satisfy the conditions of proposition 16, we could obtain a TSENI transition system whose associated net is isomorphic to that of TS. Before proving this hypothesis, we need to define the targets of transitions added in that way. A good candidate for the target of the transition associated with certain $s \in S$ and $u \subseteq E_{TS}$ would be s' such that there exists an event sequence

$$\rho_u = e_{i_1} e_{i_2} \ldots e_{i_n}, \text{ where } (e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \text{ is an enumeration of events from } u, \quad (7)$$

and $s \overset{\rho_u}{\rightsquigarrow} s'$. Notice that corollary 3 guarantees that for $u \in U$ such an event sequence always exists, but for $u \in V_{TS}$ we can only say, following corollary 4, that if it exists then the state s' is well defined as it does not depend on the chosen enumeration. Unfortunately, for some $u \in V_{TS}$ and $s \in S$ such a sequence does not exist, as shown in figure 4.



(a) $TS \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$ (b) $\mathcal{N}_{TS}$

Fig. 4. $TS \in \mathrm{TSENI}_{apost} \setminus$ TSENI and the associated net, $\mathcal{N}_{TS}$.

The regions of TS depicted in figure 4 are:

$$r_1 = \{s_{in}, s_2, s_3, s_5\} \qquad r_2 = \{s_{in}, s_1, s_3, s_6\} \qquad r_3 = \{s_{in}, s_1, s_2, s_4\}$$
$$r_4 = \{s_1, s_4, s_6\} \qquad r_5 = \{s_2, s_4, s_5\} \qquad r_6 = \{s_3, s_5, s_6\}$$

and the pre-regions, post-regions and I-regions of events are given by:

$$^\circ a = \{r_1\} \qquad a^\circ = \{r_4\} \qquad \overset{\square}{a} = \{r_5\}$$
$$^\circ b = \{r_2\} \qquad b^\circ = \{r_5\} \qquad \overset{\square}{b} = \{r_6\}$$
$$^\circ c = \{r_3\} \qquad c^\circ = \{r_6\} \qquad \overset{\square}{c} = \{r_4\}.$$

Notice that $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\} \in V_{TS}$, but $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\} \notin SV_{TS}$, because $b^\circ \cap \overset{\Box}{a} = \{r_5\} \neq \emptyset$, $a^\circ \cap \overset{\Box}{c} = \{r_4\} \neq \emptyset$ and $c^\circ \cap \overset{\Box}{b} = \{r_6\} \neq \emptyset$. The transition system TS satisfies axioms (A1)-(A7) and (A1*)-(A5*), but does not satisfy (A6*). Hence $TS \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$. The set $u = \{a,b,c\} \in V_{TS} \setminus SV_{TS}$ cannot be enumerated in any way to constitute an event sequence of three events which is enabled at $s = s_{in}$. In such a case, it is difficult to tell whether the target s' for the transition associated with $s \in S$ and $u \subseteq E_{TS}$ should be sought among the existing states of TS or a new state should be added. Foreseeing many complications if adding new states was necessary, we will only be interested in the situation when for every $s \in S$ and $u \subseteq E_{TS}$ satisfying the conditions stated in proposition 16, there is an event sequence ρ_u as in (7) with a source at s.

Let $TS = (S, U, T, s_{in})$ be a transition system in $\mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$ that satisfies the following condition.

> If $s \in S \ \wedge \ u \in V_{TS} \setminus SV_{TS} \ \wedge \ \forall e \in u: \ {}^\circ e \subseteq R_s \ \wedge \ \overset{\Box}{e} \cap R_s = \emptyset$ then there is an event sequence ρ_u (as in (7)) such that $s \overset{\rho_u}{\rightsquigarrow} s'$, for some $s' \in S$. (8)
> The target of the event sequence ρ_u, s', will be denoted by $fin(s,u)$.

We then define the *saturation* of TS as the quadruple $sat(TS) = (S', U', T', s'_{in})$ given by:

$$
\begin{aligned}
T' &= T \cup \{(s,u,fin(s,u)) \mid s \in S \ \wedge \ u \in V_{TS} \setminus SV_{TS} \ \wedge \ \forall e \in u: \ {}^\circ e \subseteq R_s \ \wedge \ \overset{\Box}{e} \cap R_s = \emptyset\},\\
U' &= U \cup \{u \subseteq E_{TS} \mid \exists s \in S: \ (s,u,fin(s,u)) \in T' \setminus T\},\\
S' &= S,\\
s'_{in} &= s_{in}.
\end{aligned}
$$

It is immediate to see that $sat(TS)$ is a transition system, i.e. it satisfies (TS1)-(TS4). Before showing that $sat(TS)$ is a TSENI transition system, we need to prove some properties which relate the regions of TS with the regions of $sat(TS)$.

Proposition 17. If $r \in R_{TS}$ then $r \in R_{sat(TS)}$.

Proof. We prove the first part of definition 1. Let $s \overset{u}{\Leftrightarrow\!\!\!\rightarrow} s'$ and $s \in r$ and $s' \notin r$ in $sat(TS)$.
Case 1: $u \in U$.
Hence $s \overset{u}{\Leftrightarrow\!\!\!\rightarrow} s'$ and $s \in r$ and $s' \notin r$ in TS. Since $r \in R_{TS}$ there exists an r-crossing event e in u, in TS. We will show that e is the r-crossing event in u in $sat(TS)$ as well.

Let $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Leftrightarrow\!\!\!\rightarrow} s''$ in $sat(TS)$. Notice that $u' \in U$ since $u' \neq \emptyset$ and u' is a subset of u (proposition 11). Since r is a region in TS we have $s'' \in r$. Let $q \overset{v}{\Leftrightarrow\!\!\!\rightarrow} q'$ and $e \in v$ in $sat(TS)$ (note that e is the r-crossing event in u, in TS). We need to consider two cases.

1. If $v \in U$ then from definition 1, for r in TS, we have $q \in r$ and $q' \notin r$.
2. If $v \in U' \setminus U$ then, from the definition of U', for every $f \in v$, ${}^\circ f \subseteq R_q$ and $\overset{\Box}{f} \cap R_q = \emptyset$ (in TS). Hence from axiom (A7) for TS we have, for every $f \in v$, $q \overset{\{f\}}{\Leftrightarrow\!\!\!\rightarrow} q^f$ for some $q^f \in S$. In particular, $q \overset{\{e\}}{\Leftrightarrow\!\!\!\rightarrow} q^e$, where $q \in r$ and $q^e \notin r$ as e is the r-crossing event in u in TS. Since $v \in U' \setminus U$, we have $v \in V_{TS}$, which together with $r \in {}^\circ e$ and $q \in r$ gives $q^f \in r$, for all $f \neq e$, $f \in v$ (in TS). From (8) we have that q' is the target of some event sequence ρ_v (as in (7)), such that $q \overset{\rho_v}{\rightsquigarrow} q'$ in TS. Since none of the transitions associated with ρ_v except the one labelled with e crosses the border of r, we have $q' \notin r$.

Case 2: $u \in U' \setminus U$.

From the definition of U', we have that for every $f \in u$, ${}^\circ f \subseteq R_s$ and $\overset{\square}{f} \cap R_s = \emptyset$ (in TS). Hence from axiom (A7) for TS we have, for every $f \in u$, $s \overset{\{f\}}{\Longleftrightarrow} s^f$ for some $s^f \in S$. From (8) we have that s' is the target of some event sequence ρ_u (as in (7)), such that $s \overset{\rho_u}{\rightsquigarrow} s'$ in TS. Hence there exists $e \in u$ such that the transitions labelled with it leave r. So, for $s \overset{\{e\}}{\Longleftrightarrow} s^e$, we have $s^e \notin r$. Since $u \in V_{TS}$ and $s \in r$ and $r \in {}^\circ e$ in TS we have $s^f \in r$, for all $f \neq e$, $f \in u$. We will prove that e is the r-crossing event in u, in $sat(TS)$.

Let $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Longleftrightarrow} s''$ in $sat(TS)$. Since s'' is the target of some event sequence $\rho_{u'}$ (as in (7)), such that $s \overset{\rho_{u'}}{\rightsquigarrow} s''$ in TS, and all the events from u' are enabled at s and none of the transitions labelled with them crosses the border of r, we have $s'' \in r$. Let $q \overset{v}{\Longleftrightarrow} q'$ and $e \in v$ in $sat(TS)$ (note that e is the event from u for whom $s \overset{\{e\}}{\Longleftrightarrow} s^e$ and $s \in r$ and $s^e \notin r$). We consider two cases.

1. If $v \in U$ then from the fact that r is a region in TS and $r \in {}^\circ e$ we have $q \in r$ and $q' \notin r$.
2. If $v \in U' \setminus U$ then from the definition of U' we have, ${}^\circ f \subseteq R_q$ and $\overset{\square}{f} \cap R_q = \emptyset$, for every $f \in v$ (in TS). Hence from axiom (A7) for TS we have, for every $f \in v$, $q \overset{\{f\}}{\Longleftrightarrow} q^f$ for some $q^f \in S$. From the fact that $r \in {}^\circ e$ in TS we have $q \in r$ and $q^e \notin r$. Since $v \in V_{TS}$ we obtain $q^f \in r$, for all $f \neq e$, $f \in v$. From (8) we have that q' is the target of some event sequence ρ_v (as in (7)), such that $q \overset{\rho_v}{\rightsquigarrow} q'$ in TS. Since none of the transitions associated with ρ_v except the one labelled with e crosses the border of r, we have $q' \notin r$.

Now we prove the second part of definition 1. Let $s \overset{u}{\Longleftrightarrow} s'$ and $s \notin r$ and $s' \in r$ in $sat(TS)$.
Case 1: $u \in U$.
Hence we have $s \overset{u}{\Longleftrightarrow} s'$ and $s \notin r$ and $s' \in r$ in TS. Since $r \in R_{TS}$ there exists an r-crossing event e in u, in TS. We will show that e is the r-crossing event in u in $sat(TS)$ as well.

Let $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Longleftrightarrow} s''$ in $sat(TS)$. Notice that $u' \in U$ since $u' \neq \emptyset$ and u' is a subset of u (proposition 11). Since r is a region in TS we have $s'' \notin r$. Let $q \overset{v}{\Longleftrightarrow} q'$ and $e \in v$ in $sat(TS)$ (note that e is the r-crossing event in u, in TS). We need to consider two cases.

1. If $v \in U$ then from definition 1, for r in TS, we have $q \notin r$ and $q' \in r$.
2. If $v \in U' \setminus U$ then, from the definition of U', for every $f \in v$, ${}^\circ f \subseteq R_q$ and $\overset{\square}{f} \cap R_q = \emptyset$ (in TS). Hence from axiom (A7) for TS we have, for every $f \in v$, $q \overset{\{f\}}{\Longleftrightarrow} q^f$ for some $q^f \in S$. In particular, $q \overset{\{e\}}{\Longleftrightarrow} q^e$, where $q \notin r$ and $q^e \in r$ as e is the r-crossing event in u in TS. Since $v \in U' \setminus U$, we have $v \in V_{TS}$, which together with $r \in e^\circ$ and $q \notin r$ gives $q^f \notin r$, for all $f \neq e$, $f \in v$ (in TS). From (8) we have that q' is the target of some event sequence ρ_v (as in (7)), such that $q \overset{\rho_v}{\rightsquigarrow} q'$ in TS. Since none of the transitions associated with ρ_v except the one labelled with e crosses the border of r, we have $q' \in r$.

Case 2: $u \in U' \setminus U$.

From the definition of U', we have that for every $f \in u$, ${}^\circ f \subseteq R_s$ and $\overset{\square}{f} \cap R_s = \emptyset$ (in TS). Hence from axiom (A7) for TS we have, for every $f \in u$, $s \overset{\{f\}}{\Longleftrightarrow} s^f$ for some $s^f \in S$. From (8) we have that s' is the target of some event sequence ρ_u (as in (7)), such that $s \overset{\rho_u}{\rightsquigarrow} s'$ in TS.

Hence there exists $e \in u$ such that the transitions labelled with it enter r. So, for $s \overset{\{e\}}{\Leftrightarrow\!\!\!\to} s^e$, we have $s^e \in r$. Since $u \in V_{TS}$ and $s \notin r$ and $r \in e^\circ$ in TS we have $s^f \notin r$, for all $f \neq e$, $f \in u$. We will prove that e is the r-crossing event in u, in $sat(TS)$.

Let $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Leftrightarrow\!\!\!\to} s''$ in $sat(TS)$. Since s'' is the target of some event sequence $\rho_{u'}$ (as in (7)), such that $s \overset{\rho_{u'}}{\rightsquigarrow} s''$ in TS, and all the events from u' are enabled at s and none of the transitions labelled with them crosses the border of r, we have $s'' \notin r$. Let $q \overset{v}{\Leftrightarrow\!\!\!\to} q'$ and $e \in v$ in $sat(TS)$ (note that e is the event from u for whom $s \overset{\{e\}}{\Leftrightarrow\!\!\!\to} s^e$ and $s \notin r$ and $s^e \in r$). We consider two cases.

1. If $v \in U$ then from the fact that r is a region in TS and $r \in e^\circ$ we have $q \notin r$ and $q' \in r$.
2. If $v \in U' \setminus U$ then from the definition of U' we have, ${}^\circ f \subseteq R_q$ and $\overset{\square}{f} \cap R_q = \emptyset$, for every $f \in v$ (in TS). Hence from axiom (A7) for TS we have, for every $f \in v$, $q \overset{\{f\}}{\Leftrightarrow\!\!\!\to} q^f$ for some $q^f \in S$. From the fact that $r \in e^\circ$ in TS we have $q \notin r$ and $q^e \in r$. Since $v \in V_{TS}$ we obtain $q^f \notin r$, for all $f \neq e$, $f \in v$. From (8) we have that q' is the target of some event sequence ρ_v (as in (7)), such that $q \overset{\rho_v}{\rightsquigarrow} q'$ in TS. Since none of the transitions associated with ρ_v except the one labelled with e crosses the border of r, we have $q' \in r$. □

Proposition 18. If $r \in R_{sat(TS)}$ then $r \in R_{TS}$.

Proof. Follows easily from the construction of $sat(TS)$. Specifically, from the fact that $S = S'$ and $T \subset T'$. □

Corollary 5. Let TS be a transition system in TSENI$_{apost}\setminus$ TSENI that satisfies (8). Then

1.	$E_{TS} = E_{sat(TS)}$.			
2.	For every $e \in E_{TS}$:	$r \in {}^\circ e$ (in TS)	$\Leftrightarrow$	$r \in {}^\circ e$ (in $sat(TS)$).
3.	For every $e \in E_{TS}$:	$r \in e^\circ$ (in TS)	$\Leftrightarrow$	$r \in e^\circ$ (in $sat(TS)$).
4.	For every $e \in E_{TS}$:	$r \in \overset{\square}{e}$ (in TS)	$\Leftrightarrow$	$r \in \overset{\square}{e}$ (in $sat(TS)$).
5.	For every $s \in S$:	$r \in R_s$ (in TS)	$\Leftrightarrow$	$r \in R_s$ (in $sat(TS)$).
6.	$V_{TS} = V_{sat(TS)}$.			

Proof. Follows directly from propositions 17 and 18, and the construction of the transition system $sat(TS)$. □

Proposition 19. $sat(TS)$ is a TSENI transition system.

Proof. (A1*) Let $(s, u, s') \in T'$. If $u \in U$ then $s \neq s'$ follows from (A1) which is satisfied for $TS \in$ TSENI$_{apost}$. Suppose now that $u \in U' \setminus U$ and $s = s'$. From (8) we have that there exists an enumeration of the events from u, $(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$, and an event sequence $\sigma = e_{i_1} e_{i_2} \ldots e_{i_n}$ such that $s \overset{\sigma}{\rightsquigarrow} s'$. Since $u \in U' \setminus U$, we have for every $e \in u$, ${}^\circ e \subseteq R_s$ in TS. In particular, ${}^\circ e_{i_n} \subseteq R_s$. Let $r \in {}^\circ e_{i_n}$ (${}^\circ e_{i_n} \neq \emptyset$, by proposition 9). From $\overset{\{e_{i_n}\}}{\Leftrightarrow\!\!\!\to} s$ and proposition 1 we obtain $s \notin r$. Hence $r \notin R_s$, a contradiction.
(A2*) and (A3*) Follow directly from the construction of $sat(TS)$ and the fact that $TS \in$

TSENI_{apost}.
(A4*) If $u \in U$ then this axiom is satisfied since (A4) is satisfied for TS. Let $u \in U' \setminus U$ and $s \overset{u}{\Leftrightarrow}$. From the definition of U' we have that for all $e \in u$, $^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$ in TS. Since $TS \in \text{TSENI}_{apost}$ and (A7) is satisfied we obtain that $s \overset{\{e\}}{\Leftrightarrow}$ in TS, and so in $sat(TS)$.
(A5*) Follows from corollary 5(5) and axiom (A6) for TS.
(A6*) From corollary 5(6) and 5(5), we have that $V_{TS} = V_{sat(TS)}$ and that the sets of regions containing some $s \in S = S'$ are the same for TS and $sat(TS)$. Hence, in the antedescant of the implication of (A6*) we have that: $s \in S$, $u \in V_{TS}$, and for every $e \in u$, $^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$ in TS. We need to show that $s \overset{u}{\Leftrightarrow}$ in $sat(TS)$. If $u \in SV_{TS}$ then, since (A7) is satisfied for TS, we have $s \overset{u}{\Leftrightarrow}$ in TS and thus $s \overset{u}{\Leftrightarrow}$ in $sat(TS)$. If $u \notin SV_{TS}$ then $u \in V_{TS} \setminus SV_{TS}$. Since $^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for every $e \in u$, we have from (8) and the construction of $sat(TS)$ that $(s, u, fin(s, u)) \in T' \setminus T$. So in this case u is enabled at s in $sat(TS)$ as well. □

Theorem 4. Let TS be a transition system in $\text{TSENI}_{apost} \setminus$ TSENI which satisfies (8). Then there is a transition system $sat(TS) \in$ TSENI such that $\mathcal{N}_{TS} \cong \mathcal{N}_{sat(TS)}$.

Proof. Follows from propositions 17, 18, 19 and corollary 5. □

Proposition 20. Let TS be a transition system in $\text{TSENI}_{apost} \setminus$ TSENI which satisfies (8). Then $sat(TS) \in \text{TSENI} \setminus \text{TSENI}_{apost}$.

Proof. We show that $sat(TS)$ does not satisfy (A5). From proposition 16 we have that there exists $u \in V_{TS} \setminus SV_{TS}$ and $s \in S$ such that for every $e \in u$, $^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$. Since $u \notin SV_{TS}$ there are $f_1, f_2 \in u$ such that $f_1 \neq f_2$ and $f_1{}^\circ \cap \overset{\square}{f_2} \neq \emptyset$. From $^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for $e \in \{f_1, f_2\}$, and (A7) we have $s \overset{\{f_1\}}{\Leftrightarrow}$ and $s \overset{\{f_2\}}{\Leftrightarrow}$. These transitions are in $sat(TS)$ as well, together with $(s, \{f_1, f_2\}, s')$, where $s' = fin(s, \{f_1, f_2\})$. If $\overset{\{f_1\}}{\not\rightarrow} s'$ then $sat(TS)$ does not satisfy (A5). Let $\overset{\{f_1\}}{\Leftrightarrow} s'$. Hence $f_1{}^\circ \subseteq R_{s'}$. Suppose $\overset{\{f_2\}}{\Leftrightarrow} s'$. Since $f_1{}^\circ \cap \overset{\square}{f_2} \neq \emptyset$ there exists $r \in f_1{}^\circ \cap \overset{\square}{f_2}$ and $s' \in r$. From $\overset{\{f_2\}}{\Leftrightarrow} s'$ and $r \in \overset{\square}{f_2}$ and proposition 6 (or 8[7]), we have $s' \notin r$, a contradiction. Thus, $\overset{\{f_2\}}{\not\rightarrow} s'$ and, as a consequence, $sat(TS)$ does not satisfy (A5). □

We now give sufficient and necessary conditions for (8) to be satisfied. First we introduce the idea of a 'blocking' relationship for the events of TS. Let $\{e, f\} \in V_{TS}$. We will say that e *blocks* f if $e^\circ \cap \overset{\square}{f} \neq \emptyset$, and denote this by $e \dashv f$. Let $u \in V_{TS}$. A directed graph of the relation $\dashv$ on the events u will be called the *blocking graph* of u, i.e. it is defined as follows:

$$BG(u) = (u, \{(e, f) \in u \times u \mid e \dashv f\}).$$

The vertices of the graph are labelled with the events from u and an arc from $e \in u$ to $f \in u$ means that e blocks f. If TS is not clear from the context, we will use $BG_{TS}(u)$ to denote $BG(u)$.

Let $G = (V, A)$ be a directed graph. A *directed circuit* is a sequence $v_1, v_2, \ldots, v_n$ $(n \geq 1)$ of distinct vertices of G such that $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in A$. A directed graph that has no directed circuit is called *acyclic*.
The *adjacency matrix* $X = [x_{ij}]$ of G is a $|V| \times |V|$ binary matrix whose element

$$x_{ij} = \begin{cases} 1 & \text{if there is an arc from } i\text{th vertex to } j\text{th vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

An adjacency matrix X is called *lower triangular* if $x_{ij} = 0$, for $i \leq j$.
We will need the following theorem from [3].

Theorem 5. [3] A directed graph G is acyclic if and only if its vertices can be ordered such that the adjacency matrix X is a lower triangular matrix. □

Proposition 21. Let $TS \in \mathrm{TSENI}_{apost}$. Suppose that there are $u \in V_{TS}$ and $s \in S$ such that $s \overset{\{f\}}{\Leftrightarrow}$, for every $f \in u$. Then there is no enumeration of events from u which can be executed in a sequence from s if and only if $BG(u)$ contains a directed circuit.

Proof. ($\Rightarrow$) To prove this implication we assume that $BG(u)$ contains no directed circuit and show how to order events from $u = \{f_1, \ldots, f_n\}$ to build an event sequence which is enabled at s. Since $s \overset{\{f_i\}}{\Leftrightarrow}$, we have ${}^\circ f_i \subseteq R_s$, $f_i{}^\circ \cap R_s = \emptyset$ and $\overset{\square}{f_i} \cap R_s = \emptyset$, for $i = 1, \ldots, n$ (see propositions 4(3) and 8). Suppose an event sequence $\sigma_i = f_1 f_2 \ldots f_i$, where $1 \leq i < n$, is enabled at s. Hence there is a sequence of transitions $(s_0, f_1, s_1), \ldots, (s_{i-1}, f_i, s_i)$ where $s_0 = s$ and $s_k \in S$ $(k = 1, \ldots, i)$. From proposition 4(3) we have $R_{s_k} = (R_{s_{k-1}} \setminus {}^\circ f_k) \cup f_k{}^\circ$ for $k = 1, \ldots, i$. Since $u \in V_{TS}$,

$$R_{s_i} = \Big(R_s \setminus ({}^\circ f_1 \cup \ldots \cup {}^\circ f_i)\Big) \cup (f_1{}^\circ \cup \ldots \cup f_i{}^\circ).$$

For f_{i+1} to be enabled at s_i we need to ensure that two conditions of (A7) are satisfied. The first one, ${}^\circ f_{i+1} \subseteq R_{s_i}$, is satisfied as $u \in V_{TS}$ and ${}^\circ f_{i+1} \subseteq R_s$. The second one, $\overset{\square}{f_{i+1}} \cap R_{s_i} = \emptyset$, can only be violated if $f_k{}^\circ \cap \overset{\square}{f_{i+1}} \neq \emptyset$ for some $1 \leq k \leq i$. Hence an event sequence $\sigma = f_1 f_2 \ldots f_n$, for an enumeration $(f_1, f_2, \ldots, f_n)$ of events of u, would be enabled at s if $f_i{}^\circ \cap \overset{\square}{f_j} = \emptyset$ $(f_i \not\dashv f_j)$ for every $i < j$, where $i, j = 1, \ldots, n$. The following shows it is possible. Since $BG(u)$ contains no directed circuit we have, by theorem 5, that its vertices can be ordered such that the adjacency matrix X is a lower triangular matrix. Let an enumeration $(f_{i_1}, \ldots, f_{i_n})$ be ordered in this way. Hence, in matrix X, we have $x_{f_{i_k}, f_{i_l}} = 0$, for $k \leq l$. This guarantees that in the event sequence $\sigma_X = f_{i_1} \ldots f_{i_n}$, $f_{i_k} \not\dashv f_{i_l}$ if $k < l$, where $k, l = 1, \ldots, n$. Hence, $s \overset{\sigma_X}{\rightsquigarrow}$.
($\Leftarrow$) Suppose there is an enumeration of events from u, $(f_1, f_2, \ldots, f_n)$, such that an event sequence $\sigma = f_1 f_2 \ldots f_n$ is enabled at s. We will write $f_i \overset{\sigma}{\prec} f_j$ if f_i precedes f_j, directly or indirectly, in the event sequence σ. We now show that the following holds for σ.

$$\text{For all } 1 \leq i, j \leq n,\ i \neq j: \quad \text{if } f_i \dashv f_j \text{ then } f_j \overset{\sigma}{\prec} f_i. \tag{9}$$

Let $f_i \dashv f_j$ and $f_i \overset{\sigma}{\precsim} f_j$ for some $1 \leq i,j \leq n$, $i \neq j$. Then we have $f_i{}^\circ \cap \overset{\square}{f_j} \neq \emptyset$ and a sequence of transitions in TS,

$$(s_0, f_1, s_1), \ldots, (s_{i-1}, f_i, s_i), \ldots, (s_{j-1}, f_j, s_j), \ldots, (s_{n-1}, f_n, s_n),$$

where $s_0 = s$. From proposition 4(3) we have for every transition (s_{k-1}, f_k, s_k) $(k = 1, \ldots, n)$, $R_{s_k} = (R_{s_{k-1}} \setminus {}^\circ f_k) \cup f_k{}^\circ$. Since $u \in V_{TS}$, ${}^\circ f_k \cap f_i{}^\circ = \emptyset$, for every $k \geq i+1$. Hence $f_i{}^\circ \subseteq R_{s_{j-1}}$. From $(s_{j-1}, f_j, s_j) \in T$ and proposition 8 we have $\overset{\square}{f_j} \cap R_{s_{j-1}} = \emptyset$. But $f_i{}^\circ \cap \overset{\square}{f_j} \neq \emptyset$, a contradiction. Thus (9) holds.
Since $BG(u)$ contains a directed circuit, there are events $f_{i_1}, \ldots, f_{i_k} \in u$ $(2 \leq k \leq n)$ such that $f_{i_1} \dashv f_{i_2} \dashv \ldots \dashv f_{i_k} \dashv f_{i_1}$. From (9) we have $f_{i_1} \overset{\sigma}{\precsim} f_{i_k} \overset{\sigma}{\precsim} \ldots \overset{\sigma}{\precsim} f_{i_2} \overset{\sigma}{\precsim} f_{i_1}$. Notice that while $\dashv$ relation is not a transitive relation, $\overset{\sigma}{\precsim}$ is. So, we obtain $f_{i_1} \overset{\sigma}{\precsim} f_{i_1}$, a contradiction. $\square$

The blocking graph $BG(u)$ for $u = \{a, b, c\}$, for transition system TS in figure 4, is depicted in figure 5. We can observe that, since $BG(u)$ contains a directed circuit, this TS does not satisfy condition (8).

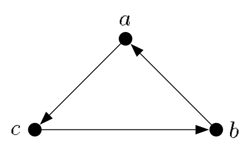


Fig. 5. $BG(\{a, b, c\})$ for the transition system TS in figure 4.

8.2 Part 2

In this section, we consider a transition system $TS \in \text{TSENI} \setminus \text{TSENI}_{apost}$ and try to determine whether it is possible to find a TSENI_{apost} transition system whose associated net would be isomorphic to that of TS.

Proposition 22. Let $TS \in \text{TSENI} \setminus \text{TSENI}_{apost}$. Then TS does not satisfy (A5).

Proof. Since $TS \in \text{TSENI} \setminus \text{TSENI}_{apost}$, it satisfies axioms (A1*)-(A6*) and, as a consequence, axioms (A1)-(A4) and (A6). The only axioms which might not be satisfied by TS are (A5) or (A7). Suppose (A7) is not satisfied. We introduce some symbols for the subformulae appearing in (A7) and (A6*), where $u \subseteq E_{TS}$ and $s \in S$ are such that (A7) fails to hold:

$$\begin{array}{ll} \alpha & u \in SV_{TS} \\ \beta & u \in V_{TS} \\ \gamma & \forall e \in u: \ {}^\circ e \subseteq R_s \wedge \overset{\square}{e} \cap R_s = \emptyset \\ \delta & s \overset{u}{\not\Leftrightarrow} \end{array}$$

(A7) is not satisfied, so α is true (and so is β), γ is true and δ is false. Then $\beta \wedge \gamma$ is true and δ is false contradicting (A6*). That means (A7) is satisfied and the only axiom which can fail for TS is (A5). □

We now need a couple of results concerning TSENI transition systems.

Proposition 23. Let $TS \in$ TSENI and $u, v, w \in U$ be steps such that $u = v \cup w$ and $v \cap w = \emptyset$. If $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$ and $s \overset{v}{\Leftrightarrow\!\!\rightarrow} s''$ and $s'' \overset{w}{\Leftrightarrow\!\!\rightarrow} s'''$ are transitions in TS, then $s' = s'''$.

Proof. From $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$, $s \overset{v}{\Leftrightarrow\!\!\rightarrow} s''$, $s'' \overset{w}{\Leftrightarrow\!\!\rightarrow} s'''$ and proposition 6[7] we have:

$$\begin{aligned} R_{s'} &= (R_s \setminus {}^\circ u) \cup u^\circ, \\ R_{s''} &= (R_s \setminus {}^\circ v) \cup v^\circ, \\ R_{s'''} &= (R_{s''} \setminus {}^\circ w) \cup w^\circ. \end{aligned}$$

Hence,

$$R_{s'''} = \Big(\big((R_s \setminus {}^\circ v) \cup v^\circ \big) \setminus {}^\circ w \Big) \cup w^\circ .$$

Since $u \in V_{TS}$ (proposition 5[7]), ${}^\circ u \subseteq R_s$ and $u^\circ \cap R_s = \emptyset$ (proposition 6[7]), and $u = v \cup w$ we obtain

$$R_{s'''} = \Big(R_s \setminus ({}^\circ v \cup {}^\circ w) \Big) \cup (v^\circ \cup w^\circ).$$

Proposition 3[7] for u, v and w implies ${}^\circ u = {}^\circ v \cup {}^\circ w$ and $u^\circ = v^\circ \cup w^\circ$. Hence $R_{s'''} = R_{s'}$. Then, since TS satisfies (A5*) as a TSENI transition system, we obtain $s' = s'''$. □

Proposition 24. Let $TS \in$ TSENI and there exists a transition $s_0 \overset{u}{\Leftrightarrow\!\!\rightarrow} s$ such that $\overset{\{e\}}{\not\rightarrow} s$ for some $e \in u$. Then there is $f \in u$ such that $f \neq e$ and $f^\circ \cap \overset{\square}{e} \neq \emptyset$.

Proof. From axiom (A4*) we have $s_0 \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}$. Hence ${}^\circ e \subseteq R_{s_0}$ and $\overset{\square}{e} \cap R_{s_0} = \emptyset$ (see propositions 6,9[7]). From proposition 12[7] we have that $s_0 \overset{u \setminus \{e\}}{\Leftrightarrow\!\!\rightarrow} s'$, for some $s' \in S$. So,

$$R_{s'} \overset{prop.\ 6[7]}{=} \Big(R_{s_0} \setminus {}^\circ(u \setminus \{e\}) \Big) \cup (u \setminus \{e\})^\circ .$$

Since $u \in V_{TS}$ (see proposition 5[7]), ${}^\circ e \subseteq R_{s'}$. Suppose $\overset{\square}{e} \cap R_{s'} = \emptyset$. Then, by (A6*), we have $s' \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow}$, which by proposition 23 implies $s' \overset{\{e\}}{\Leftrightarrow\!\!\rightarrow} s$. But $\overset{\{e\}}{\not\rightarrow} s$, a contradiction. Hence, $\overset{\square}{e} \cap R_{s'} \neq \emptyset$. This and $\overset{\square}{e} \cap R_{s_0} = \emptyset$ imply that there is $f \in u$ such that $f \neq e$ and $f^\circ \cap \overset{\square}{e} \neq \emptyset$. □

Corollary 6. Let $TS \in$ TSENI and there exist $s \in S$ and $u \in U$ such that $\overset{u}{\Leftrightarrow\!\!\rightarrow} s$ and $\overset{\{e\}}{\not\rightarrow} s$, for some $e \in u$. Then for every $s' \in S$, if $\overset{u}{\Leftrightarrow\!\!\rightarrow} s'$ then there exists $e' \in u$ such that $\overset{\{e'\}}{\not\rightarrow} s'$.

Proof. Let $s' \in S$ be such that $\overset{u}{\Leftrightarrow\!\!\!\rightarrow} s'$ and $\overset{\{e'\}}{\Leftrightarrow\!\!\!\rightarrow} s'$, for every $e' \in u$. From $\overset{u}{\Leftrightarrow\!\!\!\rightarrow} s$ and $\overset{\{e\}}{\not\rightarrow} s$ and proposition 24 we have that there is $f \in u$ such that $f \neq e$ and $f^\circ \cap \overset{\Box}{e} \neq \emptyset$. Hence there is $r \in R_{TS}$ such that $r \in f^\circ \cap \overset{\Box}{e}$. Since $\overset{\{f\}}{\Leftrightarrow\!\!\!\rightarrow} s'$ $(e' = f)$ and $r \in f^\circ$ and proposition 1[7], we have $s' \in r$. But, $\overset{\{e\}}{\Leftrightarrow\!\!\!\rightarrow} s'$ $(e' = e)$ and $r \in \overset{\Box}{e}$ and proposition 8[7] imply $s' \notin r$, a contradiction. □

Observe that, according to the above corollary, if (A5) is satisfied for a step $u \in U$ at some $s \in S$ then it will be satisfied for u at any state $s \in S$. So, we can say that 'a step u satisfies (A5)' without mentioning the state at which it is satisfied.

Proposition 25. Let $TS \in$ TSENI and there is $u \in U$ which satisfies (A5). Then for every $\emptyset \neq u' \subset u$, u' satisfies (A5).

Proof. From (A2*) we have $s \overset{u}{\Leftrightarrow\!\!\!\rightarrow} s'$, for some $s, s' \in S$. From proposition 12[7], $u' \in U$. Suppose u' does not satisfy (A5). Then from proposition 24 there are $e, f \in u'$ such that $f \neq e$ and $f^\circ \cap \overset{\Box}{e} \neq \emptyset$. Hence, there is $r \in R_{TS}$ such that $r \in f^\circ \cap \overset{\Box}{e}$. Since $TS \in$ TSENI we have from proposition 3[7] that $u^\circ = \bigcup_{e \in u} e^\circ$. So, $r \in u^\circ$ and hence $s' \in r$. But this and $r \in \overset{\Box}{e}$ implies $\overset{\{e\}}{\not\rightarrow} s'$, contradicting the fact that u satisfies (A5). □

Let $TS = (S, U, T, s_{in})$ be a transition system in TSENI $\setminus$ TSENI$_{apost}$ which satisfies the following condition.

$$\begin{array}{l}\text{If } (s, u, s') \in T \text{ and } u \text{ does not satisfy (A5) then there is} \\ \text{an event sequence } \rho_u \text{ (as in (7)) such that } s \overset{\rho_u}{\rightsquigarrow} s'.\end{array} \qquad (10)$$

We then define the *pruning* of TS as the quadruple $prun(TS) = (S', U', T', s'_{in})$ given by:

$$\begin{aligned} T' &= T \setminus \{(s, u, s') \in T \mid (s, u, s') \text{ does not satisfy (A5)}\}, \\ U' &= U \setminus \{u \in U \mid \exists (s, u, s') \in T \setminus T'\}, \\ S' &= S, \\ s'_{in} &= s_{in}. \end{aligned}$$

Notice that the condition (10) allows safe removal of transitions from TS without creating isolated (or non-reachable) states in $prun(TS)$. Corollary 6 guarantees, on the other hand, that U' is well defined. It is immediate to see that $prun(TS)$ is a transition system, i.e. it satisfies (TS1)-(TS4).

Before we show that $prun(TS)$ is a TSENI$_{apost}$ transition system, we need to prove some properties which relate the regions of TS with those of $prun(TS)$.

Proposition 26. If $r \in R_{TS}$ then $r \in R_{prun(TS)}$.

Proof. Follows easily from the construction of $prun(TS)$. Specifically, from the fact that $S = S'$ and $T' \subset T$. □

Proposition 27. If $r \in R_{prun(TS)}$ then $r \in R_{TS}$.

Proof. Let r be a region in $prun(TS)$. We need to show that it is a region in TS. Suppose $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$ and $s \in r$ and $s' \notin r$ in TS. We consider two cases.
Case 1: $u \in U'$.
Since $r \in R_{prun(TS)}$ there exists $e \in u$ such that the following are satisfied in $prun(TS)$:

(a) if $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Leftrightarrow\!\!\rightarrow} s''$ then $s'' \in r$,
(b) if $q \overset{v}{\Leftrightarrow\!\!\rightarrow} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.

We need to show that the above is true in TS as well. We will show that e is the r-crossing event in u in TS. Let $u' \subseteq u \setminus \{e\}$ and $s \overset{u'}{\Leftrightarrow\!\!\rightarrow} s''$ in TS. Since u satisfies (A5), u' satisfies (A5) as well (see proposition 25). So $s'' \in r$, as $u' \in U'$ and (a) is satisfied in $prun(TS)$. Let $q \overset{v}{\Leftrightarrow\!\!\rightarrow} q'$ and $e \in v$ in TS. If $v \in U'$ then $q \in r$ and $q' \notin r$ follow from the fact that (b) is satisfied in $prun(TS)$. If $v \in U \setminus U'$ then by (10) there exists in TS an event sequence $\rho_v = e_1 e_2 \ldots e_n$, where $(e_1, e_2, \ldots, e_n)$ is an enumeration of the events from v, such that $q \overset{\rho_v}{\rightsquigarrow} q'$ and $e_k = e$ for some $1 \leq k \leq n$. This event sequence is in $prun(TS)$ as well. For every event in ρ_v there is a transition $t_i = (q_{i-1}, e_i, q_i)$, where $i = 1, \ldots, n$ and $q_0 = q$, $q_n = q'$. Since $TS \in$ TSENI and satisfies (A4*) we have $q \overset{\{e_i\}}{\Leftrightarrow\!\!\rightarrow}$ for $i = 1, \ldots, n$. Moreover, since $r \in {}^{\circ}e_k$ (in $prun(TS)$) we have $q_{k-1} \in r$ and $q_k \notin r$, and $q \in r$. Since $r \in R_q$ in $prun(TS)$ and $q \overset{\{e_i\}}{\Leftrightarrow\!\!\rightarrow}$ $(i = 1, \ldots, n)$, we deduce that none of the transitions t_i $(i = 1, \ldots, n)$ enters into region r. Hence, since $q_k \notin r$, we have that $q_i \notin r$ for $i = k+1, \ldots, n$, as otherwise some t_i would need to enter into r. Thus $q' \notin r$.
Case 2: $u \in U \setminus U'$.
Then, $u \in U$ and u does not satisfy (A5). By (10), there exists in TS an event sequence $\rho_u = e_1 e_2 \ldots e_n$, where $(e_1, e_2, \ldots, e_n)$ is an enumeration of the events from u, such that $s \overset{\rho_u}{\rightsquigarrow} s'$. This event sequence is in $prun(TS)$ as well. For every event in ρ_u, there is a transition $t_i = (s_{i-1}, e_i, s_i)$, where $i = 1, \ldots, n$ and $s_0 = s$ and $s_n = s'$. Since $s \in r$ and $s' \notin r$, there is $1 \leq k \leq n$ such that $s_{k-1} \in r$ and $s_k \notin r$. Since $TS \in$ TSENI and satisfies (A4*) we have $s \overset{\{e_i\}}{\Leftrightarrow\!\!\rightarrow}$ for $i = 1, \ldots, n$. From the fact that $r \in R_s$ in $prun(TS)$ and $s \overset{\{e_i\}}{\Leftrightarrow\!\!\rightarrow}$ $(i = 1, \ldots, n)$ we deduce that none of the transitions t_i $(i = 1, \ldots, n)$ enters into r. Hence, since $s, s_{k-1} \in r$ and $s_k \notin r$, t_k is the only transition among the t_i's which crosses the border of r. We need to prove that e_k is the r-crossing event in u in TS.

Let $u' \subseteq u \setminus \{e_k\}$ and $s \overset{u'}{\Leftrightarrow\!\!\rightarrow} s''$ in TS. We need to show that $s'' \in r$. If $u' \in U'$ then $s'' \in r$ follows from $r \in R_{prun(TS)}$ and the fact that transitions labelled with the events from u' do not cross the border of r. If $u' \in U \setminus U'$ then by (10) there is an event sequence in TS, $\rho_{u'}$ (as in (7)), such that $s \overset{\rho_{u'}}{\rightsquigarrow} s''$. Since $s \in r$ and none of the transitions associated with the events in $\rho_{u'}$ crosses the border of r, we have $s'' \in r$. Suppose now that $q \overset{v}{\Leftrightarrow\!\!\rightarrow} q'$ and $e_k \in v$ in TS. We need to show that $q \in r$ and $q' \notin r$. If $v \in U'$ then this follows from $r \in R_{prun(TS)}$ and the fact that $s_{k-1} \overset{\{e_k\}}{\Leftrightarrow\!\!\rightarrow} s_k$, $s_{k-1} \in r$ and $s_k \notin r$. If $v \in U \setminus U'$ then we can apply similar reasoning as the one used in Case 1.

The second part of definition 1 for r in TS can be shown in a similar way. Hence r is a region in TS. Moreover, it is non-trivial since $r \in R_{prun(TS)}$ and $S = S'$. □

Corollary 7. Let TS be a transition system in TSENI $\setminus$ TSENI$_{apost}$ which satisfies (10). Then

1. $E_{TS} = E_{prun(TS)}$.
2. For every $e \in E_{TS}$: $r \in {}^{\circ}e$ (in TS) $\Leftrightarrow$ $r \in {}^{\circ}e$ (in $prun(TS)$).
3. For every $e \in E_{TS}$: $r \in e^{\circ}$ (in TS) $\Leftrightarrow$ $r \in e^{\circ}$ (in $prun(TS)$).
4. For every $e \in E_{TS}$: $r \in \overset{\square}{e}$ (in TS) $\Leftrightarrow$ $r \in \overset{\square}{e}$ (in $prun(TS)$).
5. For every $s \in S$: $r \in R_s$ (in TS) $\Leftrightarrow$ $r \in R_s$ (in $prun(TS)$).
6. $V_{TS} = V_{prun(TS)}$.
7. $SV_{TS} = SV_{prun(TS)}$.

Proof. Follows directly from propositions 26 and 27, and the construction of the transition system $prun(TS)$. □

Proposition 28. $prun(TS)$ is a TSENI$_{apost}$ transition system.

Proof. (A1),(A2) follow from $TS \in$ TSENI and the construction of $prun(TS)$.
(A3) follows from (A3*) for TS, the construction of $prun(TS)$ and (10).
(A4) holds due to the construction of $prun(TS)$ and the fact that TS satisfies (A4*).
(A5) follows from proposition 22 and the fact that the construction of $prun(TS)$ removes all the steps u which violate this axiom.
(A6) follows from corollary 7(5) and axiom (A5*) for TS.
(A7) is satisfied for TS as it is shown in the proof of proposition 22. The construction of $prun(TS)$ removes steps which do not satisfy (A5) in TS. From proposition 24 we have that such steps of TS are not potential steps in $prun(TS)$, $u \notin SV_{TS} \overset{coro.\ 7(7)}{=} SV_{prun(TS)}$. Hence the implication in the axiom (A7) holds for $prun(TS)$ as well. □

Theorem 6. Let TS be a transition system in TSENI $\setminus$ TSENI$_{apost}$ which satisfies (10). Then there is a transition system $prun(TS) \in$ TSENI$_{apost}$ such that $\mathcal{N}_{TS} \cong \mathcal{N}_{prun(TS)}$.

Proof. Follows from propositions 26, 27, 28 and corollary 7. □

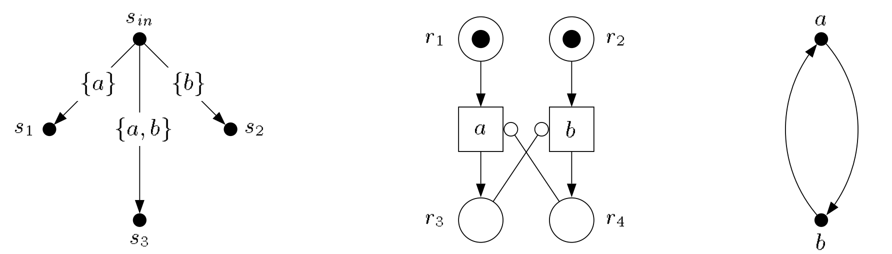
Proposition 29. Let TS be a transition system in TSENI$\setminus$TSENI$_{apost}$ which satisfies (10). Then $prun(TS) \in$ TSENI$_{apost}\setminus$ TSENI.

Proof. We need to show that $prun(TS) \notin$ TSENI. From proposition 22 we have that TS does not satisfy (A5). Therefore, there is a transition $(s, u, s') \in T$ for which (A5) does not hold and, according to the construction of $prun(TS)$, it is removed from TS ($(s, u, s') \notin T'$). But, from (A4*) we have $s \overset{\{e\}}{\Leftrightarrow}$ for every $e \in u$, in TS, and consequently in $prun(TS)$. By $u \in V_{TS}$ and corollary7(6), $u \in V_{prun(TS)}$. So u and s satisfy all the conditions in (A6*), but $(s, u, s') \notin T'$. Thus, $prun(TS)$ fails to satisfy (A6*), and so $prun(TS) \notin$ TSENI. □

Sufficient and necessary conditions for (10) to be satisfied are expressed using a blocking graph of a step appearing in condition (10).

Proposition 30. Let $TS \in$ TSENI and $s \overset{u}{\Leftrightarrow\!\!\rightarrow} s'$ be a transition in TS. Then there is no enumeration of events from u which can be executed in a sequence from s if and only if $BG(u)$ contains a directed circuit.

Proof. Since $u \in U$ and $TS \in$ TSENI, we have from proposition 5[7] that $u \in V_{TS}$, and from (A4*) that $s \overset{\{f\}}{\Leftrightarrow\!\!\rightarrow}$, for every $f \in u$. The rest of the proof is similar to that of proposition 21, as it uses the common properties of TSENI and TSENI$_{apost}$ transition systems. □



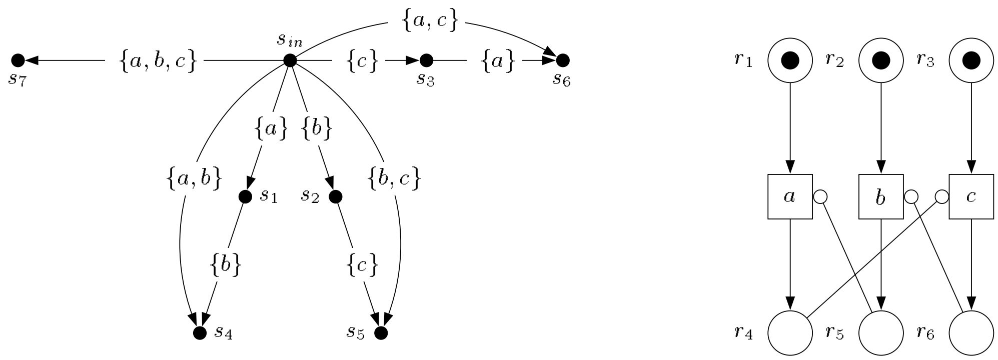
(a) $TS \in$ TSENI \ TSENI$_{apost}$ (b) $\mathcal{N}_{TS}$ (c) $BG(\{a,b\})$

Fig. 6. TSENI transition system which does not satisfy condition (10).

We observe that TS shown in figure 6 does not satisfy condition (10), since there is a step $\{a,b\} \in U$ such that $BG(\{a,b\})$ contains a directed circuit.

9 Concluding Remarks

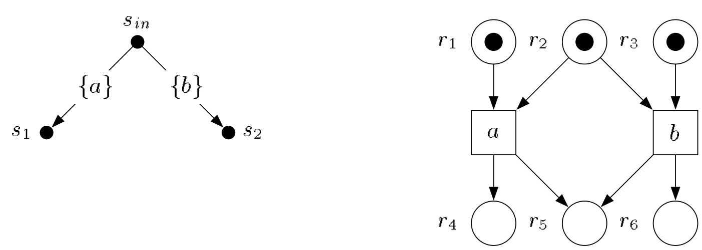
In this paper, we compared the TSENI$_{apost}$ and TSENI transition systems. It was shown that for any $TS \in$ TSENI$_{apost}$ \ TSENI satisfying the condition (8), there is a transition system $sat(TS) \in$ TSENI \ TSENI$_{apost}$, such that $\mathcal{N}_{TS} \cong \mathcal{N}_{sat(TS)}$. We mentioned that when $TS \in$ TSENI$_{apost}$ \ TSENI does not satisfy the condition (8), the problem is much more complicated. In particular, some additional states might be required to build a TSENI transition system whose associated net is isomorphic to $\mathcal{N}_{TS}$. For the TS from figure 4(a), the procedure of 'saturation' leads to the TSENI transition system depicted in figure 7(a). We can see that one extra state, s_7, was added. The number of regions of the new 'saturated' transition system will be the same as number of regions of TS, and we only need to add s_7 to the post-regions of every event. The nets associated with TS, in figure 4(b), and its 'saturated' version, in figure 7(b), are isomorphic. Notice that the transition system in figure 7(a) is not a TSENI$_{apost}$ transition system. So, by adding extra transitions, we are loosing the ability to fulfill (A5), exactly like when the process of 'saturation' is applied to the TSENI$_{apost}$ transition system satisfying the condition (8). The generalisation of the process of 'saturation' for TSENI$_{apost}$ (but not TSENI) transition systems which are not satisfying the condition (8) looks promising. One only needs to ensure that by adding extra states, we do not violate the state separation property, (A5*), of the TSENI transition system we create.



(a) $TS \in$ TSENI $\setminus$ TSENI$_{apost}$ (b) $\mathcal{N}_{TS}$

Fig. 7. $TS \in$ TSENI $\setminus$ TSENI$_{apost}$ and the net associated with it, $\mathcal{N}_{TS}$.

The generalisation of the 'pruning' procedure for a transition system $TS \in$ TSENI $\setminus$ TSENI$_{apost}$, which does not satisfy the condition (10), to obtain a TSENI$_{apost}$ transition system with isomorphic net, will certainly fail. Take, for example, the TSENI transition system in figure 6(a). After deleting transition $(s_{in}, \{a,b\}, s_3)$, for which (A5) is not satisfied, we obtain a transition system which is both TSENI and TSENI$_{apost}$ transition system (see figure 8(a)), and the net associated with it (see figure 8(b)) is not isomorphic to that of the transition system in figure 6.



(a) $TS \in$ TSENI $\cap$ TSENI$_{apost}$ (b) $\mathcal{N}_{TS}$

Fig. 8. $TS \in$ TSENI $\cap$ TSENI$_{apost}$ and the net associated with it, $\mathcal{N}_{TS}$.

Acknowledgements

I would like to thank Alex Yakovlev for several inspiring discussions on the synthesis problem. This work has been supported by EPSRC through a PhD studentship No. 96701454.

References

1. Badouel E., Darondeau Ph.: *Theory of regions.* Third Advanced Course on Petri Nets, Springer-Verlag, LNCS (1997), to appear.
2. Chiola G., Donatelli S., Francheschinis G.: *Priorities, inhibitor arcs and concurrency in P/T nets.* Proc. of 12th Intern. Conf. on Appl. and Theory of Petri Nets, Gjern, Denmark (1991), 182-205.
3. Deo N.: *Graph Theory with Applications to Engineering and Computer Science.* Prentice Hall International (1974).
4. Janicki R., Koutny M.: *Semantics of inhibitor nets.* Information and Computation, Vol. 123, No. 1 (1995), 1-16.
5. Mukund M.: *Petri nets and step transition systems.* International Journal of Foundations of Computer Science, Vol. 3, No. 4 (1992), 443-478.
6. Nielsen M., Rozenberg G., Thiagarajan P.S.: *Elementary transition systems.* Theoretical Computer Science 96 (1992), 3-33.
7. Pietkiewicz-Koutny M.: *Transition systems of elementary net systems with inhibitor arcs.* Proc. of ICATPN'97, P. Azema and G. Balbo (Eds.), Springer-Verlag, Lecture Notes in Computer Science 1248 (1997), 310-327.