

Bulk quantum effects for de Sitter branes in AdS₅

Ian G. Moss [35]^a, Wade Naylor [36]^b, Wenceslao Santiago-Germán [37]^a, Misao Sasaki [38]^b

^a*Department of Mathematics, University of Newcastle Upon Tyne,
Newcastle Upon Tyne, NE1 7RU, UK;*

^b*Department of Earth and Space Science, Graduate School of Science, Osaka University,
Toyonaka 560-0043, Japan*

We investigate some issues regarding quantum corrections for de Sitter branes in a bulk AdS₅ spacetime. The one-loop effective action for a Majorana spinor field is evaluated and compared with the scalar field result. We also evaluate the cocycle function for various boundary conditions, finding that the quantum corrections naturally induce higher order curvature terms in the original action and, in general, it is not possible to eliminate the cocycle function by renormalisation. In the one brane limit care must be taken on how one extracts physical results. The effective potential is found to be zero on the conformally related cylinder. However, using the actual metric, the contribution from the cocycle function is non-zero and must be included. Subtleties with any zero modes are also discussed.

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I. INTRODUCTION

From within the circumference of brane world cosmology, many ideas and proposals relate to the work of Randall and Sundrum [1], who initially implemented their two brane model to try and solve the hierarchy problem. In the one brane variant of this model [2] it was also shown that the extra dimension could be of infinite extent. In most contexts, the second model (and its extensions to include curved branes) has been of more interest to the cosmology community because gravity behaves essentially like 4-dimensional gravity localised near the brane [2, 3, 4].

An interesting brane world scenario (BWS) has been developed in [5, 6], known as the bulk inflaton model, where it is possible to obtain inflation on a single positive tension brane solely due to the presence of a bulk gravitational scalar field. In this regard, the vacuum energy of the bulk scalar field could have some affect on the cosmological evolution of the brane, (depending on the size and sign of this quantity). The importance of vacuum energy in BWS has also been highlighted in [7] where it has been suggested that it may be possible to solve the hierarchy problem with two positive tension branes, if the back-reaction of the Casimir effect is included (also see [8]).

In this article we evaluate the one-loop effective action for one and two de Sitter brane configurations in a bulk 5-dimensional anti-de Sitter spacetime. We focus on Majorana spinor fields and compare with the scalar field results. This article continues from a previous work [9] (also see [10] for the one brane case) for scalar fields. As before, we employ ζ -function regularisation [11], which is unusual as compared to most regularisation schemes, in that it does not require infinite counter terms. Of interest is the recent article by Elizalde et. al. [12], who also considered one and two brane set ups for scalar fields in dS₅ as well as AdS₅ backgrounds. Small mass perturbations from the massless conformally coupled case were also studied. However, here we include the cocycle function (because these techniques rely on conformal transformations) which can be related to the B_5 heat kernel coefficient [14, 15, 16].

We briefly mention the work done for a bulk AdS₅ background bounded by flat branes. In [15] ζ -function regularisation was employed to evaluate the effective potential for a massless conformally coupled scalar and fermion field and some special techniques were used for the graviton field. The possibility of radion stabilisation due to quantum effects was also studied, which gave a negative result. The dimensional reduction procedure was used in [17, 18, 19] to obtain an effective four dimensional field with an infinite KK tower. The effective potential was then regularised using dimensional regularisation. Fermion fields have also been considered in [20, 21] where the main emphasis was on fermion representations and topological symmetry breaking. Some recent work concerning the radion with bulk gauge fields is given in [22], & for higher dimensional models, see [23].

In the next section we discuss the Euclidean de Sitter metric for the bulk inflaton model and the wave equation for scalar and spinor fields. In section III we make a detailed analysis of the effective potential for a conformally coupled Majorana fermion field. In section IV we evaluate the cocycle function for spin zero fields with Dirichlet or Robin boundary conditions and a spin half field with mixed boundary conditions. This is also evaluated for the delta-function contribution in the potential. We discuss the renormalisation of these terms. In section V we look at

the flat space limit and the Casimir energy. In section VI we draw conclusions. In Appendix A we look at the case of twisted fields for the Majorana fermion. In Appendix B we discuss zero modes.

II. DE SITTER BRANES AND CONFORMALLY COUPLED FIELDS

We begin with two de Sitter branes embedded in five dimensional anti de Sitter space, placed symmetrically to preserve the de Sitter isometry group and forming the boundary of the region which we will consider. We will calculate the vacuum polarization on a Euclideanised form of the metric. On the Euclidean section the anti de Sitter metric becomes a hyperboloid and the de Sitter branes become concentric four spheres [27],

$$ds^2 = dr^2 + \ell^2 \sinh^2(r/\ell) d\Omega_4^2, \quad (1)$$

where $\ell = (-6/\Lambda_5)^{1/2}$ is the anti-de Sitter radius and $d\Omega_4^2$ is the metric on the unit 4-sphere. We use r_- for the location of the negative tension brane and r_+ for the location of the positive tension brane ($r_- < r_+$). At the classical level, boundary or junction conditions relate the locations of the branes to the brane tensions σ_{\pm} according to [27, 28]

$$\sigma_{\pm} = \pm \frac{3}{4\pi G_5 \ell} \coth(r_{\pm}/\ell). \quad (2)$$

The quantum corrections will introduce other sources of stress-energy on the brane which modify these relations.

The metric is conformal to a cylinder $I \times S^4$ [27, 28]. However, it may also be of interest to consider the compact space $S^1 \times S^4$ without boundaries, in the one brane limit used in the bulk inflaton model [6], and in this case all results are simply multiplied by a factor of two. Thus,

$$ds^2 = a^2(z)(dz^2 + d\Omega_4^2) \quad a(z) = \frac{\ell}{\sinh(z_0 + |z|)}, \quad (3)$$

where the coordinates are chosen to have the positive tension brane at $z = 0$ and the negative tension brane at $z = L$, so that the one brane configuration is given by $L \rightarrow \infty$. The non-dimensional length L is given by

$$L = \int_{r_-}^{r_+} \frac{dr}{\ell \sinh(r/\ell)} = \log \coth \frac{r_-}{2\ell} - \log \coth \frac{r_+}{2\ell}. \quad (4)$$

The one loop effective action for a conformal scalar field ϕ in five dimensions on a space with metric g takes the form

$$W[g] = \frac{1}{2} \log \det \Delta[g], \quad (5)$$

where $\Delta[g]$ is the conformal operator,

$$\Delta[g] = -\nabla^2 + \frac{3}{16} {}^5R \quad (6)$$

with Ricci scalar 5R . Conformal symmetry also restricts the boundary conditions to be either Dirichlet $\phi = 0$ or Robin $(-\frac{3}{8}\theta + \partial_n)\phi = 0$, where θ is the trace of the extrinsic curvature.

If the determinant is defined using zeta-functions, the effective actions on conformally related spaces with metrics g and g_{ω} are related by

$$W[g_{\omega}] = C[\Delta, \omega] + W[g], \quad (7)$$

where the cocycle function $C[\Delta, \omega]$ can be expressed in terms of local tensors [13]. This enables us to reduce the present problem to the calculation of the cocycle function and the effective action on the cylinder.

For the cylinder, ${}^5R = 12$ and

$$\Delta = \left(-\partial_z^2 - \Delta^{(4)} + \frac{9}{4} \right), \quad (8)$$

where $\Delta^{(4)}$ is the Laplacian on the four sphere. The conformal symmetry allows either Dirichlet or Neumann boundary conditions and both are consistent with a Z_2 reflection symmetry about either brane. We shall take the same boundary conditions on either brane. The spectrum of $-\partial_z^2$ in Eq. (8) is simply given by $\pi n/L$, $n = 0, 1, 2, \dots$ for Neumann boundary conditions and $n = 1, 2, \dots$ for Dirichlet boundary conditions. (Choosing different boundary conditions on either brane would give a spectrum $\pi(n + \frac{1}{2})/L$, see appendix A.)

We stress that in a more general setting, i.e. $\xi \neq \frac{3}{16}$, conformal transformations introduce delta functions in the potential, Eq. (8), because the scale factor $a(z)$, Eq. (3), depends on the absolute value $|z|$. Only conformal coupling to the curvature, $\xi = \frac{3}{16}$, irrespective of any mass term cancels out any distributional sources. This is for the same reason that Neumann boundary conditions remain Neumann after performing a conformal transformation of the field. In terms of the bulk inflaton model [6] for $\xi = \frac{3}{16}$ there is no bound state and gravity cannot be localised on the brane. However, the cocycle function interpolates between the original metric and the scaled one and therefore introduces distributional terms.

The eigenvalues of the Laplacian $\Delta^{(4)}$ are $m(m+2)$ with degeneracy

$$d(m) = \frac{1}{3}(m+2)(m+3/2)(m+1) = \frac{1}{3} \left[(m+3/2)^3 - \frac{1}{4}(m+3/2) \right], \quad (9)$$

where we have rearranged this expression for later convenience. Thus, the eigenvalues for the Klein-Gordon operator, Eq. (8), are

$$\lambda_{n,m} = \left(\frac{\pi n}{L} \right)^2 + (m+3/2)^2. \quad (10)$$

For spin-1/2 fermion fields ψ , the massless Dirac equation is automatically conformally covariant. The boundary conditions force exactly half of the fermion components to vanish at the brane, and generally take the form

$$\frac{1}{2}(1+S)\psi = 0, \quad (11)$$

where S is a spinor transformation representing reflection about the boundary. We shall concentrate on Majorana spinors, which have 8 real components. Reflections about the boundaries can be represented by using 8×8 Γ matrices, $S = i\Gamma_5\Gamma_6$. The details can be found in reference [20].

It is convenient to take the square of the Dirac operator when evaluating the effective action

$$W = -\frac{1}{4} \log \det \Delta \quad (12)$$

for Majorana fermions, where

$$\Delta = -\nabla^2 + \frac{1}{4}{}^5R. \quad (13)$$

The squared operator is not conformally invariant, but it is nevertheless still possible to relate the effective actions of fermions with conformally related metrics [31].

On the cylinder,

$$\Delta = \left(-\partial_z^2 - \Delta_f^{(4)} + 3 \right). \quad (14)$$

The eigenvalues of the spinor Laplacian, $\Delta_f^{(4)}$, on the 4-sphere are well known (e.g., see [30]) and are given by $(m+2)^2 - 3$. Half of the field components satisfy Dirichlet boundary conditions and half satisfy Neumann boundary conditions, and the eigenvalues of Δ are

$$\lambda_{n,m}^M = \left(\frac{\pi n}{L} \right)^2 + (m+2)^2. \quad (15)$$

The degeneracy for the 8 component Majorana spinors is given by [30]

$$d^M(m) = 8 \times \frac{1}{6}(m+1)(m+2)(m+3) = \frac{4}{3} [(m+2)^3 - (m+2)], \quad (16)$$

where we take $n \in \mathbb{Z}$.

III. CONFORMALLY FLAT SCALAR AND SPINOR FIELDS

A. The effective action and ζ functions

We can employ the ζ -function method to find the contribution to the effective action from the cylinder. The generalised zeta function is given by

$$\zeta(s) = \sum_{m,n=0}^{\infty} d(m) \lambda_{m,n}^{-s} \quad (17)$$

with the one-loop effective action related to $\zeta(s)$ by (e.g., see [29, 30])

$$W = -\frac{1}{2} \zeta'(0) - \frac{1}{2} \zeta(0) \log \mu^2 \quad (18)$$

for scalar fields and

$$W = \frac{1}{4} \zeta'(0) + \frac{1}{4} \zeta(0) \log \mu^2 \quad (19)$$

for Majorana fermions, where μ is the renormalisation scale.

The scalar field result is contained in previous work [9] and in the early work of Dowker and Apps [14]. In what follows, we therefore concentrate on the spinor field. The scalar field results are quoted for comparison (see appendix B).

B. One brane

We first evaluate the effective potential for a one brane configuration on the cylinder, where $L \rightarrow \infty$ and the discrete sum over n becomes an integral. It is then simple to see that the zeta function $\zeta^M(s)$ becomes

$$\zeta^M(s) = \frac{2L}{\pi} \int_0^\infty dk \sum_{m=0}^\infty d^M(m) (k^2 + (m+2)^2)^{-s}, \quad (20)$$

where the factor of 2 is because there are two copies of the bulk space on either side of the brane, essentially we assume an $S^1 \times S^4$ topology. For large s we can interchange the order of the sum and the integral and perform the k integration, using the well known identity

$$\int_0^\infty dk k^\alpha (k^2 + a)^{-s} = \frac{1}{2} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{\Gamma(s-1/2-\alpha/2)}{\Gamma(s)} a^{\alpha/2+1/2-s}, \quad (21)$$

implying

$$\begin{aligned} \zeta^M(s) &= \frac{2L}{\pi} \frac{\sqrt{\pi}}{2} \sum_{m=0}^\infty \frac{\Gamma(s-1/2)}{\Gamma(s)} d^M(m) (m+2)^{1-2s}, \\ &= \frac{L}{\pi} \frac{4}{3} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} (\zeta(2s-4, 2) - \zeta(2s-2, 2)), \end{aligned} \quad (22)$$

where in the second step we have used simple algebra to rewrite the equation in terms of generalised (Hurwitz) ζ -functions. For $s=0$ it is clear that $\zeta^M(0) = 0$ because

$$\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3), \quad (23)$$

where γ is Euler's constant. Thus,

$$\zeta^M(0) = \frac{L}{\pi} \frac{4}{3} \sqrt{\pi} \Gamma(-1/2) (\zeta(-4, 2) - \zeta(-2, 2)) = 0, \quad (24)$$

which is zero because the combination of ζ -functions in Eq. (24) cancel, as can be verified by employing the relation

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}, \quad (25)$$

where $B_n(x)$ is a Bernoulli polynomial. Therefore, as for scalar fields [9, 10], the one-loop effective potential on the cylinder is zero in the one brane case. We discuss the cocycle contribution later.

C. Two branes

We now consider the two brane configuration which consists of two infinite summations, which can be expressed in terms of generalised ζ -functions. In fact, there are many subtle issues regarding the correct analytic continuation of such functions upon interchanging the order of the summations (see [11] for a detailed discussion). As we shall see,

the one brane result, above, appears as one of the terms in the two brane case. Here assume the topology is $S^1 \times S^4$ rather than $I \times S^4$, to compare with the result of the last section. The function $\zeta^M(s)$ now becomes

$$\begin{aligned}\zeta^M(s) &= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} d^M(m) (c^2 n^2 + (m+2)^2)^{-s}, \\ &= \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} d^M(m) \int_0^{\infty} dt t^{s-1} \exp\{-t[c^2 n^2 + (m+2)^2]\},\end{aligned}\quad (26)$$

where $c = \pi/L$ and in the second step we have made use of the Mellin transform. The above case refers to an untwisted spinor field, the twisted case is discussed in the appendix. This formula is almost identical in form to the 2-dimensional Epstein-Hurwitz ζ -function studied by Elizalde in [24] (see also [11]), apart from the spinor degeneracy factor $d^M(m)$. This work suggests using the transformation formula of Jacobi's θ_3 function [25], equivalent to a Poisson resummation,

$$\sum_{n=-\infty}^{\infty} e^{-c^2 n^2} = \theta_3(0, e^{-c^2}) = \sqrt{\frac{\pi}{c^2}} \theta_3(0, e^{-\pi^2/c^2}) = \sqrt{\frac{\pi}{c^2}} \sum_{n=-\infty}^{\infty} \exp[-\pi^2 n^2/c^2].\quad (27)$$

Substitution of the above equation into (26) allows us to interchange the order of the summations, giving

$$\begin{aligned}\zeta^M(s) &= \sqrt{\frac{\pi}{c^2}} \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} d^M(m) \int_0^{\infty} dt t^{s-3/2} \exp\{-t(m+2)^2\} \\ &\quad + 2\sqrt{\frac{\pi}{c^2}} \frac{1}{\Gamma(s)} \sum_{n=1, m=0}^{\infty} d^M(m) \int_0^{\infty} dt t^{s-3/2} \exp\left[-\frac{\pi^2 n^2}{c^2 t} - (m+2)^2 t\right].\end{aligned}\quad (28)$$

After integrating with respect to t (using standard relations) we obtain

$$\begin{aligned}\zeta^M(s) &= \frac{4}{3} \sqrt{\frac{\pi}{c^2}} \frac{\Gamma(s-1/2)}{\Gamma(s)} (\zeta(2s-4, 2) - \zeta(2s-2, 2)) \\ &\quad + \frac{4\pi^s}{\Gamma(s)} c^{-s-1/2} \sum_{n=1, m=0}^{\infty} d^M(m) n^{s-1/2} (m+2)^{-s+1/2} K_{s-1/2}[2\pi c^{-1} n(m+2)],\end{aligned}\quad (29)$$

where K_ν is a modified Bessel function of the second kind. It is easy to verify that $\zeta^M(0) = 0$, using Eq. (23). Furthermore, taking the derivative of $\zeta^M(s)$ with respect to s (and leaving only terms that remain independent of $\Gamma(s)$ near $s = 0$, see Eq. (23)) gives

$$\begin{aligned}\zeta^{M'}(0) &= \frac{4}{3} \sqrt{\frac{\pi}{c^2}} \Gamma(-1/2) (\zeta(-4, 2) - \zeta(-2, 2)) \\ &\quad + 4c^{-1/2} \sum_{n=1, m=0}^{\infty} d^M(m) n^{-1/2} (m+2)^{1/2} K_{-1/2}[2\pi c^{-1} n(m+2)],\end{aligned}\quad (30)$$

where a prime denotes differentiation with respect to s . Interestingly, the first term is exactly the contribution from the one brane configuration Eq. (24), which is zero. As mentioned in [9] this is similar to finite temperature field theory where the $n = 0$ mode gives the zero temperature contribution. The similarity to finite temperatures is due to the compact topology in the fifth dimension.

The second term is non-zero and depends on the value of $c = \pi/L$. Simplifying the above equation we find

$$\frac{1}{4} \zeta^{M'}(0) = \sum_{n=1, m=0}^{\infty} \frac{2}{3} \frac{(m+1)(m+2)(m+3)}{n} \exp[-2Ln(m+2)],\quad (31)$$

where we have used the relation $K_{-1/2}(z) = \sqrt{\pi/(2z)} e^{-z}$. After evaluating the summation over m we have an expression for the Majorana spinor one-loop effective action on the cylinder (for $I \times S^4$)

$$W_{I \times S^4}^M = \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n(\sinh nL)^4},\quad (32)$$

where we have divided by two, since $S^1 \times S^4$ doubles the modes.

For the scalar field we only need to make simple replacements in the degeneracy and eigenvalues, see [9]. Dowker and Aapps gave the first results for scalar fields on a cylinder [14], including the zero mode (see appendix B). Their results extend trivially to five dimensions, with

$$W_{I \times S_4}^{D,N} = \pm \frac{1}{4} \zeta'_{S_4}(0) \pm \frac{1}{4} \zeta_{S_4}(0) \log \mu^2 - \frac{1}{16} \sum_{n=1}^{\infty} \frac{\cosh nL}{n(\sinh nL)^4} \quad (33)$$

for Dirichlet (D) and Neumann (N) boundary conditions, where ζ_{S_4} is the zeta function for the operator $-\Delta^{(4)} + \frac{3}{16} {}^5R$ on S_4 . Explicitly,

$$\zeta_{S_4}(0) = -\frac{17}{2880}, \quad \zeta'_{S_4}(0) = \left(-\frac{7}{24} \zeta'_R(-3) + \frac{1}{24} \zeta'_R(-1) + \frac{1}{24} \zeta_R(-3) \log 2 - \frac{1}{24} \zeta_R(-1) \log 2 \right), \quad (34)$$

where ζ_R is the Riemann zeta function (see appendix B).

IV. THE COCYCLE FUNCTION

Calculating the cocycle function is heavily dependent on knowing appropriate heat kernel coefficients. These heat kernel coefficients are defined in d dimensions by an asymptotic expansion,

$$\text{tr}(\omega e^{-\Delta t}) \sim t^{-d/2} \sum_{n=0}^{\infty} B_n[\Delta, \omega] t^{n/2}. \quad (35)$$

In five dimensions we require $B_5[\Delta, \omega]$, which can be found in the literature [26]. It takes the form of a surface integral of scalar invariants with a maximum of four derivatives of the metric.

Given a sequence of metrics $g_\epsilon = e^{-2\epsilon\omega}g$, and operators Δ_ϵ , it can be shown that [13]

$$W[g_1] - W[g] = C[\Delta, \omega] = \int_0^1 d\epsilon B_d[\Delta_\epsilon, \omega]. \quad (36)$$

To keep things simple, we have restricted attention to the class of metrics of the form

$$g_\epsilon = e^{-2\epsilon\omega(z)} (dz^2 + d\Omega_4^2) \quad (37)$$

with boundaries at $z = 0$ and $z = L$. The cocycle function then takes a generic form

$$C[\Delta, \omega] = \frac{1}{16\pi^2} \int_{S_4} \{ \alpha_0 \omega + \alpha_1 \omega_{zzzz} + \alpha_2 \omega_{zzz} \omega_z + \alpha_3 \omega_{zz}^2 + \alpha_4 \omega_z^2 \omega_{zz} + \alpha_5 \omega_z^4 + \alpha_6 \omega_z^2 + \alpha_7 \omega_{zz} \}. \quad (38)$$

The coefficients are tabulated in table I. In particular, notice that we have evaluated the contribution to the cocycle function for the delta function background. This is simply obtained by adding the Dirichlet contribution to the Robin contribution, see [32].

Anti de Sitter space has

$$\omega = \log \frac{\sinh(|z| + z_0)}{\ell} = -\log(\ell \sinh(r/\ell)), \quad (39)$$

where we ignore absolute value of z , $|z|$, because we have already included its effect (delta function terms) in the cocycle function. Using the relation $z = -\log \tanh(r/2\ell)$ to transform back to the spherical coordinate system, the cocycle function is then

$$C[\Delta, \omega] = \frac{1}{6} \sum_{\pm} \{ c_0 + c_2 \sinh^2(r_{\pm}/\ell) + c_4 \sinh^4(r_{\pm}/\ell) - \alpha_0 \log(\ell \sinh(r_{\pm}/\ell)) \}. \quad (40)$$

The coefficients are tabulated in table II. Results for scalar fields with Dirichlet boundary conditions have also been obtained by Garriga et. al. [16].

It is important to discuss whether any of the terms in the cocycle function can be regarded as renormalisations of the original action. Consider first of all the terms with coefficient c_4 . These terms are proportional to the volumes of the branes since, for example, the positive tension brane is a four-sphere of radius $\ell \sinh(r_+/\ell)$. We have only considered bulk fields, but in a more realistic theory there would also be matter fields on the brane which could give infinite renormalisations of the brane tensions. It would then make sense to absorb c_4 into a finite renormalisation. However, when there are two branes, the new terms take the same sign on the positive tension and the negative

TABLE I: Coefficients of the terms in the cocycle function for conformal fields on Anti de Sitter space with spin 0 and 1/2 with Dirichlet (D), Robin (R) and mixed (M) boundary conditions. The spinors are Majorana. δ is the contribution due to the delta-function potential instead of a boundary.

spin	background	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
0	D	$\frac{17}{1920}$	$-\frac{1}{128}$	$\frac{5}{192}$	$\frac{1}{48}$	$-\frac{9}{768}$	$-\frac{11}{3072}$	$-\frac{1}{16}$	$\frac{1}{64}$
0	R	$-\frac{17}{1920}$	$\frac{1}{128}$	$-\frac{1}{48}$	$-\frac{1}{48}$	$\frac{5}{768}$	$\frac{1}{15360}$	$\frac{1}{32}$	$-\frac{1}{64}$
1/2	M	0	0	$\frac{1}{48}$	0	$-\frac{1}{96}$	$-\frac{11}{1280}$	$-\frac{3}{32}$	0
0	δ	0	0	$\frac{1}{192}$	0	$-\frac{1}{192}$	$-\frac{9}{2560}$	$-\frac{1}{32}$	0

TABLE II: Coefficients of the terms in the cocycle function for conformal fields on Anti de Sitter space with spin 0 and 1/2 with Dirichlet (D), Robin (R) and mixed (M) boundary conditions. The spinors are Majorana. δ is the contribution due to the delta-function potential instead of a boundary.

spin	background	c_0	c_2	c_4
0	D	$-\frac{203}{3072}$	$\frac{30}{3072}$	$\frac{393}{3072}$
0	R	$\frac{481}{15360}$	$-\frac{498}{15360}$	$-\frac{1779}{15360}$
1/2	M	$-\frac{393}{3840}$	$-\frac{226}{3840}$	$\frac{167}{3840}$
0	δ	$-\frac{89}{2560}$	$-\frac{58}{2560}$	$\frac{31}{2560}$

tension brane and it would be very unnatural to then suppose that the renormalised brane tensions could be exactly equal and opposite sign as they are in the Randall-Sundrum model.

The terms with coefficients c_2 and c_0 are similar to brane curvature terms 4R and curvature squared terms in the Lagrangian. Such terms may be desirable to renormalise away divergences from the brane fields, but they represent a significant departure from the models which we mentioned in the introduction. Both the classical field equations and the quantisation would be modified. A possible resolution would be to regard the theory as a low energy approximation and restrict attention to theories in which the divergences from the brane fields vanish at one loop order.

Another feature becomes apparent when we examine the coefficients in equation (38), which imply that the cocycle function combines brane curvature with extrinsic curvature terms. These can only be expressed in terms of brane curvature terms (by the Gauss-Codacci relations) in special cases. We conclude that the cocycle function should not be eliminated by renormalisation.

Combining the cocycle function with the cylinder result gives the total one loop effective action for the Majorana fermion,

$$W^M = \frac{1}{6} \sum_{\pm} \{c_0^M + c_2^M \sinh^2(r_{\pm}/\ell) + c_4^M \sinh^4(r_{\pm}/\ell)\} + \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n(\sinh nL)^4}. \quad (41)$$

where the coefficients are tabulated in table II. For comparison, the conformal scalar effective actions are

$$W^{D,R} = \frac{1}{6} \sum_{\pm} \left\{ c_0^{D,R} + c_2^{D,R} \sinh^2(r_{\pm}/\ell) + c_4^{D,R} \sinh^4(r_{\pm}/\ell) - \alpha_0^{D,R} \log(\ell\mu \sinh(r_{\pm}/\ell)) \right\} \\ \pm \frac{1}{4} \zeta'_{S_4}(0) - \frac{1}{16} \sum_{n=1}^{\infty} \frac{\cosh nL}{n(\sinh nL)^4}. \quad (42)$$

for Dirichlet and Robin boundary conditions, where $\zeta'_{S_4}(0)$ was given in equation (34).

V. FLAT SPACE LIMIT AND CASIMIR ENERGY DENSITY

The results can be simplified in a variety of limiting cases. The curvatures of the branes are small when $r_{\pm} \gg \ell$ and they approximate the flat branes of the original Randall-Sundrum model. The conformal distance L becomes small,

$$L \sim 2 \left(e^{-r_-/\ell} - e^{-r_+/\ell} \right) \quad (43)$$

The effective action for spin 1/2 fields, including the cocycle function, is then

$$W \sim \frac{1}{96} c_4 \left(e^{4r_+/\ell} + e^{4r_-/\ell} \right) - \frac{\zeta_R(5)}{16} L^{-4}. \quad (44)$$

In effect, there is an effective potential on the negative tension brane defined by

$$V = \frac{3W}{8\pi^2\ell^4 \sinh^4(r_-/\ell)} \quad (45)$$

We can express this potential in terms of a radion field $\sigma = (r_+ - r_-)/\ell$,

$$V \sim \frac{1}{16\ell^4} c_4 (1 + e^{4\sigma}) - A (e^\sigma - 1)^{-4} \quad (46)$$

where

$$A = -\frac{3\zeta_R(5)}{128\ell^4} \quad (47)$$

The result agrees with the flat space Casimir energy for spin 1/2 fields calculated previously [15, 20, 21]. Some of these old results absorbed the first two terms into shifts in the brane tension.

Another limiting case is when the brane separation is small, $(r_+ - r_-) \ll \ell \sinh(r_-/\ell)$. The conformal separation becomes

$$L \sim \frac{r_+ - r_-}{\ell \sinh(r_-/\ell)} \quad (48)$$

and is small in this limit. The Casimir energy

$$V \sim -A\sigma^{-4} \quad (49)$$

is independent of the five dimensional cosmological constant.

Finally, the limit $r_- \ll \ell$ corresponds to large conformal separation and approaches the single brane limit. In this case

$$L \sim -\log\left(\frac{r_-}{2\ell}\right) - \log \coth \frac{r_+}{2\ell} \quad (50)$$

The contribution to the effective action from the cylinder disappears as r_-^4 and we are left only with the cocycle function on the positive tension brane. The effective potential is then obtained by dividing by the finite 5-dimensional volume (using the actual metric) and this contribution still remains. Note, on the cylinder the effective potential is zero, in the one brane limit, because the conformal volume is infinite, i.e., the 5D volume is proportional to $1/L$.

VI. CONCLUSION

We have discussed the effective action on curved brane backgrounds for conformally coupled scalar and spinor fields. The final formulae, which can be found at the end of section 4, consist of a contribution from a region of the cylinder and a cocycle function resulting from a conformal rescaling of the background.

Perturbations around the conformal case from a mass have been considered in [12] on the cylinder background. These terms make small corrections to the effective action. A mass, or small correction to the conformal coupling, will also contribute to the cocycle function. However, its generic form does not change from the case presented here.

As we have seen, the cocycle function induces curvature terms and, in general, it is not possible to eliminate the cocycle function by renormalisation. However, some parts can be absorbed by renormalisation, for example as finite renormalisation of the brane tensions. Other parts of the cocycle function are additional curvature terms. It has been known for some time that quantum fields on curved brane backgrounds give induced curvature terms (for e.g., see [19] for induced terms in the RS model).

On the conformally related cylinder, the effective potential vanishes in the single brane limit. This is expected because, as first argued in [10], the background becomes a half-cylinder with infinite volume. However, using the actual metric the effective potential includes a contribution from the cocycle function, as we have shown. Note that the volume of the spacetime (after the usual Wick rotation of the time) is finite. Therefore, we must be careful in interpreting any result on the conformally related cylinder. For a two brane configuration, the effective action on the cylinder is non-zero, because the cylindrical region is compact, and the cocycle function does not vanish.

In tables I & II we also separated out the part due to the delta function in the background potential. A puzzling point is that the result is not the same as what one would obtain for a thin but finite domain wall, for which the cocycle function would vanish. Hence the thin wall limit will not agree with the case of the delta function potential. The physical reason behind this discrepancy is unclear. We leave this issue for future study. Until it is resolved, the result for the delta function potential should perhaps be taken with care. In this connection, we note that the thin wall limit in bubble nucleation was analyzed in [33], and no anomalous behaviour was found.

When considering backreaction for conformally coupled fields, the semi-classical Einstein equations are exact and the Casimir energy can play an important role in BWS [7, 8], including inflating branes. For flat branes this only depends on the difference between the number of bosons (N_b) and fermions (N_f), where $N_b = N_f$ gives the usual RS model and $N_b > N_f$ gives two positive tension branes, see [7]. For de Sitter branes we clearly see (compare Eq. (41) with Eq. (42)) that in general the contribution from scalar and fermion fields are not the same, as was argued in [9]. Thus, in this case, the bulk will not remain pure anti-de Sitter under back-reaction even for $N_b = N_f$.

As well as untwisted fields we also considered twisted fields (see appendix A). This case occurs when one brane obeys Dirichlet and the other Neumann boundary conditions. In fact, *a priori*, there seems to be no requirement that a scalar field should only satisfy untwisted boundary conditions. One could then envisage a two brane set up with the positive tension brane satisfying Neumann and the other negative tension brane satisfying Dirichlet boundary conditions. If this were the case then it should be possible to have a bound state localised on the positive tension brane even in a two brane set up.

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APPENDIX A: TWISTED FIELDS

For a spinor field it is also possible to have twisted as well as untwisted field configurations (see [20] for the flat brane case). In Section IIIC we considered the untwisted case. Here, we will double the modes, i.e. working on the topology $S^1 \times S^4$. In any case the zero modes cancel for mixed boundary conditions (see appendix B). The function $\zeta^{TM}(s)$, Eq. (26), now becomes (with mixed boundary conditions)

$$\begin{aligned} \zeta^{TM}(s) &= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} d(m) (c^2(n+1/2)^2 + (m+2)^2)^{-s}, \\ &= \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} d^M(m) \int_0^{\infty} dt t^{s-1} \exp\{-t[c^2(n+1/2)^2 + (m+2)^2]\}. \end{aligned} \quad (\text{A1})$$

We can express the twisted result in terms of the known result for the untwisted case by using the identity [25]

$$\theta_2(0, e^{-t}) = \theta_3(0, e^{-t/4}) - \theta_3(0, e^{-t}), \quad (\text{A2})$$

where $\theta_3(0, e^{-t})$ is defined in Eq. (B2). (This procedure is very similar to that used in finite temperature field theory, when considering thermal bosons and fermions, e.g., see [34].) Thus, applying the same steps as we did for the untwisted case (see the steps following Eq. (26)) we obtain

$$\begin{aligned} \zeta^{TM}(0) &= \frac{4}{3} \sqrt{\frac{\pi}{c^2}} \Gamma(-1/2) (\zeta(-4, 2) - \zeta(-2, 2)) \\ &\quad - 4c^{-1/2} \sum_{n=1, m=0}^{\infty} d^M(m) n^{-1/2} (m+2)^{1/2} K_{-1/2}[2\pi c^{-1} n(m+2)], \\ &\quad + 4\sqrt{2}c^{-1/2} \sum_{n=1, m=0}^{\infty} d^M(m) n^{-1/2} (m+2)^{1/2} K_{-1/2}[4\pi c^{-1} n(m+2)]. \end{aligned} \quad (\text{A3})$$

The one brane contribution is zero and the effective action is given by

$$W_{I \times S^4}^{TM} = \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n(\sinh 2nL)^4} - \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n(\sinh nL)^4}, \quad (\text{A4})$$

where we have divided by two to obtain the result on $I \times S^4$. It is simple to show that this result agrees with the flat brane result given in [20] for a twisted spinor field, by simply taking the small L limit.

APPENDIX B: ZERO MODES

Here we discuss a slight subtlety with boundary conditions. Note, by zero modes we mean the $n = 0$ modes in our mode sum and not null eigenvectors. In [9] we considered the case of a scalar field, but neglected the zero modes because from the one brane point of view it is more convenient to consider $S^1 \times S^4$, which has no boundaries. For the mode sum, over n , we argued that we should double the modes given that there are two copies of the bulk on either side of the brane, required to obtain a compact spacetime. However, from Elizalde [24],

$$\sum_{n=0}^{\infty} e^{-c^2 n^2} = \mp \frac{1}{2} + \frac{1}{2} \sqrt{\frac{\pi}{c^2}} + \sqrt{\frac{\pi}{c^2}} \sum_{n=1}^{\infty} \exp[-\pi^2 n^2 / c^2], \quad (\text{B1})$$

where the \mp refers to Dirichlet and Neumann boundary conditions respectively. For the case of a spinor field with mixed boundary conditions (with topology $I \times S^4$) these modes cancel each other out. Thus, doubling the modes in the above equation we obtain

$$\sum_{n=-\infty}^{\infty} e^{-c^2 n^2} = \mp 1 + \sqrt{\frac{\pi}{c^2}} \sum_{n=-\infty}^{\infty} \exp[-\pi^2 n^2 / c^2]. \quad (\text{B2})$$

This corresponds to $2I \times S^4$. In the case of $S^1 \times S^4$ the above equation reduces to Eq. (27), i.e. no zero modes.

As before, using Eq. (26),

$$\begin{aligned} \zeta(s) &= \mp \sum_{m=0}^{\infty} d(m) (m+3/2)^{-2s} + \sqrt{\frac{\pi}{c^2}} \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} d(m) \int_0^{\infty} dt t^{s-3/2} \exp\{-t(m+3/2)^2\} \\ &+ 2\sqrt{\frac{\pi}{c^2}} \frac{1}{\Gamma(s)} \sum_{n=1, m=0}^{\infty} d(m) \int_0^{\infty} dt t^{s-3/2} \exp\left[-\frac{\pi^2 n^2}{c^2 t} - (m+3/2)^2 t\right], \end{aligned} \quad (\text{B3})$$

where in the first term, we do not need to make use of the Mellin transform. Then integrating with respect to t

$$\begin{aligned} \zeta(s) &= \mp \zeta_{S_4}(s) + \sqrt{\frac{\pi}{c^2}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_{S_4}(s-1/2) \\ &+ \frac{4\pi^s}{\Gamma(s)} c^{-s-1/2} \sum_{n=1, m=0}^{\infty} d(m) n^{s-1/2} (m+3/2)^{-s+1/2} K_{s-1/2}[2\pi c^{-1} n(m+3/2)], \end{aligned} \quad (\text{B4})$$

Again, apart from $\zeta_{S_4}(s)$, it is easy to verify that $\zeta(0) = 0$. Whence,

$$\begin{aligned} \zeta'(0) &= \mp \zeta_{S_4}'(0) + \frac{1}{3} \sqrt{\frac{\pi}{c^2}} \Gamma(-1/2) (\zeta(-4, 3/2) - \zeta(-2, 3/2)) \\ &+ 4c^{-1/2} \sum_{n=1, m=0}^{\infty} d(m) n^{-1/2} (m+2)^{1/2} K_{-1/2}[2\pi c^{-1} n(m+3/2)], \end{aligned} \quad (\text{B5})$$

where a prime denotes differentiation with respect to s . As expected the second term is exactly the contribution from the one brane configuration, where we have used the identity,

$$\frac{d}{ds} \left[\frac{2}{\sqrt{4\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_{S_4}(s-1/2) \right]_{s=0} = \frac{d}{ds} \left[\frac{1}{3} \frac{L}{\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \left(\zeta(2s-4, 3/2) - \frac{1}{4} \zeta(2s-2, 3/2) \right) \right]_{s=0} = 0, \quad (\text{B6})$$

with $c = \pi/L$. Thus, in the one brane limit the effective potential reduces to the Casimir energy on S^4 , as pointed out in [10]. This relation can also be found in [14], except for a factor of 2 because of our mode doubling.

The third term is non-zero and depends on the value of $c = \pi/L$,

$$\frac{1}{2} \zeta'(0) = \sum_{n=1, m=0}^{\infty} \frac{1}{3} \frac{(m+1)(m+2)(m+3/2)}{n} \exp[-2Ln(m+3/2)], \quad (\text{B7})$$

where we have used the relation $K_{-1/2}(z) = \sqrt{\pi/(2z)} e^{-z}$ and Eq. (9). After evaluating the summation over m we have an expression for the scalar one-loop effective action on the cylinder,

$$W_{2I \times S_4} = \frac{1}{8} \sum_{n=1}^{\infty} \frac{\cosh nL}{n(\sinh nL)^4}. \quad (\text{B8})$$

Thus, since the one brane contribution is zero,

$$W_{2I \times S_4}^{D,N} = \pm \frac{1}{2} \zeta'_{S_4}(0) \pm \frac{1}{2} \zeta_{S_4}(0) \log \mu^2 - \frac{1}{8} \sum_{n=1}^{\infty} \frac{\cosh nL}{n(\sinh nL)^4} = 2 W_{I \times S_4}^{D,N}, \quad (\text{B9})$$

with Dirichlet (D) and Neumann (N) boundary conditions, with

$$\zeta_{S_4}(s) = \frac{1}{3} \left(\zeta(2s-3, 3/2) - \frac{1}{4} \zeta(2s-1, 3/2) \right), \quad (\text{B10})$$

where $\zeta(s, a)$ is a Hurwitz ζ -function. Thus,

$$\zeta_{S_4}(0) = \frac{1}{3} \left(\zeta(-3, 3/2) - \frac{1}{4} \zeta(-1, 3/2) \right) = -\frac{17}{2880} \quad (\text{B11})$$

and using the standard relation

$$\zeta'(s, 3/2) = 2^s \log 2 \zeta(s) + (2^s - 1) \zeta'(s) - 2^s \log 2 \quad (\text{B12})$$

we find that

$$\begin{aligned} \zeta'_{S_4}(0) &= \frac{1}{3} \left(\zeta'(-3, 3/2) - \frac{1}{4} \zeta'(-1, 3/2) \right), \\ &= \frac{1}{24} \log 2 \zeta(-3) - \frac{7}{24} \zeta'(-3) - \frac{1}{24} \log 2 - \frac{1}{24} \log 2 \zeta(-1) + \frac{1}{24} \zeta'(-1) + \frac{1}{24} \log 2, \\ &= -\frac{7}{24} \zeta'_R(-3) + \frac{1}{24} \zeta'_R(-1) + \frac{1}{24} \zeta_R(-3) \log 2 - \frac{1}{24} \zeta_R(-1) \log 2, \end{aligned} \quad (\text{B13})$$

where ζ_R is the Riemann zeta function.

This is the first proper treatment of the zero modes for S^4 branes in an AdS_5 bulk. Similar results have been obtained in lower dimensions by Dowker and Apps [14].

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- [35] E-mail: ian.moss@ncl.ac.uk
- [36] Current e-mail address: wade@yukawa.kyoto-u.ac.jp
- [37] E-mail: g.w.santiago-german@ncl.ac.uk
- [38] Current e-mail address: misao@yukawa.kyoto-u.ac.jp