Close approach of a spherical particle and a quantised vortex in helium II

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To interpret the results of recent experiments which used the Particle Velocimetry (PIV) method it is necessary important to understand the interaction of the particles with the quantised vortices. We present analytical and numerical calculations of the close approach of a small spherical particle to a vortex line. The present results of a dynamically self–consistent numerical calculation of the close approach of a small particle to a vortex line. The trapping time scale compares will with approximate analytic results.

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1. INTRODUCTION

Until recently the study of superfluid turbulence[1] has been held back by the lack of direct visualisation methods which are routine in ordinary turbulence at room temperature. A recent breakthrough has been the implementation of the Particle Image Velocimetry (PIV) method [2]. In this method, images of the positions of micron–size tracer particles at successive times are taken using lasers, and software reconstruct the flow pattern.

In a previous paper[3] we showed that, depending on the parameter regime, the small particles trace either the normal fluid, or the superfluid, or neither; we also showed that the particles can become trapped in quantised vortex lines. Clearly, the interpretation of PIV images is very different if what is detected is the superfluid velocity, the normal fluid velocity or the quantised vortices. Our concern in this paper is the trapping process in the simplest possible situation: zero temperature (so that viscous drag with the normal fluid can be neglected), neutrally buoyant particles (so that Archimedean forces can be neglected), particles and vortex initially at rest.

2. ANALYTICAL MODEL

At zero temperature, the equations of motion[3] of a small, spherical, neutrally buoyant particle of radius a (smaller than any flow scale), position $\mathbf{r}_p(t)$, velocity $\mathbf{u}_p(t)$ are $d\mathbf{r}_p/dt = \mathbf{u}_p$ and

$$\frac{d\mathbf{u}_p}{dt} = \frac{\partial \mathbf{V}_s}{\partial t} + (\mathbf{V}_s \cdot \nabla)\mathbf{V}_s, \tag{1}$$

where t is time and $\mathbf{V}_s$ the superfluid velocity. In the case of a straight vortex line, using cylinder coordinates (r, ϕ, z), $\mathbf{V}_s = (0, \kappa/(2\pi r), 0)$ where κ is the quantum of circulation. If the vortex remains straight and stationary we have

$$\frac{du_p}{dt} = -\frac{\kappa^2}{4\pi^2 r^3}, \tag{2}$$

where $r_p(t)$ and $u_p(t)$ are the particle's radial position and velocity. The solution corresponding to the initial condition $r_p(0) = r_0$ and $u_p(0) = 0$ is

$$r_p(t) = \sqrt{r_0^2 - \frac{\kappa^2 t^2}{4\pi^2 r_0^2}}, \tag{3}$$

hence the sphere reaches the vortex at the trapping time

$$t_{trap} = \frac{2\pi r_0^2}{\kappa}. \tag{4}$$

3. NUMERICAL MODEL

Our numerical model computes self–consistently the motion of the sphere and the vortex, taking into account (unlike the previous analytical model) the finite size of the sphere as well as the bending and the motion of the vortex. Some of the details of the numerical analysis are subtle and will be described elsewhere[4] - only the main features are described here.

Let $\mathbf{X}_s(\xi, t)$ be the position vector along the vortex where ξ is arclength. The equation of motion is[5]

$$\frac{\partial \mathbf{X}_s}{\partial t} = \mathbf{V}_s + \mathbf{V}_b + \mathbf{V}_\phi, \tag{5}$$

The first contribution at the RHS is the superfluid velocity $\mathbf{V}_s$ which is induced at the point $\mathbf{X}_s$ by the curvature according to the Biot–Savart law:

$$\mathbf{V}_s(\mathbf{X}_s) = -\frac{\kappa}{4\pi} \int_{\mathcal{L}} d\xi \frac{\mathbf{X}_s' \times (\mathbf{X}_s - \mathbf{x})}{|\mathbf{X}_s - \mathbf{x}|^3}. \tag{6}$$

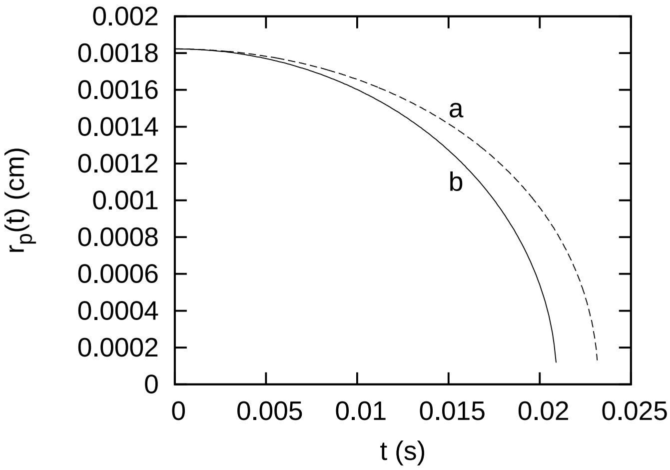

Fig. 1. Distance between sphere and vortex versus time. (a): analytical model; (b): numerical model (what is plotted is actually y_p, not r_p, but the difference is negligible until the distance between sphere and vortex is of the order of the sphere's diameter 0.002083 cm).

where the integral extends along the vortex line and $\mathbf{X}'_s = \partial \mathbf{X}_s/\partial\xi$ is the unit tangent vector to the vortex line. The second contribution arises because, at any boundary, the normal component of the superfluid velocity must be equal to the normal component of the velocity of the boundary. This effect is taken into account by $\mathbf{V}_b$ according to a method of Schwarz[6]: $\mathbf{V}_b = \nabla\psi$, where the Laplace equation $\nabla^2\delta\psi = 0$ determines the potential $\delta\psi$ that corresponds to the velocity induced by a vortex element $\delta\mathbf{l}$ corresponding to the discretization along the vortex; ψ is then recovered by summing over the discrete vortex elements. The third contribution, $\mathbf{V}_\phi$ is the velocity of the fluid at $\mathbf{x}$ because of the presence of a sphere of radius a at $\mathbf{r}_p$:

$$\mathbf{V}_\phi(\mathbf{x}, t|\mathbf{r}_p) = -\frac{1}{2}\left(\frac{a}{r}\right)^3 \mathbf{u}_p(\mathbf{r}_p)\cdot\mathbf{T}, \tag{7}$$

where the tensor $\mathbf{T}$ has components $T_{ij} = \delta_{ij} - 3x'_i x'_j/R^2$ where δ_{ij} is Kronecker's delta, x'_i are the components of $\mathbf{x}' = \mathbf{x} - \mathbf{r}_p$ and $R = ||\mathbf{x} - \mathbf{r}_p||^2$.

The sphere moves according to[6] $d\mathbf{r}_p/dt = \mathbf{u}_p$ and

$$m_{eff}\frac{d\mathbf{u}_p}{dt} = 2\pi\rho_s a^3 \frac{\partial \mathbf{V}_s(\mathbf{r}_p)}{\partial t} + \frac{1}{2}\int_S (\mathbf{V}_s + \mathbf{V}_b)^2 \hat{\mathbf{n}} dS, \tag{8}$$

where the integral extends over the sphere, $m_{eff} = m + 2\pi\rho_s a^3/3$ is the effective mass and $m = 4\pi\rho_s a^3/3$.

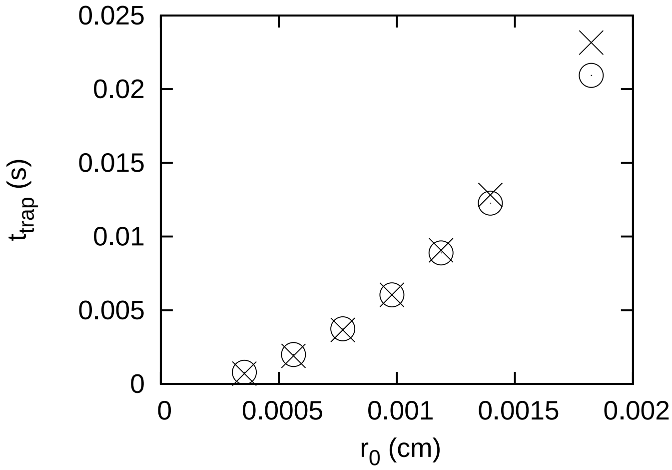

Fig. 2. Trapping time versus initial distance. crosses: numerical model; circles: Eq. (4).

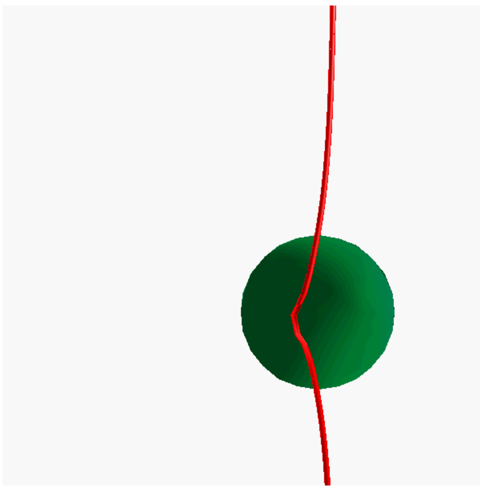

Fig. 3. Vortex line and sphere at the moment of trapping.

Close approach of a sphere and a quantised vortex

The initial condition consists of a straight vortex placed along the z axis and a sphere placed at distance r_0 from the origin along the y axis; the initial velocity of the sphere is zero. The calculation is performed in a periodic box of size 35 times the sphere's radius $a = 1.04 \times 10^{-4}$ cm (typical of PIV experiments). The number N of Legendre functions[6] used to determine $\mathbf{V}_b$ to satisfy the boundary condition is found by an adaptive algorithm which averages the angle between the normal (radial) direction and the fluid's velocity over the sphere and requires that the average angle is close to 90 degrees within one thousandth of a degree. N varies from $N \approx 100$ when the sphere is far from the vortex to $N \approx 700$ when it is close. The discretization length along the superfluid vortices is $\Delta\xi = 2.083 \times 10^{-5}$cm, which is a tenth of the sphere's diameter; this guarantees enough grid points to resolve the deformation of the vortex induced by the sphere.

Fig. (1) compares (a) the numerically computed distance between the sphere and the vortex versus time for $r_0 = 0.0018$ cm with (b) the distance from Eq. (3). Fig. (2) compares (crosses) the numerically computed trapping time with (circles) Eq. (4). The agreement is good, despite the simplicity of the analytic model. The reason is that the vortex is only slightly deformed by the sphere; the deformation is localised in the vortex region facing the sphere and occurs only when the sphere is at a distance to the vortex of the order of the sphere's diameter, in agreement with a calculation performed by Berloff and Roberts[7] of the capture of an impurity by a vortex line in a Bose–Einstein condensate using the Gross–Pitaevskii nonlinear Schroedinger equation. It is interesting to remark that, if the sphere is not allowed to move and only the velocity $\mathbf{V}_b$ is used to move the vortex, the vortex deforms locally (in the region facing the sphere) in the negative x direction and as if it tried to avoid the sphere, while the regions of the vortex far from the sphere remain unchanged. If $\mathbf{V}_b$ and $\mathbf{V}_s$ are used to move the vortex, the vortex moves towards the negative x axis travelling along an arc toward the sphere while remaining almost straight; this is because the Biot–Savart dynamics allows the relaxation of the deformation induced by $\mathbf{V}_b$.

4. CONCLUSION

In conclusion, we have studied in a dynamically self–consistent way the close approach between a quantised vortex and a spherical particle which is initially placed at some given distance from the vortex with zero initial speed; the case which we considered refers to absolute zero (to neglect the Stokes drag force with the normal fluid) and for a particle's density equal to helium's density (to neglect buoyancy force). We have found that the sphere's trajectory and trapping time scale are in good agreement with a simple analytical model which neglects motion and curvature of the vortex. Since the final deformation of the vortex is of the order of the sphere's diameter only, we conclude that the interaction is essentially geometrical in character. Work is in progress to study the interaction between sphere and vortex under different initial conditions (eg nonzero initial velocity of the sphere) and finite temperature effects.

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